Homotopy theory of compactified moduli space

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Let $\mathcal{M}_g$ denote the moduli space of genus $g$ Riemann surfaces and $\overline{\mathcal{M}}_g$ the Deligne-Mumford compactification of $\mathcal{M}_g$. Very briefly, the goal of this work (in progress) is to do for $\overline{\mathcal{M}}_g$ what Madsen and Weiss did for $\mathcal{M}_g$. I will start by a short account of Madsen-Weiss’ approach. $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ will always denotes the stacks, rather than the coarse moduli space.

1. $\mathcal{M}$ and surface bundles

By a surface bundle over a manifold $X^k$ we mean a proper submersion $f : E^{k+2} \to X^k$ with oriented fibers. Let $S(X) = \{(\text{surface bundles } f : E \to X) / \simeq\}$ denote the set of isomorphism classes of surface bundles over $X$.

Thus defined, $S$ is a set-valued functor (under pullback) from the category of smooth manifolds and homotopy classes of maps. As such, it is represented by the stack

$$\mathcal{M} = \coprod_k \left( \prod_g \mathcal{M}_g \right)^k / \Sigma_k.$$ 

Thus $H^*(\mathcal{M})$ is the set of characteristic classes of surface bundles. In particular we have the Miller-Morita-Mumford classes $\kappa_i \in H^{2i}(\mathcal{M})$.

2. Formal surface bundles and Madsen-Weiss

Madsen-Weiss’ point of view is to replace “surface bundle” by the corresponding stable normal bundle condition. This means that we consider triples $(f, L, \phi)$ consisting of a proper map $f : E^{k+2} \to X^k$, a complex line bundle $L \to E$, and a stable isomorphism $\phi : TE \oplus \mathbb{R}^j \cong f^*TX \oplus L \oplus \mathbb{R}^j$, defined for some $j \gg 0$. We call such a triple a formal surface bundle and define

$$\tilde{S}(X) = \{(\text{formal surface bundles } f : E \to X) / \simeq\}.$$ 

Here, two formal surface bundles $f_\nu : E_\nu \to X$, $\nu = 0, 1$ (suppressing the $L$’s and the $\phi$’s from the notation), are equivalent if there exists a formal surface bundle $f : W^{k+3} \to X \times \mathbb{R}$, transversal to $X \times \{0, 1\}$, whose restriction to $f^{-1}(X \times \{\nu\})$ is $f_\nu$.

If $f : E \to X$ is a surface bundle, then the differential of $f$ is an epimorphism $TE \to f^*TX$. If we let $L$ denote its kernel, we have a short exact sequence $0 \to L \to TE \to f^*(TX) \to 0$, and a choice of splitting gives an isomorphism $TE \cong f^*(TX) \oplus L$. This defines a forgetful map $S(X) \to \tilde{S}(X)$.

For many purposes, the notion of formal surface bundle is easier to understand than (honest) surface bundles, even though the definition looks more complicated.
For formal reasons (viz. Pontrjagin-Thom theory), the functor $\tilde{S}$ is part of a cohomology theory, which is represented by a Thom spectrum, often denoted $\mathbb{C}P_\infty^\infty$. It is the Thom spectrum of the map

$$BU(1) \xrightarrow{L} \mathbb{Z} \times BO,$$

classifying the virtual inverse of the canonical complex line bundle $L \to BU(1)$. Thus we have a natural isomorphism

$$\tilde{S}(X) \cong [X, \Omega^\infty \mathbb{C}P_\infty^\infty],$$

and the forgetful map $S \to \tilde{S}$ is represented by a continuous map

$$\mathcal{M} \to \Omega^\infty \mathbb{C}P_\infty^\infty.$$

Also for formal reasons (Thom isomorphism), it is easy to calculate $H^*(\mathbb{C}P_\infty^\infty)$. It is $\mathbb{Z}$ in even dimensions and vanishes in odd dimensions. The generators map under the map

$$H^*(\mathbb{C}P_\infty^\infty) \to H^*(\Omega^\infty \mathbb{C}P_\infty^\infty) \to H^*(\mathcal{M})$$

to the Miller-Morita-Mumford classes. With rational coefficients, they form polynomial generators of the cohomology ring $H^*(\Omega^\infty \mathbb{C}P_\infty^\infty)$.

Finally, the statement of Madsen-Weiss can be rephrased as follows: The restriction

$$\mathcal{M}_g \to \Omega^\infty \mathbb{C}P_\infty^\infty$$

of the forgetful map, is a homology isomorphism in degrees up to $(g-1)/2$.

