Homotopy theory of compactified moduli space

Søren Galatius

(joint work with Ya. Eliashberg)

Let $\mathcal{M}_g$ denote the moduli space of genus g Riemann surfaces and $\overline{\mathcal{M}}_g$ the Deligne-Mumford compactification of $\mathcal{M}_g$. Very briefly, the goal of this work (in progress) is to do for $\overline{\mathcal{M}}_g$ what Madsen and Weiss did for $\mathcal{M}_g$. I will start by a short account of Madsen-Weiss' approach. $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ will always denotes the *stacks*, rather than the coarse moduli space.

1. $\mathcal{M}$ AND SURFACE BUNDLES

By a *surface bundle* over a manifold X^k we mean a proper submersion $f : E^{k+2} \to X^k$ with oriented fibers. Let

$$S(X) = (\text{surface bundles } f : E \to X)/\simeq$$

denote the set of isomorphism classes of surface bundles over X.

Thus defined, S is a set-valued functor (under pullback) from the category of smooth manifolds and homotopy classes of maps. As such, it is represented by the stack

$$\mathcal{M} = \coprod_k \big(\coprod_g \mathcal{M}_g\big)^k/\Sigma_k.$$

Thus $H^*(\mathcal{M})$ is the set of characteristic classes of surface bundles. In particular we have the Miller-Morita-Mumford classes $\kappa_i \in H^{2i}(\mathcal{M})$.

2. Formal surface bundles and Madsen-Weiss

Madsen-Weiss' point of view is to replace "surface bundle" by the corresponding *stable normal bundle condition*. This means that we consider triples (f, L, ϕ) consisting of a proper map $f : E^{k+2} \to X^k$, a complex line bundle $L \to E$, and a stable isomorphism $\phi : TE \oplus \mathbb{R}^j \cong f^*TX \oplus L \oplus \mathbb{R}^j$, defined for some $j \gg 0$. We call such a triple a *formal surface bundle* and define

$$\tilde{S}(X) = (\text{formal surface bundles } E \to X)/\simeq .$$

Here, two formal surface bundles $f_\nu : E_\nu \to X$, $\nu = 0, 1$ (suppressing the L's and the ϕ's from the notation), are equivalent if there exists a formal surface bundle $f : W^{k+3} \to X \times \mathbb{R}$, transversal to $X \times \{0, 1\}$, whose restriction to $f^{-1}(X \times \{\nu\})$ is f_ν.

If $f : E \to X$ is a surface bundle, then the differential of f is an epimorphism $TE \to f^*TX$. If we let L denote its kernel, we have a short exact sequence $0 \to L \to TE \to f^*(TX) \to 0$, and a choice of splitting gives an isomorphism $TE \cong f^*(TX) \oplus L$. This defines a forgetful map $S(X) \to \tilde{S}(X)$.

For many purposes, the notion of formal surface bundle is easier to understand than (honest) surface bundles, even though the definition looks more complicated.

For formal reasons (viz. Pontrjagin-Thom theory), the functor $\tilde{S}$ is part of a *cohomology theory*, which is represented by a *Thom spectrum*, often denoted $\mathbb{C}P^\infty_{-1}$. It is the Thom spectrum of the map

$$BU(1) \xrightarrow{-L} \mathbb{Z} \times BO,$$

classifying the virtual inverse of the canonical complex line bundle $L \to BU(1)$. Thus we have a natural isomorphism

$$\tilde{S}(X) \cong [X, \Omega^\infty \mathbb{C}P^\infty_{-1}],$$

and the forgetful map $S \to \tilde{S}$ is represented by a continuous map

$$\mathcal{M} \to \Omega^\infty \mathbb{C}P^\infty_{-1}.$$

Also for formal reasons (Thom isomorphism), it is easy to calculate $H^*(\mathbb{C}P^\infty_{-1})$. It is $\mathbb{Z}$ in even dimensions and vanishes in odd dimensions. The generators map under the map

$$H^*(\mathbb{C}P^\infty_{-1}) \to H^*(\Omega^\infty \mathbb{C}P^\infty_{-1}) \to H^*(\mathcal{M})$$

to the Miller-Morita-Mumford classes. With rational coefficients, they form polynomial generators of the cohomology ring $H^*(\Omega^\infty \mathbb{C}P^\infty_{-1})$.

Finally, the statement of Madsen-Weiss can be rephrased as follows: The restriction

$$\mathcal{M}_g \to \Omega^\infty \mathbb{C}P^\infty_{-1}$$

of the forgetful map, is a homology isomorphism in degrees up to $(g-1)/2$.

