SUMMER SCHOOL ON HOMOTOPY THEORY OF MODULI SPACES

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Introduction

The focus of this summer school is the homotopy theoretic approach to the study of moduli spaces. We will focus on the developments that followed Ib Madsen and Michael Weiss' proof in 2002 of a generalized version of a conjecture of Mumford.

What is a moduli space?

Talk 1. Introduction Given by the organizers.

Talk 2. Teichmüller theory and moduli spaces This talk should provide an overview of Teichmüller theory, following [5] (without any attempt to give proofs). This serves two purposes. First, Teichmüller theory provides the link between differential topology and homotopy theory on one side (the main subject of this summer school) with the more classical theory of Riemann surfaces (which belongs to algebraic geometry and complex analysis). A more technical aspect is that [5] shows that the components of the diffeomorphism group of a surface are contractible; an essential ingredient for the Mumford conjecture.

Introduce the main players: the mapping class group Γ_g, the Teichmüller space $\mathcal{T}_g$ and Riemann's moduli space. Then describe the main result of [5]. An important consequence is the existence of maps $B\,\mathrm{Diff}(F_g) \to B\Gamma_g \to \mathfrak{M}_g$; the first one is a homotopy equivalence and the second one a rational homology equivalence.

We will also need the mapping class groups of surfaces with boundary. $\Gamma_{g,b}$ is the group of components of $\mathrm{Diff}(F_{g,n}, \partial F_{g,n})$, where $F_{g,n}$ is a surface of genus g with n boundary components. The statement that the identity component of $\mathrm{Diff}(F, \partial F)$ is contractible can be deduced from the closed case by the arguments of [15], section 2 (more topological), alternatively, [6] gives an analytic proof.

Finally, you should introduce the Morita-Miller-Mumford classes of surface bundles [19] and state the Mumford conjecture.

Talk 3. Pontryagin-Thom theory I: The goal of this talk is to explain the classical Pontryagin-Thom isomorphism of the bordism group with the homotopy groups of Thom spectra.

This talk should review beginning of [24]. Give the definition of spectra, the Thom spectrum of a stable vector bundle, emphasizing the Thom spectrum MSO. State the isomorphism between bordism group of oriented n-manifolds and the homotopy groups $\pi_n(MSO)$, and explain how to construct maps in both directions (one direction by collapsing onto the Thom space of the normal bundle, the other uses Thom's transversality theorem). Participants will be expected to have heard some version of this (if not, read it as preparation to the summer school!), so you should explain the constructions clearly, but not necessarily say anything about proofs.

Then you should state the Thom isomorphism theorem, and explain how characteristic numbers are used to detect bordism classes. If there is time, you can finish with a computation of the rational oriented bordism ring. Reference: [24].

Talk 4. Pontryagin-Thom theory II: This talk should introduce the basic construction underlying the theory of [9]. Introduce the spectra $\mathrm{MTSO}(n)$. For any bundle $f : E \to B$ of smooth oriented closed n-manifolds, a parametrized version of the Pontryagin-Thom construction gives a map $\alpha_E : B \to \Omega^\infty \mathrm{MTSO}(n)$ which is well-defined up to homotopy.

Then describe the universal case of this construction: Define $B\,\mathrm{Diff}(M)$ as the space of embeddings $M \to \mathbb{R}^\infty$ modulo the action of $\mathrm{Diff}(M)$, and construct the map $B\,\mathrm{Diff}(M) \to \Omega^\infty \mathrm{MTSO}(n)$. You should state, but not necessarily prove, that $B\,\mathrm{Diff}(M)$ is a classifying space for smooth fiber bundles with fiber M.

Then compute the rational cohomology of $\Omega^\infty \mathrm{MTSO}(n)$ and describe how the generalized MMM-classes come from $\Omega^\infty \mathrm{MTSO}(n)$.

References: [9], [18].

Gromov's h-principle

The *h*-principle is a very important tool in topology and geometry, and features prominently in proofs of Mumford's conjecture, cf. [17], [9]. We will give a proof of a version of the h-principle due to Gromov ([10]).

Talk 5. Gromov's theorem: Statement The goal of this talk is to give the precise statement of Gromov's h-principle for open invariant differential relations (this is the "main theorem" of [11], p. 129). The paper [22] contains more details. The statement of the theorem involves sections of jet bundles of fibre bundle, so you need to discuss the definition and some important properties of jet bundles. You also have to explain the notion of *natural fibre bundles*. Jet bundles are discussed in [13], [1], [7], [10]. Don't forget that the h-principle is about *spaces* of smooth maps, so you need to discuss the topologies.

Talk 6. Gromov's theorem: Applications and the easy part of the proof First discuss some applications (or rather, special cases) of Gromov's theorem. Haefligers paper contains some of them. The most important for us is the submersion theorem. Other examples are immersions [20], p.196, symplectic structures [11], metrics with curvature bounds [22]. A funny consequence is the sphere eversion [7]. You should also give some definite examples that show the (complete) breakdown of the h-principle for closed manifolds (consider submersions to $\mathbb{R}$).

Then you should start the proof of the theorem that is written down in [11] p. 133-140. Explain what a handlebody decomposition of a manifold is, state Propositions 1, 2, 3 and deduce the theorem from them and the existence of a handlebody decomposition of open manifolds.