### 3. $\overline{\mathcal{M}}$ and Lefschetz fibrations

We now try to apply a similar analysis to the spaces $\overline{\mathcal{M}}_g$ or, more generally, the stack

$$\overline{\mathcal{M}} = \coprod_k (\coprod_g \overline{\mathcal{M}}_g)^k / \Sigma_k.$$

Points in $\overline{\mathcal{M}}$ are nodal curves, i.e. Riemann surfaces with a certain mild kind of singularities, modelled on $\{(z, w) \in \mathbb{C}^2 \mid zw = 0\}$. In nearby points, a singularity $zw = 0$ can deform into $zw = \epsilon, \epsilon \in \mathbb{C}$. The universal nodal curve is the map

$$\pi : \overline{\mathcal{C}} \to \overline{\mathcal{M}},$$

where $\overline{\mathcal{C}}$ is the stack of pairs $(\Sigma, p)$ with $\Sigma \in \overline{\mathcal{M}}$ and $p \in \Sigma$. The subspace of $\overline{\mathcal{C}}$ where $p \in \Sigma$ is a node, is a smooth substack $\Sigma \subseteq \overline{\mathcal{C}}$ of complex codimension 2, and the restriction

$$\pi|\Sigma : \Sigma \to \overline{\mathcal{M}}$$

is an immersion with normal crossings, of complex codimension 1.
If $X$ is a smooth manifold and $g : X \to \mathcal{M}$ is smooth and transverse to $\pi : \mathcal{C} \to \mathcal{M}$, then $E = g^*\mathcal{C}$ is a smooth manifold, and we have a pullback square

\[
\begin{array}{ccc}
E & \longrightarrow & \mathcal{C} \\
\downarrow \pi & \quad & \downarrow \pi \\
X & \longrightarrow & \mathcal{M}
\end{array}
\]

The map $\pi : E \to X$ is no longer a surface bundle (as it would have been with $\mathcal{M}$ in place of $\mathcal{M}$), it is a Lefschetz fibration. We recall a definition of this notion.

Imprecisely, a Lefschetz fibration is a smooth proper map $f : E^{k+2} \to X^k$, which locally in $E$ looks like

$$(x_1, \ldots, x_{k-2}, z, w) \mapsto (x_1, \ldots, x_{k-2}, zw),$$

where the $x_i$ are real parameters and $z$ and $w$ are complex parameters.

More precisely, we will by a Lefschetz fibration mean a tuple $$(f, \Sigma, U, L, q),$$

where $f : E^{k+2} \to X^k$ is a proper map, $\Sigma^{k-2} \subseteq E$ is a closed submanifold such that $f|\Sigma$ is an immersion with normal crossings. $U \to \Sigma$ is a 2-dimensional complex vector bundle, embedded as a tubular neighborhood $U \subseteq E$. $L \to \Sigma$ is a complex line bundle, immersed as a tubular neighborhood $L \to X$. $q : U \to L$ is a nondegenerate fiberwise quadratic form (i.e. $q(v) = \frac{1}{2}b(v, v)$ for a unique $b \in (S^2U)^* \otimes L$), such that the diagram

\[
\begin{array}{ccc}
U & \longrightarrow & E \\
\downarrow q & \quad & \downarrow f \\
L & \longrightarrow & X
\end{array}
\]

commutes near the zero section $\Sigma \subseteq U$. Finally, the restriction $f|(E - \Sigma)$ should be a submersion with oriented fibers, the orientation being compatible with the complex structures of $U$ and $L$ near $\Sigma$.

It is not hard to see that the map $\pi : \mathcal{C} \to \mathcal{M}$ is a Lefschetz fibration in this sense, and that it is universal: Any Lefschetz fibration $f : E \to X$ (suppressing much from the notation) is induced by a smooth map $g : X \to \mathcal{M}$, transverse to $\pi$. Thus, if we let

$$L(X) = (\text{Lefschetz fibrations } f : E \to X)/\simeq,$$

then $L$ is represented by the space $\mathcal{M}$: There is a natural isomorphism $L(X) \cong [X, \mathcal{M}]$.

4. Formal Lefschetz Fibrations

Following Madsen-Weiss, we replace “Lefschetz fibration” by the corresponding stable normal bundle condition. This leads to the notion of a Formal Lefschetz Fibration.

