Discussion 2

January 16, 2009

Exercise 0.1. Compute

$$\int_0^1 \int_{y+2}^{y^2+4} 2xy - 1 \; dx \; dy$$

and describe the region you are integrating over.

$$\begin{aligned}
&\int_0^1 \int_{y+2}^{y^2+4} 2xy - 1 \; dx \; dy \\
&= \int_0^1 x^2 y - x \Big|_{y+2}^{y^2+4} \; dy \\
&= \int_0^1 [(y^2+4)^2 y - (y^2+4)] - [(y+2)^2 y - (y+2)] \; dy \\
&= \int_0^1 y^5 + 7y^3 - 5y^2 + 13y - 2 \; dy \\
&= \frac{1}{6}y^6 + \frac{7}{4}y^4 - \frac{5}{3}y^3 + \frac{13}{2}y^2 - 2y \Big|_0^1 \\
&= \frac{1}{6} + \frac{7}{4} - \frac{5}{3} + \frac{13}{2} - 2 \\
&= \text{something.}
\end{aligned}$$

The region looks sort of like a parallelogram. Its vertices are at $(2,0)$, $(4,0)$, $(3,1)$, and $(5,1)$. The edges are straight lines from $(2,0)$ to $(4,0)$, from $(2,0)$ to $(3,1)$, from $(3,1)$ to $(5,1)$, and a curve from $(5,1)$ to $(4,0)$ that is part of the graph of $x = y^2$.

Exercise 0.2. Evaluate

$$\iint\limits_R \frac{1}{\sqrt{xy+4}}\, dA$$

where R is the region bounded by $x = 0$, $x = 5$, $y = 0$, and $y = x + 4$.

Okay, this region is type 3. In fact it is a quadrilateral with vertices at $(0,0)$, $(5,0)$, $(0,4)$, and $(5,9)$. It is clear that the inequalities this region satisfy are $x \geq 0$, $x \leq 5$, $y \geq 0$, $y \leq x+4$. You can probably just eyeball this one, but lets try being more systematic. Suppose we try doing horizontal slices. Then we want to parametrize the horizontal endpoints by y and we see they are of the form $[0,5]$ or $[x+4,5]$ depending on whether 0 or $y-4$ is larger (which changes around $y = 4$). The shadow/projection onto the y-axis is found by writing down all the above inequalities which involve y and not x and also considering modifications of the ones that involve both x and y as follows. For an example, the inequalities we would have are $y \geq 0$ (involves y and not x) and then $y-4 \leq x \leq 5 \rightsquigarrow y-4 \leq 5$ (which is the same as $y \leq 9$). So our integral would be

$$\int_0^9 \int_0^{x(y)} \frac{1}{\sqrt{xy+4}}\, dx\; dy.$$

where $x(y)$ is a piecewise defined function, so for computations sake we'd have to break this up into two separate integrals:

$$= \int_0^4 \int_0^5 \frac{1}{\sqrt{xy+4}}\, dx\; dy \;+\; \int_4^9 \int_{x+4}^5 \frac{1}{\sqrt{xy+4}}\, dx\; dy.$$

Instead we could integrate using vertical slices. In that case, we would get instead

$$\int_0^5 \int_0^{x+4} \frac{1}{\sqrt{xy+4}}\, dy\; dx$$

$$= \int_0^5 \frac{2\sqrt{xy+4}}{x}\Big|_0^{x+4}\, dx$$

$$= \int_0^5 \frac{2\sqrt{x^2+4x+4}}{x} - \frac{4}{x}\, dx$$

$$= \int_0^5 \frac{2\sqrt{(x+2)^2}}{x} - \frac{4}{x}\, dx$$

$$= \int_0^5 \frac{2|x+2|}{x} - \frac{4}{x}\, dx$$

and since $x > -2$ over this domain of integration the absolute value is superfluous (thanks to Rui for pointing this out)

$$= \int_0^5 \frac{2x+24}{x} - \frac{4}{x}\, dx$$

$$= \int_0^5 2\, dx = 10.$$

Exercise 0.3. Set up an integral to find the area of the region enclosed in $y = 5 - x^2$, $y = -2$, and $y = x - 1$ that contains the point $(0, 0)$.

Okay, so in the morning section I had an extra function listed which made the problem too hard, and even by the afternoon section I was ambiguous about the region because I failed to say something like "the region must contain $(0, 0)$". Otherwise there are a number of regions that are cut out by these equations....

To find the area, we want to integrate the unit function over R. When you draw the region out, you will see that it is both vertically and horizontally sliceable. If we use vertical slices we get

$$\int_{-\sqrt{7}}^{-1} \int_{-2}^{5-x^2} dy\, dx \;+\; \int_{-1}^{2} \int_{x-1}^{5-x^2} dy\, dx$$

If we use horizontal slices we get

$$\int_{-2}^{1} \int_{-\sqrt{5-y}}^{y+1} dx\, dy \;+\; \int_{1}^{5} \int_{-\sqrt{5-y}}^{\sqrt{5-y}} dx\, dy$$

Exercise 0.4. Compute

$$\int_0^\pi \int_y^\pi \frac{\sin x}{x}\, dx\, dy$$

Good luck trying to integrate $\frac{\sin x}{x}$ directly with respect to x. The trick here is to switch the order of integration. We were using horizontal slices, but lets see what happens when we switch to vertical slices. To do this, we must figure out the region we are integrating over. It is a triangle with vertices at $(0, 0)$, $(\pi, 0)$, and (π, π). Then Fubini's theorem allows us to change the order of integration:

$$= \int_0^\pi \int_0^x \frac{\sin x}{x}\, dy\, dx$$

$$
\begin{aligned}
&= \int_0^\pi \left(\frac{y\text{sin}x}{x}\Big|_{y=0}^{x}\right) \, dx \\
&= \int_0^\pi \frac{x\text{sin}x}{x} - 0 \, dx \\
&= \int_0^\pi \text{sin}x \, dx \\
&= 2.
\end{aligned}
$$