MATH 283A: TOPICS IN TOPOLOGY
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# 1 January 8

We'll start by discussing Milnor's construction of what is now called Milnor $K$-theory, done in the 1970 paper [12].

Let $F$ be a field, with unit group $F^\times$. Consider the free associative algebra on $F^\times$, i.e. the tensor algebra: $T(F^\times) = \bigoplus_{n=0}^\infty \left(F^\times\right)^{\otimes_{\mathbf{Z}} n}$. We define the Milnor $K$-theory (which from now on, we will just refer to as "$K$-theory") of $F$:

**Definition 1.1.** For a field $F$, the *Milnor $K$-theory* of $F$ is:

$$K_*^M(F) = T(F^\times)/\langle a \otimes b \mid a, b \in F^\times, a + b = 1\rangle$$

Since we quotient $T(F^\times)$ by a homogeneous ideal, $K_*^M(F)$ is naturally a graded associative algebra, with graded pieces denoted $K_n^M(F)$. The relation $a \cdot b = 0$ for $a + b = 1$ is called the *Steinberg relation*.

There is a canonical isomorphism $\ell\colon F^\times \to K_1^M(F)$ [1] sending $a \in F$ to the class of $a \in T_1(F) = F$. This satisfies:

- $\ell(1) = 0$
- $\ell(ab) = \ell(a) + \ell(b)$.
- $\ell(a) \cdot \ell(b) = 0$ whenever $a + b = 1$.

**Lemma 1.2.** If $a + b \in \{0, 1\}$, $\ell(a) \cdot \ell(b) = 0 \in K_2^M(F)$.

*Proof.* If $a + b = 1$, this is one of the properties of $\ell$ mentioned above. If $a + b = 0$, then if $a = 1$, $\ell(a) = 0$, so $\ell(a) \cdot \ell(b) = 0$. If $a \neq 1$, then in $F^\times$ we have:

$$\frac{1-a}{1-a^{-1}} = -a \cdot \frac{a-1}{a-1} = -a$$

Thus, $\ell(-a) = \ell\left(\frac{1-a}{1-a^{-1}}\right) = \ell(1-a) - \ell(1-a^{-1})$, so:

$$\begin{aligned}
\ell(a) \cdot \ell(-a) &= \ell(a) \cdot \ell(1-a) + \ell(a) \cdot \ell(1-a^{-1}) \\
&= \ell(a) \cdot \ell(1-a^{-1}) \\
&= -\ell(a^{-1}) \cdot \ell(1-a^{-1}) \\
&= 0
\end{aligned}$$

□

[1] called $\ell$ for "logarithm"

We can proceed similarly to prove:

**Lemma 1.3.** If $a_1 + \cdots + a_n \in \{0, 1\}$, then $\ell(a_1) \cdot \cdots \cdot \ell(a_n) = 0 \in K_n^M(F)$.

The multiplication on $K_*^M(F)$ is graded commutative:

**Lemma 1.4.** $\ell(a)\ell(b) = -\ell(b)\ell(a)$ for all $a, b \in F^\times$

*Proof.*

$$\begin{aligned}
\ell(a)\ell(b) + \ell(b)\ell(a) &= \ell(a)\left(\ell(-a) + \ell(b)\right) + \ell(b)\left(\ell(a) + \ell(-b)\right) \\
&= \ell(a)\ell(-ab) + \ell(b)\ell(-ab) \\
&= \left(\ell(a) + \ell(b)\right)\ell(-ab) \\
&= \ell(ab)\ell(-ab) \\
&= 0
\end{aligned}$$

□

By graded commutativity, $\ell(a) \cdot \ell(a) = -\ell(a) \cdot \ell(a)$, so $\ell(a) \cdot \ell(a)$ is 2-torsion. We can ask if it is actually $0$, i.e. if multiplication is alternating. It turns out that we have:

**Lemma 1.5.** $\ell(a) \cdot \ell(a) = \ell(-1) \cdot \ell(a)$

*Proof.* $\ell(a) \cdot \ell(a) = \ell(a)\left(\ell(-1) + \ell(-a)\right) = \ell(-1) \cdot \ell(a)$ □

**Example 1.6.** We have a non-trivial ring homomorphism $K_*^M(\mathbf{R}) \to \mathbf{F}_2$ sending $\ell(a)$ to 0 if $a > 0$ and 1 if $a < 0$. This sends $\ell(-1)^n$ to $1^n = 1$, so we get $\ell(a) \cdot \ell(a) = \ell(a)$ and thus the multiplication is not alternating.

Now, we want to indicate why Milnor $K$-theory is a useful invariant of the field $F$. We recall the state of algebraic $K$-theory in 1970. If $A$ is a ring, there were definitions available for $K_0(A)$, $K_1(A)$, $K_2(A)$, and $K_3(A)$:

For each $n$, we may define groups $\mathrm{St}_n(A)$ called the $n$-th *Steinberg group* of $A$. This has generators $x_{ij}(\lambda)$ for $1 \leq i \neq j \leq n$, $\lambda \in A$, and relations $x_{ij}(\lambda)x_{ij}(\mu) = x_{ij}(\lambda + \mu)$ and

$$[x_{ij}(\lambda), x_{kl}(\mu)] = \begin{cases} x_{il}(\lambda\mu) & j = k \\ 1 & j \neq k \end{cases}$$

There is a homomorphism from $\mathrm{St}_n(A)$ to $\mathrm{GL}_n(A)$ sending $x_{ij}(\lambda)$ to the elementary matrix

$$e_{ij}(\lambda) = \begin{pmatrix} 1 & \lambda & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

with the $\lambda$ in position $i, j$.

Then, we may define:

**Definition 1.7.** The *Steinberg group of $K$* is $\mathrm{St}(A) = \varinjlim_n \mathrm{St}_n(A)$.

The maps $\mathrm{St}_n(A) \to \mathrm{GL}_n(A)$ patch together to a map $\Phi\colon \mathrm{St}(A) \to \mathrm{GL}(A) := \varinjlim_n \mathrm{GL}_n(A)$. Then we define:

**Definition 1.8.**
- $K_1(A) = \mathrm{GL}(A)/\Phi(\mathrm{St}(A))$ When $A$ is a field[2], there is an isomorphism $K_1(A) = \mathrm{GL}(A)/\Phi(\mathrm{St}(A)) \xrightarrow{\sim} A^\times$ given by the determinant. In particular, $K_1(F) \simeq K_1^M(F)$ for a field $F$.
- $K_2(A) = \ker(\Phi)$.
- $K_3(A) = H_3(\mathrm{St}(A))$.

In 1973, Quillen extended this by defining *(Quillen) algebraic $K$-theory* $K_*^Q(A) = \pi_*(K(A))$ (i.e. homotopy groups) for a particular topological space $K(A)$. This agrees with the definitions in terms of the Steinberg group for degrees 0 to 3.

Now, assume that $A = F$ is a field. For $\lambda, \mu \in F^\times$, pick elements $D_\mu, D_\lambda \in \mathrm{St}_3(F)$ mapping to $\begin{pmatrix} \mu & & \\ & 1 & \\ & & \mu^{-1} \end{pmatrix}$, $\begin{pmatrix} \lambda & & \\ & \lambda^{-1} & \\ & & 1 \end{pmatrix}$ in $\mathrm{GL}_3(F)$. Then $[D_\lambda, D_\mu] \in \ker(\Phi)$. We denote by $\{\lambda, \mu\}$ the corresponding class in $K_2(F)$.

We have:

**Lemma 1.9.** $\{\lambda, \mu\} = 0$ if $\lambda + \mu = 1$

This is the motivating appearance of the Steinberg relation, and gives a well-defined map $K_2^M(F) \to K_2(F)$ sending $\ell(a)\ell(b)$ to $\{a, b\}$. Milnor's definition, which came before Quillen's definition of higher algebraic $K$-theory, was motivated by the following theorem:

**Theorem 1.10** (Matsumoto)**.** This is an isomorphism from $K_2^M(F)$ to $K_2(F)$.

Another place where the Steinberg relation appears is in the study of *Kähler differentials*. Recall that for a field $F$, we have the $F$-vector space $\Omega^1_{F/\mathbf{Z}}$ of Kähler differentials of $F$. This is defined by:

$$\Omega^1_{F/\mathbf{Z}} = \bigoplus_{a \in F} F \cdot da / \big(d(a+b) = da + db, d(ab) = adb + bda, d(1) = 0\big)$$

From this, we obtain the *algebraic de Rham complex* $\Omega^*_{F/\mathbf{Z}} = \bigwedge^*_F \Omega^1_{F/\mathbf{Z}}$. This is an alternating graded ring.

There is a map $\mathrm{dlog}\colon K_1^M(F) \simeq F^* \to \Omega^1_{F/\mathbf{Z}}$ given by $a \mapsto a^{-1}\, da$. If $a + b = 1$ for $a, b \in F^*$, we have $da + db = 0 \in \Omega^1_{F/\mathbf{Z}}$, so:

$$\frac{da}{a} \wedge \frac{db}{b} = a^{-1}b^{-1}(da) \wedge (-da) = 0$$

Thus, $\mathrm{dlog}(a) \wedge \mathrm{dlog}(b) = 0$ for $a + b = 1$, and so we get a (unique) map of rings $K_*^M(F) \to \Omega^*_{F/\mathbf{Z}}$ sending $\ell(a)$ to $\mathrm{dlog}(a)$.

When $F = \mathbf{Q}(x_1, \ldots, x_n)$ is a purely transcendental extension of $\mathbf{Q}$ of transcendence degree $n$, then $\Omega^1_{F/\mathbf{Z}}$ has basis $dx_1, \ldots, dx_n$. This implies that $\Omega^k_{F/\mathbf{Z}}$ has basis $\{dx_{i_1} \wedge \cdots \wedge dx_{i_k} \mid 1 \le i_1 <$

[2]this holds for certain other types of rings as well, but is a harder result

$i_2 < \cdots < i_k \leq n\}$. This makes sense when we think of $F$ as the field of meromorphic functions on $\mathbf{A}^n_{\mathbf{Z}} = \operatorname{Spec}\big(\mathbf{Z}[x_1, \ldots, x_n]\big)$.

