Discussion 1

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1 Wtf is an integral in higher dimensional (loose version)

From hisicool we learned how to take the integral of a real valued function of one variable. The definition of the integral $\int_a^b f\ dx$ was given by approximation of Riemann sums. We split the interval $[a, b]$ into N smaller intervals $[a, a + \frac{b-a}{N}], [a + \frac{b-a}{N}, a + 2\frac{b-a}{N}] ..., [b - \frac{b-a}{N}, b]$ and considered the sum

$$\sum_{j=1}^{N} \frac{1}{N}\ f(x_j).$$

The x_j is a point chosen at random that lies in the j^{th} interval. Instead of writing $\frac{1}{N}$ we could have written Length(i^{th} interval).

The integral $\int_a^b f\ dx$ is defined to be the limit of this sum as we force all the boxes to become smaller and smaller. As a definition, this only makes sense if you always get the same answer regardless of which points x_j you pick and how you let the boxes get smaller and smaller. Mathematicians spent lots of time fretting over this stuff. In fact, there are many, many functions for which the integral is not well defined, meaning that if you were to have picked different points x_j the Riemannian approximation would have limited to a different number, or no number at all (it is conceivable that the numbers you get from calculating the Riemann approximations for finer and finer interval meshes are 1,-1,1,-1,... which doesn't have a limit).

Anyway, the good news is that many of the functions you come across in your everyday life are integrable. In particular, the integral is well-defined for any piece-wise continuous function over any interval of *finite* length.

From the definition of Riemann approximations, we can see that the integral tells us about the signed area under the graph of f.

Now lets roll up to the higher dimensional world and think about the integral of a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ over a rectangle $R \subset \mathbb{R}^2$. Again, the way to define the integral is by chopping R up into tiny smaller rectangles $R_1, ..., R_N$ and looking at

$$\sum_{j=1}^{N} \text{Area of } j^{th} \text{ rectangle} \times f(x_j).$$

where x_j is a point chosen at random in the j^{th} rectangle. The integral of f over R is the limit (provided it exists) of this sum as we let N get large and the mesh of rectangles get finer and finer.

This higher dimensional integral behaves much like its lower dimensional counterpart. Here are some basic facts about the higher dimensional integral. Notice these facts are shared with the integrals little one dimensional brethren.

· It represents the signed volume under the graph of f

· There are crazy functions for which the Riemann sums don't approximate anything, so the integral doesn't make sense.

· The higher dimensional integral satisfies the same linearity properities as the 1-dimensional integral.

· The higher dimensional definite integral is well-defined for any piece-wise countinuous function.

2 The Iterated Integral

Given a real valued function of n real variables $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and a rectangular region $[a_1, b_1] \times \cdots \times [a_n, b_n]$ we may calculate the iterated integral of f over this region. The calculation will involve taking n regular old high school integrals of one variable.

Lets see how the iterated integral works with an example. when $n = 2$.

Example 2.1. Let

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$(x, y) \mapsto x^2 - 3y^2$$

and let R be the rectangular subset of $\mathbb{R}^2$ defined by

$$R = \{\ (x, y) \mid -1 \leq x \leq 1 \text{ and } -2 \leq y \leq 2\ \}.$$

For each $x_0 \in [-1, 1]$ there is a slice, which we call R_{x_0}, consisting of those points $(x_0, y) \in R$ i.e. the first component is fixed at x_0 and the second component may range from -2 to 2. The function f restricted to R_{x_0} is now a function of one variable, since the x coordinate is fixed at the constant value x_0. We write this like:

$$f_{x_0} : [-2, 2] \longrightarrow \mathbb{R}$$
$$y \mapsto f(x_0, y) = x_0^2 - 3y^2.$$

As f_{x_0} is a function of one variable, we may take its integral on $[-2, 2]$ which of course gives us the signed area under the graph of f_{x_0}.

$$\int_{-2}^{2} f_{x_0}\, dy$$
$$= \int_{-2}^{2} x_0^2 - 3y^2\, dy$$
$$= x_0^2 y - y^3 \Big|_{y=-2}^{2}$$
$$= (2x_0^2 - 8) - (-2x_0^2 + 8)$$
$$= 4x_0^2 - 16.$$

The main point here is that x_0 is a constant, not a variable in this calculation. Now we can repeat this calculation for any $x_0 \in [-1, 1]$. Doing this gives us a function.

$$G : [-1, 1] \longrightarrow \mathbb{R}$$
$$x \mapsto \int_{-2}^{2} f_{x_0}\, dy = 4x^2 - 16$$

Since x is no longer fixed, we stripped it of $_0$ status.

