Discussion 1

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1 How to write down an integral over a complicated region that is too hard to draw

Lets just do this example-style. Let R be the region defined by the inequalities

$$y - 4 \leq x$$
$$y \leq 9 - x^2$$
$$8x \geq y^2 - 40$$
$$x \leq 3$$

We'll integrate a function f over this region in the order $dy\,dx$. This means we'll need to find the projection of R on the x-axis and give the upper and lower bounds of integration in the y variable in terms of x. Here is the recipe. Just as R is given as the set of solutions to the inequalities written above, we'll write the projection of R on the x-axis as a set of solutions to certain inequalities. Here are the inequalities that we'll need:

1) All the inequalities for R that DON'T involve the variable we're integrating. In our case the only equation that does not involve y is $x \leq 3$. In some problems there will be no inequalities that fall into this category.

2) A monster inequality that is related to the inequalities for R that DO involve the variable we're integrating (in this case y).

Here is how we find out what the monster inequality is. First solve all the inequalities that involve y for y. Doing this yields:

$$y \leq x + 4$$

$$y \leq 9 - x^2$$
$$y \leq \sqrt{8x + 40}$$
$$-\sqrt{8x + 40} \leq y$$

Notice that solving $8x \geq y^2 + 40$ for y actually yields two inequalities. The monster inequality will now be

$$Max(-\sqrt{8x + 40}) \leq Min(x + 4, 9 - x^2, \sqrt{8x + 40}).$$

The lefthand side should be the Maximum over all the functions of variables other than y that are less than y. The righthand side should be the Minimum of the functions that are greater than y. The lefthand side should be $\leq$ than the right. In some cases, there will be only one function you're taking the Maximum or Minimum of. Don't worry, that's fine, it just means the problem will be easier! However, you should be worried if there are no functions you're taking the Max of or Min of. In that case the wheels fell off somewhere earlier on (or perhaps you are integrating over a region which is not bounded).

Ok, so now to summarize, the projection of R onto the x-axis is given as the solution set to the equations

$$x \leq 3 \text{ and } Max(-\sqrt{8x + 40}) \leq Min(x + 4, 9 - x^2, \sqrt{8x + 40}).$$

The monster inequality is needed not just for describing the projection of R on the x-axis, but also shows up in the upper and lower bounds of integration over the y variable. In fact our integral can be written as

$$\int_{\text{Proj. of } R \text{ on x-axis}} \int_{Max(-\sqrt{8x+40})}^{Min(x+4, 9-x^2, \sqrt{8x+40})} f \; dy \; dx$$

In this particular case, since the lower bound involves the Max over just one function we can simplify to

$$\int_{\text{Proj. of } R \text{ on x-axis}} \int_{-\sqrt{8x+40}}^{Min(x+4, 9-x^2, \sqrt{8x+40})} f \; dy \; dx$$

As an aside to the thinkers among you, it might be nice to realize that since all points in the projection of W on the x-axis satisfy the monster inequality, the bounds for integration in the y-variable make sense (i.e. the upper bound is indeed bigger than the lower bound).

Okay, now the bounds on the inner integral abstractly make sense, but are of little use to actually compute with. We need to figure out where the Minimum and Maximum are attained. The maximum is over just one function so we don't have to worry about him. The minimum on the other hand is another story. Lets start by finding where $x + 4 \le 9 - x^2$.

$$x^2 + x - 5 \le 0$$

$$x \in [\frac{-1 - \sqrt{21}}{2}, \frac{-1 + \sqrt{21}}{2}].$$

Okay, it remains to compare $x+4$ with $\sqrt{8x + 40}$ when $x \in [\frac{-1-\sqrt{21}}{2}, \frac{-1+\sqrt{21}}{2}]$ and we must compare $9 - x^2$ with $\sqrt{8x + 40}$ when $x \notin [\frac{-1-\sqrt{21}}{2}, \frac{-1+\sqrt{21}}{2}]$.

In the first case, $x + 4$ is positive, so it suffices to compare the square of $x + 4$ with the square of $\sqrt{8x + 40}$

$$x^2 + 8x + 16 \le 8x + 40$$

$$x^2 - 24 \le 0$$

which is satisfied when $x \in [-\sqrt{24}, \sqrt{24}]$. The intersection of this interval with $[\frac{-1-\sqrt{21}}{2}, \frac{-1+\sqrt{21}}{2}]$ is $[\frac{-1-\sqrt{21}}{2}, \frac{-1+\sqrt{21}}{2}]$ again, so on the interval $[\frac{-1-\sqrt{21}}{2}, \frac{-1+\sqrt{21}}{2}]$ the minimum is attained by $x + 4$.

In the second case, we may assume that $x \ge -5$ because the monster inequality fails to make sense outside this region. Also we may only bother to check when $x \le 3$ because that was one of our original constraints on R. For $x \in [-5, -3]$, $9 - x^2$ is negative, so it is clearly smaller than $\sqrt{8x + 40}$. Otherwise, when $x \in [-3, 3]$ both functions are positive and we compare their squares:

$$x^4 - 18x^2 + 81 \le 8x + 40$$

$$x^4 - 18x^2 - 8x + 41 \le 0$$

One may check that this inequality holds when $x \in [-3, \frac{-1-\sqrt{21}}{2}]$ and when $x \in [\frac{-1+\sqrt{21}}{2}, 3]$ so in fact $9 - x^2$ is always smaller than $\sqrt{8x + 40}$ when $x \in [-5, \frac{-1-\sqrt{21}}{2}] \cup [\frac{-1+\sqrt{21}}{2}, 3]$, so $9 - x^2$ attains the minimum.

$$\int_{-5}^{\frac{-1-\sqrt{21}}{2}} \int_{-\sqrt{8x+40}}^{9-x^2} f\ dy\ dx + \int_{\frac{-1-\sqrt{21}}{2}}^{\frac{-1+\sqrt{21}}{2}} \int_{-\sqrt{8x+40}}^{x+4} f\ dy\ dx$$

$$+\int_{\frac{-1+\sqrt{21}}{2}}^{3}\int_{-\sqrt{8x+40}}^{9-x^2} f\, dy\, dx$$

Whew. If you're up for a challenge, try integrating over this region in the order $dx\, dy$. Let me know if you need help....warning, the numbers will be as ugly as they were here.