Another fact[3] is that if $\mathbf{Q} \hookrightarrow F$ is a finite extension, then $K_i^M(F) \simeq \bigoplus_{F \hookrightarrow \mathbf{R}} \mathbf{Z}/2$ for $i \geq 3$. These facts indicate that the study of the higher $K_i^M(F)$ is interesting mostly for function fields of varieties of sufficiently large dimension relative to $i$.

# 2 January 10

Last time, we defined Milnor $K$-theory of a field $F$:

$$K_*^M(F) = \mathbf{Z}\langle \ell(a)) \mid a \in F^\times \rangle / (\ell(a) + \ell(b) \mid a + b = 1)$$

We saw the defining Steinberg relation appear in de Rham cohomology and in algebraic $K$-theory, and today we will discuss another manifestation of this relation: quadratic forms. Appearing in the title of Milnor's paper, these are important in Milnor's original conception of Milnor $K$-theory.

**Definition 2.1.** Let $F$ be a field of characteristic different from 2. We define a *quadratic form* $(M, q)$ over $F$ to be an $F$-vector space $M$ together with a function $q\colon M \to F$ which is a homogeneous polynomial of degree two with respect to a basis of $M$.

Given a quadratic form $(M, q)$, we may define a function

$$b(x, y) = q(x + y) - q(x) - q(y)$$

this function is bilinear and symmetric, and we may recover $q$ by the relation $q(x) = b(x, x)/2$. Thus, there is a one-to-one correspondence between quadratic forms and symmetric bilinear forms.[4]

Now, assume that the map $b\colon\ : M \to M^\vee = \operatorname{Hom}_F(M, F)$ sending $x$ to $b(x, \cdot)$ is an isomorphism. In this case, we say that $(M, q)$ is *non-degenerate*. With respect to a basis, we may write $b(x, y) = x^t\, A\, y$ for a matrix $A$, and non-degeneracy says exactly that $\det(A) \neq 0$.

Two quadratic forms $(M, q), (M', q')$ are equivalent if there exists an $F$-linear isomorphism from $M$ to $M'$ which pulls back $q'$ to $q$. In terms of matrices, this equivalence relation replaces the matrix $A$ with $B^t\, A\, B$ for some $B \in \mathrm{GL}_n(F)$.

In dimension $n = 1$, we may write a quadratic form as $\langle a \rangle = (F, q(x) = ax^2)$ for $a \in F^\times$. Then $\langle a \rangle \simeq \langle b \rangle$ iff $a/b \in (F^\times)^2$.

There are binary operations on the set of non-degenerate quadratic forms defined by

$$(M, q) \oplus (M', q') = (M \oplus M', q + q')$$

where $(q + q')(m, m') = q(m) + q'(m')$ and

$$(M, q) \otimes (M', q') = (M \otimes M', qq')$$

Now, using these operations, we may show that the set of equivalence classes of non-degenerate quadratic forms $(M, q)$ over $F$ is a ring without subtraction. We define:

[3] Bass, Tate: *The Milnor ring of a global field*

[4] This is not true when 2 is not invertible!

**Definition 2.2.** The *Grothendieck-Witt ring* of $F$, denoted $\mathrm{GW}(F)$, is the ring obtained by formally adjoining additive inverses to the set of non-degenerate quadratic forms over $F$ with these operations.

We have the following easy proposition:

**Proposition 2.3.** Any $(M, q)$ with $\dim M = n$ may be diagonalized as:

$$(M, q) \simeq \langle a_1 \rangle \oplus \cdots \oplus \langle a_n \rangle$$

for some $a_i \in F$.

This leads to the question of when $\langle a_1 \rangle \oplus \langle a_2 \rangle \simeq \langle a_1'' \rangle \oplus \langle a_2' \rangle$.

First, consider $\langle 1 \rangle + \langle -1 \rangle \in \mathrm{GW}(F)$, i.e. the (class of the) quadratic form defined by $q(x, y) = x^2 - y^2$. Since $x^2 - y^2 = (x + y)(x - y)$, we may make an invertible change of variables to turn this into the form $q'(x, y) = xy$. This form is called the *hyperbolic plane*.

Now, we obtain the following diagram:

$$\begin{tikzcd} & \mathrm{GW}(F) \arrow[d, "\dim"] \\ \mathbf{Z} \arrow[ur, "\langle 1\rangle + \langle -1\rangle"] \arrow[r, "2"] & \mathbf{Z} \end{tikzcd}$$

The image of the diagonal map is an ideal, since $\langle a \rangle \big(\langle 1 \rangle + \langle -1 \rangle\big) = \langle a \rangle + \langle -a \rangle$. This is the class of the form $(x, y) \mapsto ax^2 - ay^2 = (a(x + y))(x - y)$, so by making the change of variables $x' = a(x + y), y' = x - y$, we see that this is isomorphic to the hyperbolic plane.

This allows us to define:

**Definition 2.4.** The *Witt ring* of $F$, denoted $\mathrm{W}(F)$, is the ring $\mathrm{GW}(F)/\mathbf{Z}\big(\langle 1 \rangle + \langle -1 \rangle\big)$.

By considering the dimension maps, we obtain a commutative diagram:

$$\begin{tikzcd} \widehat{I} \arrow[r, "\simeq"] \arrow[d] & I \arrow[d] \\ \mathrm{GW}(F) \arrow[r] \arrow[d, "\dim"] & \mathrm{W}(F) \arrow[d, "\dim \pmod 2"] \\ \mathbf{Z} \arrow[r] & \mathbf{Z}/2 \end{tikzcd}$$

Here, the vertical rows are exact, i.e. $I, \widehat{I}$ are the respective kernels of the dimension maps. We note that $I \cap \mathbf{Z}\big(\langle 1 \rangle + \langle -1 \rangle\big) = 0$.

Now, what does all this have to do with Milnor $K$-theory? We have a map $K_1^M(F) = F^\times \twoheadrightarrow I$ given by $a \mapsto \langle a \rangle - \langle 1 \rangle$. Then $a^2 \mapsto \langle a^2 \rangle - \langle 1 \rangle = 0$, since $\langle a^2 \rangle \simeq \langle 1 \rangle$. Since $\ell(a^2) = 2\ell(a)$, we think of this as a map $s\colon K_1^M(F)/2 \twoheadrightarrow I$.

Note that:

$$\begin{aligned} s(ab) - s(a) - s(b) &= \langle ab \rangle - \langle a \rangle - \langle b \rangle + \langle 1 \rangle \\ &= \big(\langle a \rangle - \langle 1 \rangle\big)\big(\langle b \rangle - \langle 1 \rangle\big) \in I^2 \end{aligned}$$

This gives a corollary:

**Corollary 2.5.** The map $s\colon K_1^M(F)/2 \twoheadrightarrow I/I^2$ sending $\ell(a)$ to $\big(\langle a\rangle - \langle 1\rangle\big)$ is a homomorphism.

We may tensor this map with itself to get a map $F^\times \otimes F^\times \to I/I^2 \otimes I/I^2$, and compose this with the multiplication map $I/I^2 \otimes I/I^2 \to I^2/I^3$. This sends $a \otimes b$ to $\big(\langle a\rangle - \langle 1\rangle)\big)\big(\langle b\rangle - \langle 1\rangle\big)$.

**Proposition 2.6.** If $a+b=1$, $a\otimes b$ maps to $0$ under this map.

*Proof.* We may write $\langle ab\rangle + \langle 1\rangle$ as:

$$\begin{aligned}\begin{pmatrix} ab & \\ & 1\end{pmatrix} &= \begin{pmatrix} a(1-a) & \\ & 1\end{pmatrix}\\ &= \begin{pmatrix} a-a^2 & \\ & 1\end{pmatrix}\\ &\sim \begin{pmatrix} a & a \\ a & 1\end{pmatrix}\\ &\sim \begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix}\end{aligned}$$

□

Thus, we obtain a map $s_2$ from $K_2^M(F) = F^\times \otimes F^\times/(a\otimes b \mid a+b) = 1$ to $I^2/I^3$. By the universal property of the construction of Milnor $K$-theory, we obtain a unique ring map:

$$s\colon K_*^M(F)/2 \twoheadrightarrow \bigoplus_{n=0}^{\infty} I^n/I^{n+1} = \mathrm{gr}_I(\mathrm{W}(F))$$

sending $\ell(a)$ to $\big(\langle a\rangle - \langle 1\rangle\big) \in I/I^2$. A natural question which appears at this point is:

**Question 1.** Is $\bigcap_{n=0}^{\infty} I^n = 0$?

This question was settled affirmatively by Arason and Pfister in 1971 [2]. Another fundamental question attached to this story turned out to be much harder:

**Theorem 2.7** (Milnor Conjecture on quadratic forms)**.** The map $s$ defines an isomorphism from $K_*^M(F)/2$ to $\mathrm{gr}_I(\mathrm{W}(F))$.

This was proved by Orlov-Vishik-Voevodsky in [10]. Much earlier, Milnor proved that the maps $s_1\colon K_1^M(F)/2 \twoheadrightarrow I/I^2$ and $s_2\colon K_2^M(F)/2 \to I^2/I^3$ are *injective*. He constructs a splitting using "Stiefel-Whitney classes": we may think of quadratic forms over $F$ as some sort of vector bundle[5] over $\operatorname{Spec} F$, and $K_*^M(F)$ as some sort of cohomology theory for $\operatorname{Spec} F$, so we might expect there to be a "characteristic class" construction connecting them.

We have a map $w_1\colon \mathrm{GW}(F) \to F^\times/(F^\times)^2 = K_1^M(F)/2$ sending $\langle a\rangle$ to the class of $\ell(a)$ in dimension $1$. More generally, if $A$ is a matrix representing a quadratic form, the class of $A$ maps

[5] or perhaps "vector bundle with inner product"

to the class of $\ell(\det(A))$. This gives a well-defined map on $\mathrm{GW}(F)$ because the determinant of $B^t\ A\ B$ differs from the determinant of $A$ by $\det(B)^2$, so they have the same class in $K_1^M(F)/2$.