Talk 7. Gromov's theorem: the difficult part of the proof This talk finishes the proof, i.e. you have to prove Propositions 1,2,3 of [11]. It won't be easy to prepare the proof Proposition 3 loc.cit., which is the heart of the whole argument but somewhat technical, but it is the core of the argument. Intelligent pictures are essential. Poenaru [22] gives the same proof and again he gives more details than [11]. Consulting [22] will be helpful, for example when you got stuck at the top of page 139 of [11].

Homological stability of mapping class groups

The goal of the day is to cover a proof of the homological stability theorems for mapping class groups. The first proof of this was given in [12], and the stability range was later improved by [14] and then [3], and a different proof by [23]. The literature on this theorem can be confusing, with several improvements on improvements, but the recent survey [26] gives a streamlined proof.

Talk 8. Homological stability You should start with a brief introduction to homological stability, describe a couple of examples (symmetric groups, braid groups, configuration spaces, ...; an introduction to stability phenomena can be found in [4]), and then state the precise theorem for mapping class groups of oriented surfaces. Then you should introduce the main ingredient in the proof given in [26]: the ordered arc complex. State the main properties ("ingredient 1-4" in [26]), and prove as many as you have time for (except for the connectivity, which we will get back to). Reference: [26, section 2].

Talk 9. The spectral sequence argument In this talk you should go through the spectral sequence argument in [26, section 3] in detail. Start explaining how group acting on a simplicial complex gives rise to a spectral sequence by filtering the complex by its skeleta, then specialize to the mapping class group acting on the ordered arc complex.

Talk 10. Connectivity arguments Sketch the proof of the connectivity of the ordered arc complex. Do in some details the contractibility of the full complex, and one of the simpler deductions, like from all arcs between two sets of points to the subcomplex of non-separating arcs. Reference: [26, section 4 (Thms 4.1 and 4.8)].

The homotopy type of the cobordism category

We now go through the main steps of the proof of the main theorem of [9], and explain how it implies Madsen-Weiss' theorem (and hence Mumford's conjecture).

Talk 11. The cobordism category Define the cobordism category $\mathcal{C}_d$ as a topological category, via spaces of submanifolds of euclidean space. Explain how the homotopy type of spaces of objects and morphisms are classifying space of diffeomorphism groups of manifolds, and that a smooth map from X to the space of objects (or morphisms) is the same as a certain kind of fiber bundle $E \to X$, where E is a submanifold of $X \times \mathbb{R}^N$ for some large N. (A very similar statement may have been mentioned previously, in talk 1 or 4.) Then define the classifying space of a category, and state the main theorem: $B\mathcal{C}_d$ is homotopy equivalent to $\Omega^{\infty-1}MTO(d)$).

You should also sketch the definition of the map $B\mathcal{C}_d \to \Omega^{\infty-1}MTO(d)$, and explain the relation to talk 4: if W is a closed d-manifold, there is a map $B\operatorname{Diff}(W) \to \Omega B\mathcal{C}_d$, and the resulting map $B\operatorname{Diff}(W) \to \Omega^\infty MTO(d)$ is homotopic to the one described in talk 4.

Reference: [9], Introduction and section 2.1.

Talk 12. Sheaves and their realization This talk should introduce the sheaf language of [17] (there's a survey in [9]). You should emphasize how one can get a topological space from a sheaf (by taking geometric realization), and vice versa (by taking continuous/smooth maps into a space). Also explain that these constructions are "inverse up to homotopy" ([9, equation (2.7)], [17, proposition A.1.1]).

After explaining the general theory, you should explain the sheaf model of $\Omega^{\infty-1}MTO(d)$, using Pontryagin-Thom theory (from day 1) and Phillips' theorem (from day 2), i.e. you should explain the proof of [9, theorem 3.4].

Talk 13. Sheaves of categories Explain how a sheaf of categories gives a topological category upon realization. The define the "cocycle sheaf" of a sheaf of categories, following [9, section 2.4] (see [17] for more detail). You should state, but not attempt to prove, [17, theorem 4.1.2]. Then define the sheaf of posets called $D^{\pitchfork}$ in [9], and prove that $\beta D^{\pitchfork}$ is homotopy equivalent to the sheaf model of $\Omega^\infty MTO(d)$. Finally, explain the zig-zag of maps that relates the nerve of this to the nerve of the category defined in talk 11, i.e. explain [9, proposition 2.9 and section 4].

This finishes the proof of the "main theorem" of [9].

Talk 14. The group completion theorem The goal of this talk is to explain how to deduce Madsen-Weiss' theorem (and hence Mumford's conjecture) from a theorem about cobordism categories. The main step was proved in [25], but use [9, section 7] as your main reference. Define the "positive boundary category" $\mathcal{C}_{d,\partial}$ and state Theorem 6.1 of [9]. The main goal of this talk is to show that there is a homology equivalence $\mathbb{Z} \times B\Gamma_\infty \to \Omega B(\mathcal{C}_{2,\partial})$, which is done in [9, section 7]. This relies on the Harer stability theorem (day 3) and the "group completion theorem", i.e. [9, prop. 7.1]. The proof uses [16, proposition 4], and if there is time, it is desirable to discuss the proof of that as well.

Talk 15. The positive boundary category This talk should outline the proof of [9, Theorem 6.1], i.e. the statement that $B\mathcal{C}_{d,\partial} \simeq B\mathcal{C}_d$. Reference: [9, chapter 6].

References

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Books that are useful to have available during the summer school

Hirsch: Differential topology

Brown: Cohomology of groups

Weibel: Introduction to homological algebra (for spectral seqences)

Hatcher: Algebraic topology

Kosinski: Differential manifolds

Stong: Notes on cobordism theory.