3. $\overline{\mathcal{M}}$ AND LEFSCHETZ FIBRATIONS

We now try to apply a similar analysis to the spaces $\overline{\mathcal{M}}_g$ or, more generally, the stack

$$\overline{\mathcal{M}} = \coprod_k \big(\coprod_g \overline{\mathcal{M}}_g\big)^k / \Sigma_k.$$

Points in $\overline{\mathcal{M}}$ are *nodal curves*, i.e. Riemann surfaces with a certain mild kind of singularities, modelled on $\{(z,w) \in \mathbb{C}^2 \mid zw = 0\}$. In nearby points, a singularity $zw = 0$ can deform into $zw = \epsilon$, $\epsilon \in \mathbb{C}$. The *universal nodal curve* is the map

$$\pi : \overline{\mathcal{C}} \to \overline{\mathcal{M}},$$

where $\overline{\mathcal{C}}$ is the stack of pairs (Σ, p) with $\Sigma \in \overline{\mathcal{M}}$ and $p \in \Sigma$. The subspace of $\overline{\mathcal{C}}$ where $p \in \Sigma$ is a node, is a smooth substack $\Sigma \subseteq \overline{\mathcal{C}}$ of complex codimension 2, and the restriction

$$\pi|\Sigma : \Sigma \to \overline{\mathcal{M}}$$

is an immersion with normal crossings, of complex codimension 1.

If X is a smooth manifold and $g : X \to \overline{\mathcal{M}}$ is smooth and transverse to $\pi : \overline{\mathcal{C}} \to \overline{\mathcal{M}}$, then $E = g^*\overline{\mathcal{C}}$ is a smooth manifold, and we have a pullback square

$$\begin{tikzcd} E \arrow[r] \arrow[d, "\pi"'] & \overline{\mathcal{C}} \arrow[d, "\pi"] \\ X \arrow[r, "g"] & \overline{\mathcal{M}} \end{tikzcd}$$

The map $\pi : E \to X$ is no longer a surface bundle (as it would have been with $\mathcal{M}$ in place of $\overline{\mathcal{M}}$), it is a *Lefschetz fibration.* We recall a definition of this notion.

Imprecisely, a Lefschetz fibration is a smooth proper map $f : E^{k+2} \to X^k$, which locally in E looks like

$$(x_1, \dots, x_{k-2}, z, w) \mapsto (x_1, \dots, x_{k-2}, zw),$$

where the x_i are real parameters and z and w are complex parameters.

More precisely, we will by a Lefschetz fibration mean a tuple (f, Σ, U, L, q), where $f : E^{k+2} \to X^k$ is a proper map, $\Sigma^{k-2} \subseteq E$ is a closed submanifold such that $f|\Sigma$ is an immersion with normal crossings. $U \to \Sigma$ is a 2-dimensional complex vector bundle, embedded as a tubular neighborhood $U \subseteq E$. $L \to \Sigma$ is a complex line bundle, immersed as a tubular neighborhood $L \to X$, $q : U \to L$ is a nondegenerate fiberwise quadratic form (i.e. $q(v) = \frac{1}{2}b(v, v)$ for a unique $b \in (S^2U)^* \otimes L$), such that the diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "q"] & E \arrow[d, "f"] \\ L \arrow[r] & X \end{tikzcd}$$

commutes near the zero section $\Sigma \subseteq U$. Finally, the restriction $f|(E - \Sigma)$ should be a submersion with oriented fibers, the orientation being compatible with the complex structures of U and L near Σ.

It is not hard to see that the map $\pi : \overline{\mathcal{C}} \to \overline{\mathcal{M}}$ is a Lefschetz fibration in this sense, and that it is universal: Any Lefschetz fibration $f : E \to X$ (suppressing much from the notation) is induced by a smooth map $g : X \to \overline{\mathcal{M}}$, transverse to π. Thus, if we let

$$L(X) = (\text{Lefschetz fibrations } f : E \to X)/\simeq,$$

then L is represented by the space $\overline{\mathcal{M}}$: There is a natural isomorphism $L(X) \cong [X, \overline{\mathcal{M}}]$.