We want to extend this to a "total Stiefel-Whitney class":

$$w = 1 + w_1 + \cdots : \mathrm{GW}(F) \to \left(\prod_{n=0}^{\infty} K_n^M(F)/2\right)^{\times}$$

sending $\langle a\rangle$ to $1 + w_1(\langle a\rangle) = 1 + \ell(a)$. If $\langle a\rangle \simeq \langle b\rangle$, then $a/b$ is a square, so the classes of $\ell(a)$ and $\ell(b)$ are the same in the codomain.

In order to show this map is really well-defined, it suffices[6] to show that if $\langle a_1\rangle + \langle a_2\rangle \simeq \langle b_1\rangle + \langle b_2\rangle$ implies that

$$\big(1 + \ell(a_1)\big)\,\big(1 + \ell(a_2)\big) = \big(1 + \ell(b_1)\big)\,\big(1 + \ell(b_2)\big) \tag{1}$$

in $K_*^M(F)$. We already proved $w_1$ is well defined, so it remains to consider the component in $K_2^M(F)$.

In other words, we assume that $a_1x^2 + a_2y^2 \sim b_1x^2 + b_2y^2$. Since $a_1 = a_1(1)^2 + a_2(0)^2$, it is in the image of this quadratic form. Thus (since equivalent forms have the same image), we may write $a_1 = b_1x^2 + b_2y^2$ for some $x, y \in F$. Then $1 = \frac{b_1x^2}{a_1} + \frac{b_2y^2}{a_1}$. The case $xy = 0$ is easy, so we assume $xy \in F^\times$ and thus, by the Steinberg relation, we have

$$0 = \ell\left(\frac{b_1x^2}{a_1}\right) \cdot \ell\left(\frac{b_2y^2}{a_1}\right)$$

In $K_2^M(F)/2$ this is $(\ell(b_1) - \ell(a_1))(\ell(b_2) - \ell(a_1))$, so $\ell(b_1)\ell(b_2) = \ell(a_1)(\ell(b_1) + \ell(b_2) - \ell(a_1))$. We already showed $w_1$ is well defined, so $\ell(a_1) + \ell(a_2) = \ell(b_1) + \ell(b_2) \in K_1^M(F)/2$, so we conclude $\ell(a_1)\ell(a_2) = \ell(b_1)\ell(b_2)$, finishing the proof of (1).

Finally, one may check that $w_1, w_2$ are left inverse to $s_1, s_2$, which shows Milnor's theorem that $s_1, s_2$ are injective.

# 3 1/12/18

Last time, we discussed the map from $K_*^M(F)/2$ to $\mathrm{gr}_I(\mathrm{W}(F)) = \oplus I^n/I^{n+1}$. That this is an isomorphism is the content of *Milnor's conjecture on quadratic forms*, proved by Orlov-Vishik-Voevodsky. Today, we will discuss another map from $K_*^M/2$, this time with target $H^*(G_F; \mathbf{Z}/2)$, i.e. the Galois cohomology of the trivial Galois module $\mathbf{Z}/2$. Here, $G_F$ is the absolute Galois group of $F$, i.e. the profinite group given by the inverse limit of $\mathrm{Gal}(L/F)$ as $L$ ranges over finite Galois field extensions.

We recall Hilbert's Theorem 90. Let $F \hookrightarrow L$ be a cyclic Galois field extension of degree $n$, i.e. a Galois field extension such that $\mathrm{Gal}(L/F) \simeq \langle\sigma\rangle \simeq \mathbf{Z}/n\mathbf{Z}$. There is a multiplicative norm map $N_{L/K} : L \to K$ defined by $a \mapsto \det_K(L \xrightarrow{\cdot a} L) = \prod_{i=0}^{n-1} \sigma^i(a)$. We think of $L^\times, K^\times$ as $\mathrm{Gal}(L/K)$-modules and write them additively, so $N_{L/K} = 1 + \sigma + \cdots + \sigma^{n-1}$. We have a complex:

$$L^\times \xrightarrow{\sigma - 1} L^\times \xrightarrow{N_{L/K}} K^\times$$

[6] see the reference in [12, Lemma 3.1]

The map on the left sends $b \in L^\times$ to $\sigma(b)/b$.

We have:

**Theorem 3.1** (Hilbert's Theorem 90)**.** The above complex is exact in the middle, i.e. the kernel of $N_{L/K}$ is equal to the image of $\sigma - 1$.

This complex is reminiscent of the resolution used to calculate group cohomology for the cyclic group $\mathrm{Gal}(L/K)$, i.e. the complex:

$$L^\times \xrightarrow{\sigma-1} L^\times \xrightarrow{1+\sigma+\cdots+\sigma^{n-1}} L^\times \xrightarrow{\sigma-1} L^\times \longrightarrow \cdots$$

This shows us that Hilbert's Theorem 90 is equivalent to the following statement, which is true for any finite Galois field extensions:

**Theorem 3.2** (Hilbert's Theorem 90, version 2)**.** For any finite Galois field extension $L/F$, $H^1(\mathrm{Gal}(L/F), L^\times) = 0$.

We can patch together the Galois cohomology of the finite extensions of $F$ to get absolute Galois cohomology of $F$: Pick a separable closure $F \hookrightarrow F_s$ and take colimits over all finite Galois sub-extensions $F \hookrightarrow L \subseteq F_s$.

We have:

**Definition 3.3.** $H^n(G_F; F_s^\times) := \varinjlim_L H^n(\mathrm{Gal}(L/F); L^\times)$

We may also define Galois cohomology intrinsically, without passing to finite sub-extensions, in the category of *discrete* $G_F$-modules by using *continuous* cochains. We get another version of Hilbert's theorem 90:

**Theorem 3.4** (Hilbert's Theorem 90, version 3)**.** $H^1(G_F; F_s^\times) = 0$

Assume that $\mathrm{char}(F) \neq 2$. We have an action of $G_F$ on $F_s^\times$, leading to the short exact Kummer sequence of $G_F$-modules:

$$1 \longrightarrow \mu_2(F_s^\times) = \{\pm 1\} \longrightarrow F_s^\times \xrightarrow{x \mapsto x^2} F_s^\times \longrightarrow 1$$

There is a similar sequence with $\mu_n$ and $n$-th powers when $\mathrm{char}(F) \nmid n$.

This gives a long exact sequence:

$$\cdots \longrightarrow H^0(G_F; F_s^\times) \longrightarrow H^0(G_F; F_s^\times) \xrightarrow{s} H^1(G_F; \mathbf{Z}/2) \longrightarrow H^1(G_F; F_s^\times) = 0$$

We may rewrite this as:

$$F^\times \xrightarrow{x \mapsto x^2} F^\times \longrightarrow H^1(G_F; \mathbf{Z}/2) \longrightarrow 0$$

Thus, we have an isomorphism $s\colon K_1^M(F)/2 = F^\times/(F^\times)^2 \xrightarrow{\sim} H^1(G_F; \mathbf{Z}/2)$. Now, there are cup products in Galois cohomology, giving a map $\smile\colon H^1(G_F; \mathbf{Z}/2) \otimes H^1(G_F; \mathbf{Z}/2) \to H^2(G_F; \mathbf{Z}/2)$. This gives us another manifestation of the Steinberg relation:

**Proposition 3.5** (Bass-Tate)**.** If $a, b \in F^\times$ satisfy $a+b=1$, then $\delta(a) \smile \delta(b) = 0 \in H^2(G_F; \mathbf{Z}/2)$.

This gives the immediate corollary.

**Corollary 3.6.** The Kummer map $s$ determines a well-defined map of rings $s\colon K_*^M(F)/2 \to H^*(G_F; \mathbf{Z}/2)$.

Now, we prove the proposition:

*Proof.* Assume that $a, b \notin (F^\times)^2$, $b = 1-a$. Pick $\alpha \in F_s$ with $\alpha^2 = a$. Let $E = F[\alpha] \simeq F[X]/(X^2 - \alpha)$ be the field extension generated by $\alpha$. This gives an embedding map $\pi\colon G_E \hookrightarrow G_F$ realizing $G_E$ as an index-two subgroup of $G_F$.

This determines a commutative diagram, with $\pi^*$ the restriction map and $\pi_*$ the co-restriction or transfer map.

$$\begin{tikzcd}
E^\times \arrow[r, "\delta"] \arrow[d, shift left, "N_{E/F}"] & H^1(G_E; \mathbf{Z}/2) \arrow[d, shift left, "\pi_*"] \\
F^\times \arrow[r, "\delta"] \arrow[u, shift left] & H^1(G_F; \mathbf{Z}/2) \arrow[u, shift left, "\pi^*"]
\end{tikzcd}$$

We can think of this transfer map topologically. Let $f\colon X \to Y$ be a finite covering space. Then there is a map $f_*\colon H^*(X; \mathbf{Z}/2) \to H^*(Y; \mathbf{Z}/2)$ (more generally, this works for any constant coefficient module or sheaves when there is a map of sheaves compatible with $f$), defined at the level of cochains by summing over lifts of chains in $Y$ to chains in $X$.

This gives the transfer map in group cohomology because (for $M$ a constant coefficient group) $H^*(G; M) = H^*(BG; M)$, and when $H \subseteq G$ is a finite-index subgroup, we obtain a finite covering map $\mathrm{B}H = \mathrm{E}G/H \to \mathrm{B}G = \mathrm{E}G/G$.[7]

Note that the transfer map is *not* multiplicative, but it is a homomorphism of modules over the ring $H^*(G_F; \mathbf{Z}/2)$, which acts on $H^*(G_E; \mathbf{Z}/2)$ via $\pi^*$. Topologically, we see this because $\pi_*(x \smile \pi^*(y)) = \pi_*(x) \smile y$.