Okay then. The iterated integral is what we get when we integrate G with respect to x.

$$\int_{-1}^{1} G\,dx$$

$$= \int_{-1}^{1} 4x^2 - 16\,dx$$

$$= \frac{4}{3}x^3 - 16x\,\Big|_{x=-1}^{1}$$

$$= 2(\frac{4}{3} - 16).$$

In summary what we have done is evaluated two integrals, one nested inside the other:

$$\int_{-1}^{1} \Big(\int_{-2}^{2} f(x,y)\,dy \Big) dx.$$

First we evaluted the inner integral treating x as a constant and integrated with respect to y, thus leaving us with a function of x. Then we integrated with respect to x.

Now, there was nothing sacrosanct about y over x. We could have just as well integrated first along horizontal slices R_{y_0}. This computation would be calculating

$$\int_{-2}^{2} \Big(\int_{-1}^{1} f(x,y)\,dx \Big) dy.$$

Lets check that we get the same answer as before. To that end we must first evaluate the inner integral with respect to x treating y as a constant.

$$= \int_{-2}^{2} \Big(2 \int_{0}^{1} f(x,y)\,dx \Big) dy.$$

because $x^2 - 3y^2$ is an even function with respect to the x variable

$$= 2 \int_{-2}^{2} \Big(\frac{1}{3}x^3 - 3y^2 x\Big|_{x=0}^{1}\Big)\,dy$$

here we integrated with respect to x and treated y as a constant.

$$= 2 \int_{-2}^{2} (\frac{1}{3} - 3y^2) - (\frac{1}{3}0 - 3y^2 0)dy$$

$$= 2 \int_{-2}^{2} \frac{1}{3} - 3y^2\,dy$$

$$= 2\times 2\int_{-2}^{2}\frac{1}{3} - 3y^2\,dy$$

because $\frac{1}{3} - 3y^2$ is an even function.

$$= 4(\frac{1}{3} - 3y^2|_{y=0}^{2})$$

$$= 4(\frac{2}{3} - 8)$$

which is the answer we got above.

Now, how (how now brown cow) do we know that this calculation we just did has any relevance to the actual integral defined using Riemannian approximation?

3 The actual integral and the iterated integral are the same....mostly

The title of this section is what is more officially referred to as Fubini's theorem.

Theorem 3.1. *Let $R = [a_1, b_1]\times\cdots\times[a_n, b_n]$ be a rectangular region in $\mathbb{R}^n$ and suppose $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a function whose integral over R exists (i.e. is well defined), then*

$$\int_R f\,dA = \int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1} f\,dx_1\ \cdots\ d_n.$$

It also doesn't matter in what way we order the variables when we compute the iterated integral.

Important! This is really, really important for us. It says that whenever we want to compute an integral, provided we know the integral is well-defined, we might as well use the much easier iterated integral to compute it.

Warning: It is a logical fallacy to compute an iterated integral, get a valid answer, and then conclude that the function was integrable to begin with. The examples where this fails is when f is one of these badly behaved functions that can't be well approximated by Riemann sums.

When R is bounded, f will have to be pretty gnarly. If R is not bounded, i.e $R = \mathbb{R}^2$, then there are many, many functions f which are not integrable because there is an infinite subregion of $\mathbb{R}^2$ on which f is both largely positive and largely negative. This includes many piece-wise continuous and even continuous functions! Lets see an example. Please try to work through this one because although we won't be spending our time in 52 worrying about non-integrable functions, the iterated integral calculations are excellent practice in visualizing functions.

Exercise 3.2. Let $R = \mathbb{R}^2$. Let f be the piece-wise continuous function

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$(x, y) \mapsto \begin{cases} 1 & \text{if } x \leq y < x + 1 \text{ and } x, y \geq 0; \\ -1 & \text{if } x + 1 \leq y < x + 2 \text{ and } x, y \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Show that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f \; dx \; dy \neq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f \; dy \; dx.$$

Okay, really you should draw out pictures to help you see this function. Lets compute $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f \; dy \; dx$. To that end we must compute the inner integral $\int_{-\infty}^{\infty} f \; dy$ along every vertical slice $x = x_0$. Check that when $x_0 \geq 0$, the function $f_{|x=x_0}$ along one of these vertical slices is given by

$$y \mapsto \begin{cases} 1 & \text{if } x_0 \leq y < x_0 + 1 \text{ and } y > 0; \\ -1 & \text{if } x_0 + 1 \leq y < x_0 + 2 \text{ and } y > 0; \\ 0 & \text{otherwise.} \end{cases}$$

and when $x_0 < 0$, $f_{|x=x_0}$ is identically zero. Either way, both kinds of functions have integrals which are equal to zero. In otherwords the function

$$G : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x_0 \mapsto \int_{-\infty}^{\infty} f_{|x=x_0} \; dy$$

is the zero function. Thus

$$\int_{-\infty}^{\infty} \Big(\int_{-\infty}^{\infty} f \; dy \Big) \; dx$$
$$= \int_{-\infty}^{\infty} 0 \; dx$$
$$= 0.$$