4. Formal Lefschetz Fibrations

Following Madsen-Weiss, we replace "Lefschetz fibration" by the corresponding stable normal bundle condition. This leads to the notion of a *Formal Lefschetz Fibration.*

A formal Lefschetz fibration is a tuple

$$(f, V_S, V_N, U, L, q, L', k, \phi, \psi)$$

where

(i) $f : E^{k+2} \to X^k$ is a proper map

(ii) $E = V_S \cup V_N$ is an open cover (S and N stands for "singular" and "nonsingular")

(iii) $U \to V_S$ and $L \to V_S$ are complex vector bundles of dimension 2 and 1, respectively

(iv) $q : U \to L$ is a fiberwise quadratic, nondegenerate form

(v) $L' \to V_N$ is a complex line bundle

(vi) k is an isomorphism of vector bundles over $V_S \cap V_N$: $L' \oplus L \cong U$

(vii) ϕ is a stable isomorphism of vector bundles over V_S: $\mathbb{R}^j \oplus TE \oplus L \cong \mathbb{R}^j \oplus f^*TX \oplus U$.

(viii) ψ is a stable isomorphism of vector bundles over V_N: $\mathbb{R}^j \oplus TE \cong \mathbb{R}^j \oplus f^*TX \oplus L'$.

ϕ, ψ and k should be compatible over $V_N \cap V_S$, in the sense that $(\psi \oplus \mathrm{id}_L) \circ (\mathrm{id}_{\mathbb{R}^j \oplus f^*TX} \oplus k) = \phi$. (Note that we require these to be *equal* as maps. Alternatively we could require them to be homotopic via a homotopy h which we should then include in the data).

Let

$$\tilde{L}(X) = (\text{Formal Lefschetz Fibrations } f : E \to X)/\simeq$$

where $\simeq$ is the equivalence relation generated by increasing j, and by *homotopy*, i.e. if $W^{k+3} \to X^k \times \mathbb{R}$ is a formal Lefschetz fibration, transverse to $X \times \{0, 1\}$, then the restriction to $X \times \{0\}$ and $X \times \{1\}$ are equivalent.

There is a forgetful map $L(X) \to \tilde{L}(X)$, defined as follows. Given a Lefschetz fibration (f, Σ, U, L, q), we let

- $V_N = E - \Sigma$, and $L' \to V_N$ is the kernel of $D(f|V_N))$.
- $V_S = U \subseteq E$.

As in the uncompactified case, the point of considering the corresponding stable normal bundle condition is, that $\tilde{L}(X)$ is for many purposes easier to understand. The "usual" cohomology classes in $\overline{\mathcal{M}}_g$, thought of as natural transformations

$$L(X) \to H^*(X; \mathbb{Q}),$$

factor through $\tilde{L}(X)$. In particular we have the Miller-Morita-Mumford classes κ_i, but also some new classes that I will call $\theta_{i,j}$, $i, j \geq 0$. For a Lefschetz fibration $f : E \to X$ they are defined as

$$\theta_{i,j} = (f|\Sigma)_!(c_1^i c_2^j(U)) \in H^{2+2i+4j}(X)$$

5. Classifying FLFs

Pontrjagin-Thom theory implies that formal Lefschetz fibrations are classified by a Thom spectrum. The general procedure is to translate the stable normal bundle condition into a map $\xi : B \to \mathbb{Z} \times BO$. The stable normal bundle of a proper map $f : E \to X$ is a map

$$Nf : E \to \mathbb{Z} \times BO,$$

whose homotopy class is $[f^*TX] - [TE] \in KO^0(E)$, and $\xi : B \to \mathbb{Z} \times BO$ should be such that the bundle conditions on f are equivalent to a lifting of Nf to a map $l : E \to B$. Then the Thom spectrum B^ξ of ξ will classify $\tilde{L}$ in the sense that there is a natural isomorphism

$$\tilde{L}(X) \cong [X, \Omega^\infty B^\xi].$$

In our case, E is a pushout $V_S \leftarrow V_S \cap V_N \to V_N$, and the space B is most easily described as a homotopy pushout of spaces over $\mathbb{Z} \times BO$:

(1)

$$\begin{tikzcd}
B_{SN} \arrow[r] \arrow[d] & B_S \arrow[d] \arrow[ddr, "L-U"] & \\
B_N \arrow[r] \arrow[drr, "-L'"'] & B & \\
& & \mathbb{Z} \times BO.
\end{tikzcd}$$

Here, $B_N = BU(1)$ models the same bundle condition as in the uncompactified case. More interestingly, B_S is the universal space carrying two complex bundles U, L of dimensions 2 and 1, respectively, equipped with a quadratic nondegenerate map $q : U \to L$. This is

$$B_S = E(U(2) \times U(1)) \times_{U(2)\times U(1)} \mathrm{Quad}(\mathbb{C}^2, \mathbb{C}^1),$$

where $\mathrm{Quad}(\mathbb{C}^2, \mathbb{C}^1)$ denotes the space of quadratic, nondegenerate maps. It turns out that this is homotopy equivalent to the classifying space of the maximal torus normalizer in $U(2)$:

$$B_S = B(\Sigma_2 \wr U(1)).$$

B_{SN} is the sphere bundle of the canonical bundle $U \to B_S$.