Now, we may compute that $N_{E/F}(\alpha) = \alpha(-\alpha) = -a$, $N_{E/F}(1-\alpha) = (1-\alpha)(1+\alpha) = 1 - \alpha^2 = 1 - a$. Then, the commutative diagram gives us:

$$\begin{aligned}
\delta_F(a) \smile \delta_F(1-a) &= \delta_F(a) \smile \delta_F(N_{E/F}(1-\alpha)) \\
&= \delta_F(a) \smile \pi_*\left(\delta_E(1-\alpha)\right) \\
&= \pi_*\left(\pi^*(\delta_F(a)) \smile \delta_E(1-\alpha)\right) \\
&= \pi_*\left(\delta_E(a) \smile \delta_E(1-\alpha)\right) \\
&= \pi_*\left(\delta_E(\alpha^2) \smile \delta_E(1-\alpha)\right) \\
&= 0
\end{aligned}$$

$\square$

[7]Note that for the purposes of group cohomology, $\mathrm{E}H$ is any contractible space with a free action of $H$; by restriction the $G$-action on $\mathrm{E}G$ to $H$, we may identify $\mathrm{E}G$ with $\mathrm{E}H$ as topological spaces.

This construction works more generally, proceeding via the degree $n$ Kummer sequence:

$$1 \longrightarrow \mu_n(F_s^\times) \longrightarrow F_s^\times \longrightarrow F_s^\times \longrightarrow 1$$

This gives a map $K_1^M(F)/n \xrightarrow{\sim} H^1(G_F; \mu(n))$. By an analogous result to the one above, this extends to a map of rings:

$$K_*^M(F)/n \to \bigoplus_i H^i(G_F; \mu_n^{\otimes_{\mathbf{Z}/n} i})$$

See [16] for details.

This leads to:

**Conjecture 1** (Milnor Conjecture)**.** The map $K_*^M(F)/2) \to H^*(G_F; \mathbf{Z}/2)$ is an isomorphism.

We have the incredible result:

**Theorem 3.7** ("Norm Residue Theorem": Voevodsky, Root)**.** For any field $F$ with $\mathrm{char}(F) \nmid n$, the map $K_*^M(F)/n \to \bigoplus_i H^i(G_F; \mu_n^{\otimes_{\mathbf{Z}/n} i})$ is an isomorphism.

Prior to its proof, this was known as the Bloch-Kato conjecture (not to be confused with the Bloch-Kato conjecture on special values of $L$-functions, which is wide open).

Already in degree 2, this is a hard result:

**Theorem 3.8** (Merkurjev)**.** $K_2^M(F)/2 \to H^2(G_F; \mathbf{Z}/2)$ is an isomorphism.

**Theorem 3.9** (Merkurjev-Souslin)**.** $K_2^M(F)/n \to H^2(G_F; \mu_n^{\otimes 2})$ is an isomorphism.

One reason to care about the result in degree 2 is that $H^2(G_F; \mathbf{Z}/2)$ is isomorphic to the 2-torsion part of the *Brauer group* of $F$, denoted ${}_2\mathrm{Br}(F)$.

This group is constructed from the set of isomorphism classes of central simple $F$-algebras $A$, i.e. associative $F$-algebras with center $F$ such that $A \otimes_F E \simeq \mathrm{M}_n(E)$ (the $n$-dimensional matrix algebra over $E$) for some $n$ and some finite extension $E/F$. We mod out this set by the equivalence relation $A \sim M_n(A)$. This has a group structure with multiplication given by tensor product over $F$ and unit element given by the trivial central simple $F$-algebra $F$. The inverse of $A$ is $A^{\mathrm{op}}$, the *opposite algebra* of $A$ (i.e. it has the same underlying $F$-vector space as $A$, but with multiplication given by $a \cdot_{A^{\mathrm{op}}} b = b \cdot_A a$). Thus, the two-torsion elements are exactly the central simple algebras $A$ with $A \simeq A^{\mathrm{op}}$.

By composing the Kummer map $K_2^M(F)/2 \to H^2(G_F; \mathbf{Z}/2)$ and the isomorphism of the latter group with ${}_2\mathrm{Br}(F)$, we may verify that $\ell(a)\ell(b)$ maps to the class of the central simple algebra $F\langle x, y\rangle/(x^2 - a, y^2 - b, xy + yx)$. The surjectivity part of Merkurjev's theorem implies that these so-called "quaternion algebras" generate the 2-torsion of the Brauer group.

## 3.1 Cheat sheet by SG on group cohomology

### 3.1.1 As a special case of $\mathrm{Ext}$

If $G$ is a (discrete) group, and $M$ is a module over the group ring $\mathbb{Z}[G]$, then group cohomology may be defined as $H^i(G; M) = \mathrm{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z}, M)$.

### 3.1.2 Via singular homology

For any group $G$ there exists a connected CW complex $BG$ with basepoint $* \in BG$, an isomorphism $\phi : \pi_1(BG, *) \cong G$, such that the universal cover of $BG$ is contractible. The triple $(BG, *, \phi)$ is unique up to homotopy equivalence in a suitable sense (and the homotopy equivalence is unique up to homotopy). If we pick such a triple for $G$, we get one for any subgroup. Indeed, if $EG \to BG$ denotes the universal cover then $G$ acts on $EG$ by deck transformations and if $H < G$ is a subgroup, then the quotient map $EG \to EG/H$ is a covering space. Hence we may take $BH = EG/H$. In this model, $BH \to BG$ is a covering space; it is a finite covering space if $|G : H| < \infty$.

Then we may define $H^i(G; \mathbb{Z}) = H^i(BG)$, so group cohomology is a special case of singular cohomology. If $M$ has non-trivial action, group cohomology becomes a special case of "cohomology with local coefficients" (see e.g. Hatcher's book).

### 3.1.3 Via an explicit cochain complex

Finally, one may define $C^n(G; M)$ as the set of all functions $f : G \times \cdots \times G \to M$, where the product has $n$ factors. There is a coboundary map $\delta : C^{n-1}(G; M) \to C^n(G; M)$ given by

$$(\delta f)(g_1, \ldots, g_n) = g_1.f(g_2, \ldots, g_n) + \sum_{i=1}^{n-1} (-1)^n f(g_1, \ldots, g_i g_{i+1}, \ldots, g_n) + (-1)^n f(g_1, \ldots, g_{n-1}),$$

where the first term involves the action of $g_1$ on $f(g_2, \ldots, g_n) \in M$.

$n = 1$ is particularly interesting: $f : G \to M$ is a cocycle if and only if it satisfies $f(g_1 g_2) = g_1.f(g_2) + f(g_1)$. If the action of $G$ on $M$ is trivial, this says precisely that $f$ is a homomorphism and in this case $H^1(G; M)$ is just $\mathrm{Hom}(G, M)$.

### 3.1.4 Transfer

For any group homomorphism $\phi : H \to G$ there is an induced $f^* : H^*(G; M) \to H^*(H; M)$, where $H$ acts on $M$ via $\phi$. If $H \subset G$ has finite index, there is a "transfer" (also called "corestriction") map $f_* : H^*(G; M) \to H^*(H; M)$ in the other direction. The most important properties to know about this construction is that it is a homomorphism of $H^*(G; M)$-modules, and that the composition $f_* \circ f^*$ is multiplication by the index of $H$ in $G$.

I find the topological description (lifting simplices in a model for $BH \to BG$ which is a finite covering space) the most intuitive, but the transfer can of course be described in any of the three equivalent models of group cohomology given above. For example, in the explicit cochain complex, one chooses a set-theoretic section $s : G/H \to G$ of the quotient map (i.e. picks a representative of each coset), defines $\phi_* : C^*(H; M) \to C^*(G; M)$ by

$$(\phi_* f)(g_1, \ldots, g_n) = \sum_{x \in G/H} (sx)^{-1}.f((sx)g_1(s(xg_1))^{-1}, \cdots (sx)g_n(s(xg_n))^{-1}),$$

and checks that this is a map of cochain complexes and has the desired properties.

### 3.1.5 Profinite topological groups

For certain kinds of topological groups $G$, there is a way to take the topology into account when defining group cohomology. If you're familiar with cohomology of discrete groups and pretend that we just take cohomology of the underlying discrete group, your intuition might not be too far off. (If on the other hand you are familiar with taking "classifying spaces" of topological groups, then you should know that this is not the correct thing to do here.)

For much more on this, see e.g. Serre's *Galois cohomology*. For even more, see Neukirch–Schmidt–Wingberg: *Cohomology of number fields*.

If $G$ is a topological group we may consider open normal subgroups $H \subset G$ of finite index. For such $H$ the quotient group $G/H$ is a finite group and the quotient topology is the discrete one. There is a canonical continuous homomorphism $G \to G/H$, and hence a continuous map

$$G \to \varprojlim_H G/H,$$

where the inverse limit runs over open normal subgroups of finite index. This inverse limit inherits a topology from the product of the discrete $G/H$, and the resulting topological group is called the *profinite completion* of $G$. The topological group $G$ is called *profinite* if the canonical continuous homomorphism from $G$ is a homeomorphism.

If $M$ is an abelian group with an action $G \times M \to M$ which is continuous in the discrete topology of $M$, then we can define the continuous cochain complex $C^*_{\mathrm{cont}}(G; M)$ using only continuous functions. One can show that the cohomology of this cochain complex is canonically isomorphic to the direct limit

$$\varinjlim_H H^*(G/H; M^H),$$

where $H$ runs through open normal subgroups of $G$ and $M^H \subset M$ is the subgroup fixed by $H$. Either can be taken as the definition of (continuous) cohomology of the profinite group $G$ with coefficients in $M$. The colimit description is useful if you are already familiar with cohomology of discrete groups and want to transfer some of your knowledge into continuous cohomology of profinite groups.

# 4 1/17/18

The goal for the next few lectures will be to develop *motivic cohomology*. This was conjectured to exist in the 1980's by Lichtenbaum ([8]) and Beilinson ([3]). Their idea is to associate cohomology groups $H^{p,q}(X)$ to a scheme $X$ satisfying certain properties, such as the Euler characteristic being related to special values of $\zeta$-functions. Voevodsky's motivation came more from the Atiyah-Hirzebruch spectral sequence, which relates topological $K$-theory to singular cohomology: motivic cohomology is supposed to have a similar relationship with Quillen's algebraic $K$-theory.