A formal Lefschetz fibration is a tuple

$$(f, V_S, V_N, U, L, q, L', k, \phi, \psi)$$
where

(i) \( f : E^{k+2} \to X^k \) is a proper map

(ii) \( E = V_S \cup V_N \) is an open cover (S and N stands for “singular” and “nonsingular”)

(iii) \( U \to V_S \) and \( L \to V_S \) are complex vector bundles of dimension 2 and 1, respectively

(iv) \( q : U \to L \) is a fiberwise quadratic, nondegenerate form

(v) \( L' \to V_N \) is a complex line bundle

(vi) \( k \) is an isomorphism of vector bundles over \( V_S \cap V_N: L' \oplus L \cong U \)

(vii) \( \phi \) is a stable isomorphism of vector bundles over \( V_S: \mathbb{R}^j \oplus TE \oplus L \cong \mathbb{R}^j \oplus f^*TX \oplus U \).

(viii) \( \psi \) is a stable isomorphism of vector bundles over \( V_N: \mathbb{R}^j \oplus TE \cong \mathbb{R}^j \oplus f^*TX \oplus L' \).

\( \phi, \psi \) and \( k \) should be compatible over \( V_N \cap V_S \), in the sense that 
\( (\psi \oplus \text{id}_L) \circ (\text{id}_{\mathbb{R}^j \oplus f^*TX} \oplus k) = \phi \). (Note that we require these to be equal as maps. Alternatively we could require them to be homotopic via a homotopy \( h \) which we should then include in the data).

Let

\[ \tilde{L}(X) = \text{(Formal Lefschetz Fibrations} \ f : E \to X) / \sim \]

where \( \sim \) is the equivalence relation generated by increasing \( j \), and by homotopy, i.e. if \( W^{k+3} \to X^k \times \mathbb{R} \) is a formal Lefschetz fibration, transverse to \( X \times \{0, 1\} \), then the restriction to \( X \times \{0\} \) and \( X \times \{1\} \) are equivalent.

There is a forgetful map \( L(X) \to \tilde{L}(X) \), defined as follows. Given a Lefschetz fibration \( (f, \Sigma, U, L, q) \), we let:

- \( V_N = E - \Sigma \), and \( L' \to V_N \) is the kernel of \( D(f|V_N) \).
- \( V_S = U \subseteq E \).

As in the uncompactified case, the point of considering the corresponding stable normal bundle condition is, that \( \tilde{L}(X) \) is for many purposes easier to understand. The “usual” cohomology classes in \( \overline{M}_g \), thought of as natural transformations

\[ L(X) \to H^*(X; \mathbb{Q}) \]

factor through \( \tilde{L}(X) \). In particular we have the Miller-Morita-Mumford classes \( \kappa_i \), but also some new classes that I will call \( \theta_{i,j}, i, j \geq 0 \). For a Lefschetz fibration \( f : E \to X \) they are defined as

\[ \theta_{i,j} = (f|\Sigma)_!(c_1^j c_2^i(U)) \in H^{2+2i+4j}(X) \]

5. CLASSIFYING FLFS

Pontrjagin-Thom theory implies that formal Lefschetz fibrations are classified by a Thom spectrum. The general procedure is to translate the stable normal bundle condition into a map \( \xi : B \to \mathbb{Z} \times BO \). The stable normal bundle of a proper map \( f : E \to X \) is a map

\[ Nf : E \to \mathbb{Z} \times BO, \]

\[ \theta \text{- } \]
whose homotopy class is \([f^*TX] - [TE] \in KO^0(E)\), and \(\xi : B \to \mathbb{Z} \times BO\) should be such that the bundle conditions on \(f\) are equivalent to a lifting of \(Nf\) to a map \(l : E \to B\). Then the Thom spectrum \(B^\xi\) of \(\xi\) will classify \(\tilde{L}\) in the sense that there is a natural isomorphism

\[\tilde{L}(X) \cong [X, \Omega^\infty B^\xi].\]

In our case, \(E\) is a pushout \(V_S \leftarrow V_{S \cap V_N} \to V_N\), and the space \(B\) is most easily described as a homotopy pushout of spaces over \(\mathbb{Z} \times BO\):

\[
\begin{array}{ccc}
B_{SN} & \to & BS \\
\downarrow & & \downarrow \\
B_N & \to & B \\
\downarrow & & \downarrow \\
\mathbb{Z} \times BO & \to & -L'
\end{array}
\]

Here, \(B_N = BU(1)\) models the same bundle condition as in the uncompactified case. More interestingly, \(BS\) is the universal space carrying two complex bundles \(U, L\) of dimensions 2 and 1, respectively, equipped with a quadratic nondegenerate map \(q : U \to L\). This is

\[BS = E(U(2) \times U(1)) \times U(2) \times U(1)\ \text{Quad}(\mathbb{C}^2, \mathbb{C}^1),\]

where \(\text{Quad}(\mathbb{C}^2, \mathbb{C}^1)\) denotes the space of quadratic, nondegenerate maps. It turns out that this is homotopy equivalent to the classifying space of the maximal torus normalizer in \(U(2)\):

\[BS = B(\Sigma_2 U(1)).\]

\(B_{SN}\) is the sphere bundle of the canonical bundle \(U \to B_S\).