The pushout diagram (1) above leads to a map $\xi : B \to \mathbb{Z} \times BO$, and thus a Thom spectrum B^ξ. By Pontrjagin-Thom theory, this will classify formal Lefschetz fibrations. Therefore we will denote it $\underline{FLF} := B^\xi$. I have sketched a proof that there is a natural isomorphism

$$\tilde{L}(X) \cong [X, \Omega^\infty \underline{FLF}].$$

6. Cohomology of $\underline{\mathrm{FLF}}$

The pushout diagram (1) of spaces over $\mathbb{Z} \times BO$ leads to a pushout diagram of Thom spectra, and in turn a cofibration sequence of spectra

$$\mathbb{C}P^\infty_{-1} \longrightarrow \underline{FLF} \longrightarrow B(\Sigma_2 \wr U(1))^L.$$

$B(\Sigma_2 \wr U(1))^L$ is the Thom spectrum (space, in fact) of the bundle $L \to B(\wr U(1))$. It is not hard to calculate the cohomology of these spectra, using Thom isomorphism. I will state the answer with rational coefficients.

As stated earlier, $H^*(\mathbb{C}P^\infty_{-1})$ is one-dimensional in each even degree. The classes correspond to the Miller-Morita-Mumford classes.

The inclusion $B(\Sigma_2 \int U(1)) \to BU(2)$ induces an isomorphism in rational cohomology (actually with coefficients in $\mathbb{Z}[1/2]$). Thus the cohomology has basis $c_1^i c_2^j$, $i, j \geq 0$. This gives rise to the characteristic classes $\theta_{i,j}$ described earlier. It is not hard to see that the κ_i classes together with the $\theta_{i,j}$ classes form a basis for $H^*(\underline{FLF}; \mathbb{Q})$. On the level of infinite loop spaces we get

$$H^*(\Omega^\infty \underline{FLF}; \mathbb{Q}) \cong \mathbb{Q}[\kappa_i | i \geq 0] \otimes \mathbb{Q}[\theta_{i,j} | i, j \geq 0].$$

Thus we know precisely what are characteristic classes of *formal* Lefschetz fibrations.

7. CONCLUDING REMARKS

The forgetful map from Lefschetz fibrations to formal Lefschetz fibrations is classified by a map

$$\overline{\mathcal{M}} \to \Omega^\infty \underline{FLF}.$$

It seems that many of the cohomology classes in $\overline{\mathcal{M}}_g$ that are "usually" considered, can be pulled back from classes in $\Omega^\infty \underline{FLF}$ (namely precisely the κ_i and the $\theta_{i,j}$ classes).

The question of understanding the intersection theory of $\overline{\mathcal{M}}_g$ can now, at least partly, be rephrased as understanding the bordism (or just homology) class of the map $\overline{\mathcal{M}}_g \to \Omega^\infty \underline{FLF}$.

A slightly weaker question is to understand the class

$$[\overline{\mathcal{M}}] \in H_*(\underline{FLF}; \mathbb{Q}). \tag{2}$$

A goal of this work (in progress) is to give a homotopy theoretic description of the class (2). In the longer term, one should of course consider Gromov-Witten theory of an arbitrary symplectic manifold X (in a way that the above would correspond to the case where X is a point). The analogue of (2) would be $[\overline{\mathcal{M}}(X)] \in H_*(\underline{FLF} \wedge X_+; \mathbb{Q})$.

A question orthogonal to that of understanding $[\overline{\mathcal{M}}]$ is to ask, what the analogue of Madsen-Weiss' theorem would be. The naive guess that $\overline{\mathcal{M}}_g \to \Omega^\infty \underline{FLF}$ might be a homology isomorphism in a range increasing with g, turns out to be wrong. Instead we propose to consider the subspace $\tilde{\mathcal{M}}_g \subseteq \overline{\mathcal{M}}_g$, consisting of *irreducible* curves. Then the composition

$$\tilde{\mathcal{M}}_g \subseteq \overline{\mathcal{M}}_g \to \Omega^\infty \underline{FLF}$$

seems to be a homology isomorphism in a stable range. Thus, in that stable range, the cohomology of $\overline{\mathcal{M}}_g$ will be the direct sum of a *stable* part, which is the polynomial algebra in the κ_i and the $\theta_{i,j}$, and an unstable part, which is the homology of $\overline{\mathcal{M}}_g - \tilde{\mathcal{M}}_g$.