We will briefly discuss (complex) *topological $K$-theory*. Let $X$ be a finite CW complex; to $X$ we may associate the group $K^0(X)$, defined as the Grothendieck group of the category of (complex) vector bundles on $X$. It turns out that this functor is representable: $K^0(X) \simeq [X, \mathbf{Z} \times \mathrm{B}U] = \pi_0\big(\mathrm{Maps}(X, \mathbf{Z} \times \mathrm{B}U)\big)$. Here $\mathrm{B}U = \varinjlim \frac{U(2n)}{U(n) \times U(n)}$.

These groups are extended to negative degrees by defining $K^{-n}(X) = K^0(S^n X)$, and the resulting groups are also representable. If we write $ku_0 = \mathbf{Z} \times \mathrm{B}U$, we have $K^{-n}(X) = [X, \Omega^n ku_0]$, where $\Omega^n$ is the $n$-th fold loop space. The space $\mathbf{Z} \times \mathrm{B}U$ admits *deloopings*, i.e. $ku_0 \simeq \Omega ku_1 \simeq \Omega^2 ku_2$, and so on. The $ku_i$ form a *spectrum*. Thus, we have homotopy equivalences:

$$\mathrm{Maps}(X, ku_0) \simeq \Omega \mathrm{Maps}(X, ku_1) \simeq \Omega^2 \mathrm{Maps}(X, ku_2)$$

We have $K^{-n}(X) = \pi_n \mathrm{Maps}(X, ku_0) = \pi_{n+k} \mathrm{Maps}(X, ku_k)$ for any $k$. Thus, we may define $K^*(X)$ in positive degree as well by defining $K^n(X) = \pi_{k-n} \mathrm{Maps}(X, ku_k)$ for sufficiently large $k$.

The Atiyah-Hirzebruch spectral sequence comes from the *Postnikov tower* of $ku_0 = \mathbf{Z} \times \mathrm{B}U$. This gives a diagram:

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & \tau_{\leq 2}(ku_0) \arrow[d] \\
 & \tau_{\leq 1}(ku_0) \arrow[d] \\
ku_0 \arrow[r] \arrow[ur] \arrow[uur] & \tau_{\leq 0}(ku_0)
\end{tikzcd}$$

Here, $\tau_{\leq n}(ku_0)$ has $\pi_k = 0$ for all $k > n$, and the vertical maps are homotopy fibrations. The homotopy fiber is an Eilenberg-Mac Lane space $K(\pi_n(ku_0), n)$. These Eilenberg-Mac Lane spaces have the property that they represent cohomology: $[X, K(A, n)] \simeq H^n(X; A)$ for any abelian group $A$.

The Postnikov tower gives a tower of maps from $X$:

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & \mathrm{Maps}(X, \tau_{\leq 2}(ku_0)) \arrow[d] \\
 & \mathrm{Maps}(X, \tau_{\leq 1}(ku_0)) \arrow[d] \\
\mathrm{Maps}(X, ku_0) \arrow[r] \arrow[ur] \arrow[uur] & \mathrm{Maps}(X, \tau_{\leq 0}(ku_0))
\end{tikzcd}$$

The homotopy fibers are $\mathrm{Maps}(X, K(\pi_n(ku_0), n))$. This tower gives a spectral sequence for homotopy groups (with bi-degrees labeled as in the cohomological Serre spectral sequence):

$$H^p(X, \pi_{-q}(ku_0)) \implies \pi_{-p-q} \mathrm{Maps}(X, ku_0) = K^{p+q}(X)$$

This is a "fourth quadrant" cohomological spectral sequence: i.e. it's supported when $p > 0, q < 0$ with differentials increasing $p$ and decreasing $q$.

Now, we know the homotopy groups of $ku_0 = \mathbf{Z} \times \mathrm{B}U$, as a result of the following theorem:

**Theorem 4.1** (Bott Periodicity)**.**

$$\pi_*(\mathbf{Z} \times \mathrm{B}U) = \begin{cases} \mathbf{Z} & * \geq 0 \text{ even} \\ 0 & \text{else} \end{cases}$$

Thus, the Atiyah-Hirzebruch spectral sequence takes the form

$$E_2^{p,q} = \begin{cases} H^p(X; \mathbf{Z}) & \text{for } q \leq 0 \text{ even} \\ 0 & \text{otherwise} \end{cases} \qquad \Longrightarrow K^{p+q}(X).$$

The first possible differentials are $d_3 : E_3^{p,-2j} \to E_2^{p+3,-2j-2}$. It turns out that they can be identified with the composition

$$H^p(X; \mathbf{Z}) \to H^p(X; \mathbf{Z}/2) \xrightarrow{\mathrm{Sq}^2} H^{p+2}(X; \mathbf{Z}/2) \xrightarrow{\beta} H^{p+3}(X; \mathbf{Z}),$$

where $\mathrm{Sq}^2$ denotes the "Steenrod square" cohomology operation (cf. e.g. Section 4.L in Hatcher's textbook, or similar accounts in other textbooks), and $\beta$ denotes the "Bockstein homomorphism" (i.e. the connecting homomorphism in the long exact sequence associated to $0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}/2 \to 0$). In particular the $d_3$ differential satisfies $2d_3 = 0$, and the higher differentials can also be shown to be torsion. As an aside, the whole spectral sequence may be equipped with "Adams operations" converging to the usual Adams operations $\psi^k : K^*(X) \to K^*(X)$, in which $\psi^k$ acts as $k^j$ on the row $E_r^{*,-2j}$.

In Quillen $K$-theory, we associate to a scheme $X$ a topological space $K(X)$ with $K_i^Q(X) := \pi_i(K(X))$. As before, this space admits arbitrarily many de-loopings. There's supposed to be a so-called "motivic spectral sequence" whose $E_2$-page is given by a cohomology groups which we can denote (in a grading convention which makes it look analogous to the AHSS, but may not be the usual grading of the motivic spectral sequence)

$$E_2^{p,q} = \begin{cases} H^p(X; \mathbf{Z}(-\frac{q}{2})) & \text{for } q \leq 0 \text{ even} \\ 0 & \text{otherwise,} \end{cases}$$
$$\Longrightarrow \pi_{-p-q}(K(X)),$$

where $H^p\Big(X; \mathbf{Z}(-\frac{q}{2})\Big)$ is for now just notation for a functor indexed by two integers $(p, -q/2)$ that we haven't defined yet. (In the actual definition $\mathbf{Z}(-\frac{q}{2})$ will be a cochain complex of sheaves of abelian groups.)

Beilinson and Lichtenbaum were able to predict some properties which these groups should have, at least when $X$ is smooth over a field:

- $H^*(X; \mathbf{Z}(0))$ should be (Zariski) sheaf cohomology with coefficients in $\mathbf{Z}$. (This is $0$ in higher degrees when $X$ is smooth.)
- $H^p(X; \mathbf{Z}(1))$ should be $H^{p-1}(X; \mathscr{O}_X^\times)$, again in Zariski sheaf cohomology.
- $H^{2q}(X; \mathbf{Z}(q))$ should be the Chow groups $\mathrm{CH}^q(X)$ which measure algebraic cycles of dimension $q$ up to rational equivalence.[8]

[8]Note that this means that the Chow groups of $X$ should, after accounting for some differentials, give a filtration of $K_0(X)$. Indeed, the construction of Chern characters gives an isomorphism from $K_0(X) \otimes_{\mathbf{Z}} \mathbf{Q}$ to $\bigoplus_q \mathrm{CH}^q(X) \otimes_{\mathbf{Z}} \mathbf{Q}$.

In the case $X = \operatorname{Spec} F$, the above properties tell us that we should have $H^1(\operatorname{Spec} F; \mathbf{Z}(1)) = H^0(\operatorname{Spec} F; F^\times) = F^\times = K_1^M(F)$. We should also have $H^0(\operatorname{Spec} F; \mathbf{Z}(0)) = \mathbf{Z} = K_0^M(F)$. More generally, we should have $H^p(\operatorname{Spec} F; \mathbf{Z}(p)) = K_p^M(F)$. Thus, this spectral sequence should give a relationship between Milnor and Quillen $K$-theory for $F$. Because Quillen $K$-theory satisfies the Steinberg relation and agrees with Milnor $K$-theory in degrees up to 2, there is a canonical map of graded rings $K_*^M(F) \to K_*^Q(F)$, and we hope for this to be an edge map in the motivic spectral sequence.

Beilinson also proposed how to find these motivic cohomology groups. $H^*(X; \mathbf{Z}(0))$ and $H^*(X; \mathbf{Z}(1))$ are both sheaf cohomology for the Zariski topology. The former is just sheaf cohomology of the constant sheaf $\mathbf{Z}$. The latter is, up to a degree shift by 1, sheaf cohomology of the sheaf $\mathbf{G}_m = \mathscr{O}_X^\times$. Thus, $H^*(X; \mathbf{Z}(1))$ is cohomology with coefficients in the object $\mathbf{G}_m[-1]$. Beilinson's proposal, which Voevodsky accomplished, is to realize the $\mathbf{Z}(q)$ as chain complexes of sheaves.