The pushout diagram (1) above leads to a map \(\xi : B \to \mathbb{Z} \times BO\), and thus a Thom spectrum \(B^\xi\). By Pontrjagin-Thom theory, this will classify formal Lefschetz fibrations. Therefore we will denote it \(FLF := B^\xi\). I have sketched a proof that there is a natural isomorphism

\[\tilde{L}(X) \cong [X, \Omega^\infty FLF].\]

6. Cohomology of \(FLF\)

The pushout diagram (1) of spaces over \(\mathbb{Z} \times BO\) leads to a pushout diagram of Thom spectra, and in turn a cofibration sequence of spectra

\[\mathbb{C}P^\infty_{-1} \longrightarrow FLF \longrightarrow B(\Sigma_2 U(1))^L.\]

\(B(\Sigma_2 U(1))^L\) is the Thom spectrum (space, in fact) of the bundle \(L \to B(fU(1))\). It is not hard to calculate the cohomology of these spectra, using Thom isomorphism. I will state the answer with rational coefficients.

As stated earlier, \(H^*(\mathbb{C}P^\infty_{-1})\) is one-dimensional in each even degree. The classes correspond to the Miller-Morita-Mumford classes.
The inclusion $B(\Sigma_2 fU(1)) \to BU(2)$ induces an isomorphism in rational cohomology (actually with coefficients in $\mathbb{Z}[1/2]$). Thus the cohomology has basis $c_i^1 c_j^2$, $i, j \geq 0$. This gives rise to the characteristic classes $\theta_{i,j}$ described earlier. It is not hard to see that the $\kappa_i$ classes together with the $\theta_{i,j}$ classes form a basis for $H^*(FLF; \mathbb{Q})$. 

Thus we know precisely what are characteristic classes of formal Lefschetz fibrations.

7. Concluding remarks

The forgetful map from Lefschetz fibrations to formal Lefschetz fibrations is classified by a map $\mathcal{M} \to \Omega^\infty \text{FLF}$. It seems that many of the cohomology classes in $\mathcal{M}_g$ that are "usually" considered, can be pulled back from classes in $\Omega^\infty \text{FLF}$ (namely precisely the $\kappa_i$ and the $\theta_{i,j}$ classes).

The question of understanding the intersection theory of $\mathcal{M}_g$ can now, at least partly, be rephrased as understanding the bordism (or just homology) class of the map $\mathcal{M}_g \to \Omega^\infty \text{FLF}$.

A slightly weaker question is to understand the class $[\mathcal{M}] \in H_4(\text{FLF}; \mathbb{Q})$. A goal of this work (in progress) is to give a homotopy theoretic description of the class (2). In the longer term, one should of course consider Gromov-Witten theory of an arbitrary symplectic manifold $X$ (in a way that the above would correspond to the case where $X$ is a point). The analogue of (2) would be $[\mathcal{M}(X)] \in H_4(\text{FLF} \wedge X; \mathbb{Q})$.

A question orthogonal to that of understanding $[\mathcal{M}]$ is to ask, what the analogue of Madsen-Weiss' theorem would be. The naive guess that $\mathcal{M}_g \to \Omega^\infty \text{FLF}$ might be a homology isomorphism in a range increasing with $g$, turns out to be wrong. Instead we propose to consider the subspace $\tilde{\mathcal{M}}_g \subseteq \mathcal{M}_g$, consisting of irreducible curves. Then the composition $\tilde{\mathcal{M}}_g \subseteq \mathcal{M}_g \to \Omega^\infty \text{FLF}$ seems to be a homology isomorphism in a stable range. Thus, in that stable range, the cohomology of $\mathcal{M}_g$ will be the direct sum of a stable part, which is the polynomial algebra in the $\kappa_i$ and the $\theta_{i,j}$, and an unstable part, which is the homology of $\mathcal{M}_g - \tilde{\mathcal{M}}_g$. 

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