Now lets compute the iterated integral in the other order. To that end we must compute the inner integral $\int_{-\infty}^{\infty} f\ dx$ along every horizontal slice $y = y_0$. Check that when $y_0 < 0$, the function $f_{|y=y_0}$ along one of these horizontal slices is identically zero. Check that when $y_0 \geq 0$ the function $f_{|y=y_0}$ is given by the formula

$$x \mapsto \begin{cases} 1 & \text{if } y_0 - 1 \leq x < y_0 \text{ and } x > 0; \\ -1 & \text{if } y_0 - 2 \leq x < y_0 - 1 \text{ and } x > 0; \\ 0 & \text{otherwise.} \end{cases}$$

both kinds of functions have integrals which are equal to zero provided $y \notin (0,2)$. In otherwords the function

$$\begin{aligned} H : \mathbb{R} &\longrightarrow \mathbb{R} \\ y_0 &\mapsto \int_{-\infty}^{\infty} f_{|y=y_0}\ dx \end{aligned}$$

is equal to zero if $y \in (-\infty, 0] \cup [2, \infty)$.

Now what is value of this function when $y \in (0,2)$? If $y_0 \in (0,1]$, then we may check that $f_{y=y_0}$ takes the value 1 on the interval $[0, y_0]$ and is zero everywhere else (There is no place where it is -1 because in those places $x < y - 1 < 0$ but we also need $x > 0$). So if $y \in (0,1]$

$$H(y_0) = \int_{-\infty}^{\infty} f_{|y=y_0}\ dx = \int_0^{y_0}\ dx = x\Big|_0^{y_0} = y_0$$

If $y_0 \in [1,2]$, then we may check that $f_{y=y_0}$ takes the value 1 on the interval $[y_0 - 1, y_0]$, takes value -1 on the interval $[0, y_0 - 1]$. So if $y \in [1,2]$

$$\begin{aligned} H(y_0) &= \int_{-\infty}^{\infty} f_{|y=y_0}\ dx \\ &= \int_0^{y_0-1} -1\ dx + \int_{y_0-1}^{y_0}\ dx = -x\Big|_0^{y_0-1} + x\Big|_{y_0-1}^{y_0} \\ &= (-y_0 + 1) - 0 + (y_0 - (y_0 - 1)) = 2 - y_0. \end{aligned}$$

Now we have computed the inner integral as a function of y:

$$H : \mathbb{R} \longrightarrow \mathbb{R}$$

$$y \mapsto \begin{cases} y & \text{if } y \in [0,1] \text{ and } x > 0; \\ 2 - y & \text{if } y \in [1,2]; \\ 0 & \text{otherwise.} \end{cases}$$

and the full iterated integral is:

$$\int_{-\infty}^{\infty} \big(\int_{-\infty}^{\infty} f \; dx \big) \; dy$$
$$= \int_{-\infty}^{\infty} H(y) \; dy$$
$$= \int_{0}^{1} y \; dy + \int_{0}^{2} 2 - y \; dy$$
$$= \frac{1}{2} + \frac{1}{2} = 1.$$

4 Some extra detritus, odds and ends

Here is a tetrad of important rules that hold for integrals over a rectangular region R.

Fubini's theorem:

$$\int\limits_R f \; dA = \int \big(\int f \; dx \big) dy = \int \big(\int f \; dy \big) dx$$

For two integrable functions f and g we have

$$\int\limits_R f + g \; dA = \int\limits_R f \; dA + \int\limits_R g \; dA.$$

Let $c \in \mathbb{R}$ be any scalar. Then

$$\int\limits_R cf \; dA = c \int\limits_R f \; dA.$$

Now suppose that R_1 and R_2 are *disjoint* rectangular regions which together comprise one larger rectangular region $R_1 \cup R_2$. Then

$$\int\limits_{R_1 \cup R_2} f \; dA = \int\limits_{R_1} f \; dA + \int\limits_{R_2} f \; dA.$$