How do we associate cohomology groups to complexes of sheaves? If we are given a cochain complex $\mathscr{F}^\bullet$ of sheaves of abelian groups, we may pick injective resolutions $\mathscr{F}^p \hookrightarrow \mathscr{I}^{\bullet,p}$ in such a way that we get the following diagram:

$$\begin{tikzcd}
\mathscr{F}^0 \arrow[r] \arrow[d] & \mathscr{F}^1 \arrow[r] \arrow[d] & \cdots \\
\mathscr{I}^{0,0} \arrow[r] \arrow[d] & \mathscr{I}^{0,1} \arrow[r] \arrow[d] & \cdots \\
\mathscr{I}^{1,0} \arrow[r] \arrow[d] & \mathscr{I}^{1,1} \arrow[r] \arrow[d] & \cdots \\
\vdots & \vdots &
\end{tikzcd}$$

Taking global sections gives us a bi-complex, and we can define *hypercohomology* $H^*(X; \mathscr{F}^\bullet) = H^*(\operatorname{Tot}(I^{*,*}))$, i.e. the cohomology groups of the total complex associated to this double complex. This is the complex with groups $\operatorname{Tot}(I^{*,*})^n = \bigoplus_{p+q=n} I^{p,q}$ with differential given (up to sign) on each $I^{p,q}$ as the sum of the horizontal and vertical differentials.[9]

# 5 1/19/2018

In order to construct motivic cohomology, we introduce a category $\mathrm{SH}(F)$, the *Morel-Voevodsky stable homotopy category of* $F$, which is supposed to be a "stable homotopy theory of smooth varieties". We expect cohomological functors to factor through a canonical functor from $\mathrm{Sm}_F$, the category of smooth schemes over $F$, to $\mathrm{SH}(F)$.

In particular, we expect to have the following diagrams of functors:

$$\begin{tikzcd}
\mathrm{Sm}_F \arrow[r] \arrow[dr, "{H^p(-,\mathbf{Z}(q)),\ K_i^Q(-)}"'] & \mathrm{SH}(F) \arrow[d] \\
& \mathbf{Ab}
\end{tikzcd}$$

[9]note: there is a natural quasi-isomorphism from $\mathscr{F}^\bullet$ to $\operatorname{Tot}(I^{\bullet,\bullet})$, so the hypercohomoogy of $\mathscr{F}^\bullet$ is also $H^i(RF)$ with $RF$ the derived functor of $F$, defined on the derived category of sheaves. See the Stacks Project, Tag 0133

We also want the vertical functors to be *representable*, as is the case with topological $K$-theory and singular homology. One construction of the motivic spectral sequence uses Voevodsky's "slice filtration" (analogous to the Postnikov filtration in usual stable homotopy theory). Just as singular cohomology of a space may be defined without first defining the stable homotopy category and Eilenberg–MacLane spectra, so may motivic cohomology be defined in a direct and "elementary" way (using only chain complexes, no fancier stable homotopy theory). Today, we will discuss the "direct" definition of $H^p(X; \mathbf{Z}(q))$ for $X$ smooth over a field $F$. Some references are the summary in Section 2.1 of Voevodsky's '96 preprint "The Milnor Conjecture", [14], and the textbook [11].

First, we discuss *cycles* on a smooth $F$-scheme $X$: compare the theory developed in [5].

**Definition 5.1.** A *cycle* on a smooth $F$-scheme $X$ is a finite linear combination $\sum n_V[V]$ with $n_V \in \mathbf{Z}$ and $V \subseteq X$ a closed subvariety (i.e. a closed irreducible topological subspace or equivalently a closed irreducible reduced subscheme, but *not* necessarily smooth).

If $f\colon X \to Y$ is a proper map, then the topological image $W = f(V) \subseteq Y$ is a closed and irreducible subvariety of $Y$. We define:

**Definition 5.2.** For $f\colon X \to Y$ a proper map between smooth $F$-schemes, the *proper pushforward* $f_*$ is defined by:

$$f_*\big([V]\big) = \begin{cases} 0 & \dim W < \dim V \\ (K(V) : K(W))[W] & \end{cases}$$

Here $(K(V) : K(W))$ is the degree of the function field extension $K(W) \hookrightarrow K(V)$, which is finite when $\dim W = \dim V$.

Cycles allow the development of *intersection theory*:

**Definition 5.3.** If $V, W \subseteq X$ are subvarieties of the smooth $F$-scheme $X$, then we say the intersection $V \cap W$ is *proper* if each irreducible component $T \subseteq V \cap W$ satisfies

$$\operatorname{codim}(T) = \operatorname{codim}(V) + \operatorname{codim}(W)$$

This is something like transversality in topology, but it is much weaker. For example, the line $Y = 0$ and the parabola $Y = X^2$ in the affine plane $\mathbf{A}^2_F = \operatorname{Spec} F[X,Y]$ have proper intersection but are not transverse since they are tangent at their intersection point $(0,0)$.

If $V, W$ have proper intersection, we want to define

$$[V] \cdot [W] = \sum_{\substack{T \subseteq V \cap W \\ \text{irreducible component}}} n_T[T]$$

for "multiplicities" $n_T$. To define this, let $\mathscr{O}_{X,T}$ be the local ring of $X$ along $T$, i.e. the stalk of $\mathscr{O}_X$ at the generic point $\operatorname{Spec} K(T) \hookrightarrow T$. (This ring consists exactly of those regular functions defined on open subsets $U \subseteq X$ such that $U \cap T \neq \emptyset$.) Let $\mathscr{I}, \mathscr{J}$ be the ideals of $\mathscr{O}_{X,T}$ corresponding to the closed subvarieties $V, W$ respectively. Now, we define:

**Definition 5.4** (Serre's formula)**.** If $V, W$ are two closed subvarieties of a smooth $F$-scheme $X$ with proper intersection, then for each irreducible component $T \subseteq V \cap W$, we have:

$$n_T = \sum_{i=0}^{\infty} (-1)^i \operatorname{length}_{\mathscr{O}_{X,T}} \operatorname{Tor}^i_{\mathscr{O}_{X,T}} \left(\mathscr{O}_{X,T}/\mathscr{I}, \mathscr{O}_{X,T}/\mathscr{J}\right)$$

Note that this formula specializes to $n_T = \mathrm{length}_{\mathscr{O}_{X,T}} \mathscr{O}_{X,T}/(\mathscr{I} + \mathscr{J})$ if we are in a setting where the higher Tor groups vanish. Note that $\mathscr{I} + \mathscr{J}$ is the ideal of $\mathscr{O}_{X,T}$ corresponding to the scheme-theoretic intersection of $V$ and $W$ at $T$.

**Example 5.5.** Let $X = \mathbf{A}^2_F = \operatorname{Spec} F[X,Y]$, $V$ the line $Y = 0$ so $V = \operatorname{Spec} F[X,Y]/(Y)$, and $W$ the parabola $Y = X^2$, so $W = \operatorname{Spec} F[X,Y]/(Y - X^2)$. Then the topological intersection of $V$ and $W$ consists of the point $(0,0)$, so we have $T = F[X,Y]/(X,Y)$ and $\mathscr{O}_{X,T} = F[X,Y]_{(X,Y)}$. Then $\mathscr{O}_{X,T}/(\mathscr{I} + \mathscr{J}) = F[X,Y]_{(X,Y)}/(Y, X^2) \simeq F[X]/(X^2)$. This has length 2 over $\mathscr{O}_{X,T}$, and $n_T = 2$. Thus, we have:

$$[V] \cdot [W] = 2[(0,0)]$$

Next, we will define the *category of correspondences* over $F$. The objects of this category will be smooth $F$-schemes, but there will be additional morphisms. There is an analogue to this construction in topology:

Let $X$ be a space. Then we can define the $n$-th symmetric power of $X$ $\mathrm{SP}^n(X) = X^n/S_n$ (where $S_n$ is the symmetric group acting on $X^n$ by permuting the factors). Then we have a diagram:

$$\begin{tikzcd} X \arrow[r] & \coprod_{n=0}^{\infty} \mathrm{SP}^n(X) \arrow[d] \\ & \mathrm{SP}(X) \end{tikzcd}$$

Here, $\coprod_{n=0}^{\infty} \mathrm{SP}^n(X)$ is the free topological abelian monoid generated by $X$, and $\mathrm{SP}(X)$ is the free topological abelian group generated by $X$. Points of $\mathrm{SP}(X)$ consist of $\mathbf{Z}$-linear combination of points of $X$.

A map $f\colon X \to \coprod_n \mathrm{SP}^n(Y)$ is like a "multi-valued function" from $X$ to $Y$, i.e. each point of $X$ maps to some finite set of points of $Y$, *possibly with multiplicity*. A map $f\colon X \to \mathrm{SP}(Y)$ sends each point of $X$ to some $\mathbf{Z}$-linear combination of points in $Y$.

Given maps $f\colon X \to \mathrm{SP}(Y)$ and $g\colon Y \to \mathrm{SP}(Z)$, we obtain a map $X \to \mathrm{SP}(Z)$:

$$X \xrightarrow{f} \mathrm{SP}(Y) \xrightarrow{\mathrm{SP}(g)} \mathrm{SP}(\mathrm{SP}(Z)) \longrightarrow \mathrm{SP}(Z)$$

Here, the final map $\mathrm{SP}(\mathrm{SP}(Z))$ is defined by "expanding" linear combinations of linear combinations of points of $Z$. Thus, we have a category whose objects are topological spaces and whose morphisms from $X$ to $Y$ consist of maps from $X$ to $\mathrm{SP}(Y)$. We want to do something similar in algebraic geometry.

**Definition 5.6.** An *elementary correspondence* $f\colon X \rightsquigarrow Y$ is a subvariety $V \subseteq X \times Y$ such that the map $V \to X$ defined by restricting the first projection $p_1\colon X \times Y \twoheadrightarrow X$ to $V$ is a finite morphism which is surjective onto a component of $X$.

The intuition is that such $V$ are the graphs of "multi-valued functions" from $X$ to $Y$.

This lets us define:

**Definition 5.7.** $\mathrm{Corr}_F(X,Y)$ is the set of $\mathbf{Z}$-linear combinations of elementary correspondences $X \rightsquigarrow Y$. This is a subset of the set of cycles on $X \times Y$.

We give some examples of correspondences:

**Example 5.8.** Given a morphism $f\colon X \to Y$, the graph $\Gamma_f \hookrightarrow X \times Y$ is a correspondence from $X$ to $Y$, which is elementary when $X$ is connected. (Otherwise it is the formal sum of one elementary correspondence for each component of $X$.)

**Example 5.9.** Consider the multi-valued function $\mathbf{C} \to \mathbf{C}$ defined by $z \mapsto \pm\sqrt{z}$. We can realize this as a correspondence from $X = \mathbf{A}^1_{\mathbf{C}}$ to $X = \mathbf{A}^1_{\mathbf{C}}$. Define $V \subseteq X \times Y = \mathbf{A}^2_{\mathbf{C}}$ by $V = \operatorname{Spec} \mathbf{C}[X, Y]/(X - Y^2)$. The projection to $X$ is a surjective finite morphism of degree 2, so this is an elementary correspondence.

We may define a bilinear composition law on the groups of correspondences, making the category of smooth $F$-schemes and correspondences into an additive category:

**Definition 5.10.** For $X, Y, Z$ smooth $F$-schemes, the composition law on $\mathrm{Corr}_F$ is the bilinear map $\mathrm{Corr}_F(X, Y) \times \mathrm{Corr}_F(Y, Z) \to \mathrm{Corr}_F(X, Z)$ defined by sending $(V, W)$ with $V \subseteq X \times Y, W \subseteq Y \times Z$ elementary correspondences to $p_*\big([V \times Z] \cdot [X \times W]\big)$, with $p\colon X \times Y \times Z \to X \times Z$ the projection map and $\cdot$ the intersection formula defined above.

**Lemma 5.11.** The above definitions give a well-defined associative composition law, making $\mathrm{Corr}_F$ into a category. Moreover, there is a faithful embedding $\mathrm{Sm}_F \hookrightarrow \mathrm{Corr}_F$ which is the identity on objects and sends $f\colon X \to Y$ to $\Gamma_f \subseteq X \times Y$.

# 6 1/22/2018

We continue to develop motivic cohomology. Last time, we considered the category $\mathrm{Sm}_F$ of smooth varieties over the field $F$, and we defined a functor from this to a category called $\mathrm{Corr}_F$, the "correspondence category". This is an additive category, with $X \sqcup Y$ the categorical co-product and product in this category. However, this category is not abelian, i.e. there are not usually kernels and cokernels.

**Remark 6.1.** Suppose a category has the properties that finite products and coproducts exist, that the canonical map from the initial object to the terminal object is an isomorphism, and that the canonical map $X \sqcup Y \to X \times Y$ from the coproduct to the product is an isomorphism for all objects $X$ and $Y$. Then all morphism sets canonically inherit the structure of commutative monoids: for $f, g\colon X \to Y$, we define $f + g\colon X \to Y$ by $\mathrm{codiag} \circ (f \times g)$, with $\mathrm{codiag}\colon Y \times Y \xrightarrow{\sim} Y \sqcup Y \to Y$ given by $\mathrm{id} \sqcup \mathrm{id}$. The category is additive if these abelian monoids are groups, in which case composition is automatically bilinear. Notice that being additive is entirely a *property* of a category (not extra data!).

There is a fully faithful embedding from $\mathrm{Corr}_F$ to the category $\mathrm{PST}$ of "presheaves with transfer" on $\mathrm{Sm}_F$. This is the category of additive functors from $\mathrm{Corr}_F^{\mathrm{op}}$ to $\mathrm{Ab}$. These functors are specified by associating an abelian group $S(X)$ to each smooth $F$-scheme $X$, together with homomorphisms $S(X) \otimes_{\mathbf{Z}} \mathrm{Corr}_F(X, Y) \to S(Y)$ which are compatible with the composition on $\mathrm{Corr}_F$.

We denote this embedding by $X \mapsto \mathbf{Z}_{\mathrm{tr}}(X) = \mathrm{Corr}_F(-, X)$. The reason for the terminology is that we obtain "transfer" maps $f_*$ for finite surjective maps $f\colon X \to Y$. This is because for such an

$f$, the graph $\Gamma_f \subseteq X \times Y$ defines a correspondence from $Y$ to $X$. Unlike $\mathrm{Corr}_F$, the category $\mathrm{PST}$ is abelian, with kernels and cokernels which may be computed "pointwise": if $S, T \in \mathrm{PST}$ and $\eta\colon S \to T$ is a morphism, $(\ker \eta)(X) = \ker\big(\eta(X)\big)$ for each $X \in \mathrm{Sm}_F$.

Consider $\mathbf{A}^1 \setminus \{0\} = \operatorname{Spec} F[t, t^{-1}]$, also called $\mathbf{G}_m$. Consider the maps $\mathrm{pt} = \operatorname{Spec} F \to \mathbf{G}_m \to \mathrm{pt}$ where the first map is the point $1$ and the second is the terminal map. We obtain a diagram:

$$\mathbf{Z}_{\mathrm{tr}}(\mathrm{pt}) \to \mathbf{Z}_{\mathrm{tr}}(\mathbf{A}^1 \setminus \{0\}) \to \mathbf{Z}_{\mathrm{tr}}(\mathrm{pt})$$

Since the composition of the two maps is the identity, we get a canonical splitting:

$$\mathbf{Z}_{\mathrm{tr}}(\mathbf{A}^1 \setminus \{0\}) \simeq \mathbf{Z}_{\mathrm{tr}}(\mathrm{pt}) \oplus \operatorname{Cok}\big(\mathbf{Z}_{\mathrm{tr}}(\mathrm{pt}) \to \mathbf{Z}_{\mathrm{tr}}(\mathbf{A}^1 \setminus \{0\})\big) =: \mathbf{Z}_{\mathrm{tr}}(\mathrm{pt}) \oplus \mathbf{Z}_{\mathrm{tr}}(\mathbf{G}_m, 1)$$

We may play a similar game and get a diagram of split exact sequences:

$$\begin{tikzcd}
\mathbf{Z}_{\mathrm{tr}}(\mathrm{pt} \times \mathrm{pt}) \arrow[r] \arrow[d] & \mathbf{Z}_{\mathrm{tr}}(\mathrm{pt} \times \mathbf{G}_m) \arrow[r] \arrow[d] & \mathbf{Z}_{\mathrm{tr}}(\mathbf{G}_m, 1) \arrow[d] \\
\mathbf{Z}_{\mathrm{tr}}(\mathbf{G}_m \times \mathrm{pt}) \arrow[r] \arrow[d] & \mathbf{Z}_{\mathrm{tr}}(\mathbf{G}_m \times \mathbf{G}_m) \arrow[r] \arrow[d] & \operatorname{Cok} \arrow[d] \\
\mathbf{Z}_{\mathrm{tr}}(\mathbf{G}_m, 1) \arrow[r] & \operatorname{Cok} \arrow[r] & \mathbf{Z}_{\mathrm{tr}}(\mathbf{G}_m, 1)^{\wedge 2}
\end{tikzcd}$$

This gives us:

$$\mathbf{Z}_{\mathrm{tr}}(\mathbf{G}_m \times \mathbf{G}_m) \simeq \mathbf{Z}_{\mathrm{tr}}(\mathrm{pt}) \oplus 2\mathbf{Z}_{\mathrm{tr}}(\mathbf{G}_m, 1) \oplus \mathbf{Z}_{\mathrm{tr}}((\mathbf{G}_m, 1)^2)$$

More generally, we have:

$$\mathbf{Z}_{\mathrm{tr}}(\mathbf{G}_m^n) \simeq \bigoplus_{i=0}^{n} \binom{n}{i} \mathbf{Z}_{\mathrm{tr}}\Big((\mathbf{G}_m, 1)^i\Big)$$

And here, $\mathbf{Z}_{\mathrm{tr}}\big((\mathbf{G}_m, 1)^n\big)$ is the cokernel of the map

$$\sum \mathbf{Z}_{\mathrm{tr}}(\mathbf{G}_m \times \mathbf{G}_m \times \cdots \times \mathrm{pt} \times \mathbf{G}_m \times \cdots \times \mathbf{G}_m) \to \mathbf{Z}_{\mathrm{tr}}(\mathbf{G}_m \times \cdots \times \mathbf{G}_m)$$

This is reminiscent of the behavior of the smash product of pointed topological spaces: if we replace $\mathbf{G}_m$ by the topological space $S^1$ and the pair $(\mathbf{G}_m, 1)$ of schemes with the pointed topological space $(S^1, 1)$, then the $n$-fold "smash product" is the $n$-sphere $(S^1, 1)^{\wedge n} \cong S^n$. On the level of singular chains, we have a splitting $C_*(S^1) \cong \mathbf{Z} \oplus C_*(S^1, 1)$ and a similar chain level "binomial formula" (up to quasi-isomorphism at least) for $C_*((S^1)^{\times n}) \simeq (C_*(S^1))^{\otimes n} \cong (\mathbf{Z} \oplus C_*(S^1, 1))^{\otimes n}$. We can't form anything like an actual smash product in the category of schemes, but in $\mathrm{PST}$ we may nevertheless form something which behaves like the chains on the smash product.

**Definition 6.2.** We say that $F \in \mathrm{PST}$ is a *homotopy invariant object* if $\pi^*\colon F(X) \to F(X \times \mathbf{A}^1)$ is an isomorphism for all $X$.

We want $X \times \mathbf{A}^1 \to X$ to be a "homotopy equivalence". How do we make a functor $F\colon \mathrm{Sm}_F^{\mathrm{op}} \to \mathbf{Ab}$ homotopy invariant?

We define:

**Definition 6.3.** The standard $n$-simplex is $\Delta^n = \operatorname{Spec} \mathbf{Z}[t_0, \ldots, t_n]/(\sum_i t_i = 1) \simeq \mathbf{A}^n_{\mathbf{Z}}$. We also define $\Delta^n_X = X \times_{\mathbf{Z}} \Delta^n$.

We have face maps $\delta^0, \cdots, \delta^n \colon \Delta^{n-1} \to \Delta^n$ and degeneracy maps in the other direction, as usual for simplicial sets. The $\Delta^i$ fit together into a *co-simplicial scheme*, i.e. a functor from the simplex category $\Delta$ into the category of schemes.

Given $F \colon \mathrm{Sm}_F^{\mathrm{op}} \to \mathbf{Ab}$, we define:

**Definition 6.4.** $(C_n F)(X) = F(X \times_{\mathbf{Z}} \Delta^n)$. This is a simplicial abelian group with maps $d_i = \delta_i^* \colon F(\Delta^n_X) \to F(\Delta^{n-1}_X)$. We define a differential $\delta = \sum_i (-1)^i d_i \colon C_n(F) \to C_{n-1}(F)$.