Exercise 4.1. Let $R = [0,1] \times [-\pi, 0]$ and let f be a continuous function such that $\int\limits_R f \; dA \;=\; 6$. Find all real numbers c which satisfy the equation

$$\int\limits_R cx \text{sin} y + c^2 f \; dA \;=\; 1.$$

We can use the linearity of integrals to evaluate this expression:

$$
\begin{aligned}
1 &= \int\limits_R cx\text{sin}y + c^2 f\ dA \\
&= \int\limits_R cx\text{sin}y\ dA + \int\limits_R c^2 f\ dA \\
&= c\int\limits_R x\text{sin}y\ dA + c^2\int\limits_R f\ dA
\end{aligned}
$$

and now by hypothesis we already know the integral of f over R.

$$
= c\int\limits_R x\text{sin}y\ dA + 6c^2
$$

So we've reduced the problem to finding all c satisfying

$$
1 = c\int\limits_R x\text{sin}y\ dA + 6c^2.
$$

If we can evaluate $\int\limits_R x\text{sin}y\ dA$ then we are left to solve a quadratic formula in c. Lets now evaluate $\int\limits_R x\text{sin}y\ dA$ using an iterated integral:

$$
\begin{aligned}
&= \int_0^1 \Big(\int_{-\pi}^0 x\text{sin}y\ dy\Big)dx \\
&= \int_0^1 \left(-x\text{cos}y\Big|_{y=-\pi}^0\right)\ dx \\
&= \int_0^1 -x - (-x(-1))\ dx
\end{aligned}
$$

(all those minus signs were fucking me up at the end...)

$$
= \int_0^1 -2xdx = -1
$$

Plugging this in to the original equation yields

$$
\begin{aligned}
1 &= -c + 6c^2 \\
0 &= (3c+1)(2c-1) \\
c &= -\frac{1}{3} \text{ or } \frac{1}{2}.
\end{aligned}
$$

5 Integrating odd and even functions

Remember that a function $f : \mathbb{R} \to \mathbb{R}$ is an even function if $f(x) = f(-x)$ for all $x \in \mathbb{R}$ and f is an odd function if $-f(x) = f(-x)$ for all $x \in \mathbb{R}$. Well, in higher dimensions, there are more ways for a function to be either odd or even. Suppose we have a function of n real variables $f : \mathbb{R}^n \to \mathbb{R}$. We say that f is **even** in the i^{th} coordinate if

$$f(x_1, .., x_i, ..., x_n) = f(x_1, ..., -x_i, ..., x_n)$$

and that f is **odd** in the i^{th} coordinate if

$$-f(x_1, ..., x_i, ..., x_n) = f(x_1, ..., -x_i, ..., x_n).$$

A region R is symmetric in the i^{th} coordinate if

$$(x_1, ..., x_i, ..., x_n) \in R \Leftrightarrow (x_1, ..., -x_i, ..., x_n) \in R.$$

Another way to say this is if you take R and reflect it across the hyperplane perpendicular to the x_i axis you get R back again.

If R is symmetric in the i^{th} coordinate and f is an odd function in the i^{th} coordinate, then

$$\int_R Rf\ dA \ = \ 0.$$

This is very useful. Slightly less cool, but still nice is:

If R is symmetric in the i^{th} coordinate and f is an even function in the i^{th} coordinate, then

$$\int_R f\ dA \ = 2\times \int_{\{x\in R | x_i \geq 0\}} f\ dA.$$

Exercise 5.1. Let $R_1 := [2,4]\times[-1,1]$ and $R_2 := [-1,1]\times[2,4]$. Evaluate $\int_{R_1} f\ dA$ and $\in_{R_2} f\ dA$ for the function $f(x,y) = x^2 y$.

For the integral over R_1, notice that x^2y is odd in the y variable and that $[2,4]\times[-1,1]$ is symmetric in the y coordinate, so

$$\int_{R_1} f\ dA = 0$$

For the integral over R_2, we can't conclude as much because R_2 is only symmetric in the x variable, and x^2y is even in the x coordinate. So

$$\int_{R_2} f\ dA = 2\times \int_{[0,1]\times[2,4]} x^2y\ dA$$

which we must now evaluate by hand using an iterated integral.

$$= 2\times \int_2^4 \int_0^1 x^2y\ dx\ dy$$

$$= 2\times \int_2^4 \left(\frac{1}{3}x^3y\Big|_{x=0}^1\right)\ dy$$

$$= 2\times \int_2^4 \frac{1}{3}y\ dy$$

$$= 2\times\left(\frac{1}{6}y^2\Big|_{y=2}^4\right)$$

$$= 4.$$