**Remark 6.5.** Sometimes, we may consider instead the normalized chain complex given by $N_n F = \bigcap_{i=1}^n \operatorname{Ker}(d_i)$, with differential induced by $d_0$. For general reasons of simplicial abelian groups, this is quasi-isomorphic to the one above.

Now, this gives us a functor $C_* F \colon \mathrm{Sm}_F^{\mathrm{op}} \to \mathrm{Ch}$, where $\mathrm{Ch}$ is the category of $\mathbf{N}$-graded chain complexes.

We have:

**Lemma 6.6.** If $i_0, i_1 \colon X \to X \times \mathbf{A}^1$ are the maps induced by the inclusion of the points $0, 1 \in \mathbf{A}^1(F)$, the corresponding maps

$$(i_0)^*, (i_1)^* \colon (C_* F)\big(X \times \mathbf{A}^1\big) \to C_* F(X)$$

are chain homotopic. In particular, they induce the same maps on homology.

These give 1-sided inverses to the map $\pi^* \colon (C_* F)(X) \to (C_* F)\big(X \times \mathbf{A}^1\big)$, and a two-sided inverse up to chain homotopy.

We have:

**Corollary 6.7.** $\pi^*$ is a quasi-isomorphism, i.e. the functor $H_n\big((C_* F)(-)\big)$ is a homotopy-invariant object.

Note that we always have a surjective map from $F(X) = F(X \times \Delta^0) = C_0 F$ (where $\Delta^0 = \operatorname{Spec} F$ is just a point) to $H_0(C_* F)$. Similarly, given $\mathscr{F}_* \colon \mathrm{Sm}_F^{\mathrm{op}} \to \mathrm{Ch}$, we obtain a map from this to the complex $\operatorname{Tot}(C_* \mathscr{F}_*)$.

Also, if $F \colon \mathrm{Corr}_F^{\mathrm{op}} \to \mathbf{Ab}$ is defined on the correspondence category, i.e. $F \in \mathrm{PST}$, then we may similarly consider $C_* F \colon \mathrm{Corr}_F^{\mathrm{op}} \to \mathrm{Ch}$.

We define:

**Definition 6.8.** $\mathbf{Z}(n) = C_* \mathbf{Z}_{\mathrm{tr}}\big((\mathbf{G}_m, 1)^n\big)[-n] \in \mathrm{Ch}(\mathrm{PST})$.

This is a cochain complex of PST's concentrated in cohomological degrees $(-\infty, n]$, i.e. it looks like:

$$0 \longleftarrow \mathbf{Z}(n)^n \longleftarrow \mathbf{Z}(n)^{n-1} \longleftarrow \cdots \longleftarrow \mathbf{Z}(n)^0 \longleftarrow \mathbf{Z}(n)^{-1} \longleftarrow \cdots$$

Now, we may define the motivic cohomology groups:

**Definition 6.9.** For $X \in \mathrm{Sm}_F$, the *motivic cohomology groups* $H^{p,q}(X) = H^{p,q}(X; \mathbf{Z})$ are defined to be the Zariski sheaf (hyper-) cohomology $H^p(X; \mathbf{Z}(q))$. More generally, for an abelian group $A$ we define $A(q) \in \mathrm{Ch}(\mathrm{PST})$ as $A \otimes_{\mathbf{Z}} \mathbf{Z}(q)$. Then we have $H^{p,q}(X; A) := H^p(X; A(q))$. (We shall see later that $\mathbf{Z}(q)$ and more generally $A(q)$ are in fact cochain complexes of *sheaves*, not just presheaves. This boils down to $\mathbf{Z}_{\mathrm{tr}}(X)$ being a Zariski sheaf for any $X$.)

The easiest case to understand is $q = 0$. We have $\mathbf{Z}_{\mathrm{tr}}\big((\mathbf{G}_m, 1)^0\big) = \mathbf{Z}_{\mathrm{tr}}(\mathrm{pt})$. As a functor, this sends $X$ to the constant sheaf $\underline{\mathbf{Z}}$ on $X$. Since this already respects the "homotopy equivalence" $X \times \mathbf{A}^1 \to X$, taking chains does nothing (because $\mathbf{A}^1$ and more generally $\Delta^p_F$ is connected, so locally constant maps out of $X \times \Delta^p_f$ are the same as locally constant maps out of $X$), i.e. the homology of $C_* \mathbf{Z}_{\mathrm{tr}}(\mathrm{pt})$ is concentrated in degree $0$. (The normalized chain complex is concentrated in degree 0 even on the chain level.) Then essentially the same computation as that of the singular cohomology of a point shows us that $H^{p,0}(X; A) = H^p_{\mathrm{Zar}}(X, \underline{A})$.

Up next, we will see that there is a quasi-isomorphism $\mathbf{Z}_{\mathrm{tr}}(\mathbf{G}_m) \xrightarrow{\sim} \mathbf{Z} \oplus \mathscr{O}^\times$, and this shows that $H^{p,1}(X) \simeq H^{p-1}_{\mathrm{Zar}}(X, \mathscr{O}^\times_X)$, as predicted. Also, we will see that for a field extension $F \hookrightarrow k$, we have an isomorphism $H^{p,p}(\mathrm{Spec}\, k) \simeq K^M_p(k)$.

# 7 1/24/18

Last time, we defined the objects $\mathbf{Z}(n) = C_* \mathbf{Z}_{\mathrm{tr}}(\mathbf{G}_m, 1)^n[-n]$. This is a cochain complex in the category $\mathrm{PST}$ of presheaves with transfer on the category of smooth $F$-schemes. Actually, the $\mathbf{Z}(n)$ are really complexes of *sheaves* for the Zariski topology. To see this, first note that a representable functor in this category, i.e. a functor $\mathrm{Corr}_F \to \mathbf{Ab}$ of the form $U \mapsto \mathrm{Corr}_F(U, Y)$ for a smooth $F$-scheme $Y$, is a Zariski sheaf. This is because an elementary correspondence $X \rightsquigarrow Y$ is uniquely determined by the restriction to any non-empty dense open subset $U \subseteq X$:

**Lemma 7.1.** The restriction map $\mathrm{Corr}_F(X, Y) \to \mathrm{Corr}_F(U, Y)$ is *injective* with free cokernel. Moreover, if $X = U \cup V$ for $U, V$ dense open subsets, then the following sequence is exact and remains exact after applying $A \otimes_{\mathbf{Z}} (-)$ for any abelian group $A$:

$$0 \longrightarrow \mathbf{Z}_{\mathrm{tr}}(Y)(X) \longrightarrow \mathbf{Z}_{\mathrm{tr}}(Y)(U) \oplus \mathbf{Z}_{\mathrm{tr}}(Y)(V) \longrightarrow \mathbf{Z}_{\mathrm{tr}}(Y)(U \cap V)$$

Since $\mathbf{Z}_{\mathrm{tr}}(Y)$ is a sheaf for any $Y \in \mathrm{Sm}_F$, so are direct summands of such presheaves such as $\mathbf{Z}_{\mathrm{tr}}(\mathbf{G}_m, 1)$. Therefore, for any abelian group $A$, $A(n)$ is a cochain complex of *sheaves*.

Now, if $\mathscr{F}$ is a sheaf of abelian groups, the complex $C_*\mathscr{F}$ is a cochain complex of sheaves with the property that $C_*\mathscr{F}(X \times \mathbf{A}^1) \to C_*\mathscr{F}(X)$ is a quasi-isomorphism. Thus, the presheaf $X \mapsto H_n((C_*\mathscr{F})(X))$ is a homotopy-invariant *presheaf*. However, it may not be a sheaf in general, and this does not in any way formally imply that the hypercohomology $H^*(X; C_*\mathscr{F})$ is also homotopy-invariant. Voevodsky proved the latter statement when $\mathscr{F}$ has transfers, but this requires a long and careful argument and is special to the Zariski (and Nisnevich) topologies.[10]

We collect some "elementary" (i.e. not Fields Medal-winning) properties of the motivic cohomology:

**Proposition 7.2.** The following properties hold for the motivic cohomology groups $H^{p,q}(X) = H^p(X; \mathbf{Z}(q))$:

[10] see the last 4 chapters of [11]

(i) There are Mayer-Vietoris sequences.

(ii) There is a Thom isomorphism.

(iii) $H^{p,q}(X) = H^p(X;\mathbf{Z}(q)) = 0$ for $p > q + \dim(X)$. (This follows easily from Grothendieck's theorem that $H^k(\mathscr{F}) = 0$ for $k > \dim(X)$ for any abelian sheaf $\mathscr{F}$ and topological space $X$).

(iv) Motivic cohomology $H^{p,q}(X)$ is functorial on $X \in \mathrm{Corr}_F$, i.e. there are pullbacks $f^*$ for any $f\colon X \to Y$ and transfers $f_*$ for any finite $f\colon X \to Y$.

(v) Motivic cohomology is "independent of $F$": this makes sense since $\mathbf{G}_m$ is defined over $\mathbf{Z}$ and $X \times_F (\mathbf{G}_m)_F = X \times_{\mathbf{Z}} \mathbf{G}_m$, so $\mathrm{Corr}_F(X, \mathbf{G}_m)$ does not depend on $F$.

We also have products. There is a map

$$\mathrm{Corr}_F(X,Y) \otimes \mathrm{Corr}_F(X',Y') \to \mathrm{Corr}_F(X \times X', Y \times Y')$$

defined by $Z \otimes Z' \to Z \times Z'$.

Taking $X = X'$, we get a map of presheaves:

$$\mathbf{Z}_{\mathrm{tr}}(Y) \otimes_{\mathbf{Z}} \mathbf{Z}_{\mathrm{tr}}(Y') \to \mathbf{Z}_{\mathrm{tr}}(Y \times Y')$$

where the tensor product is taken (objectwise) in the category PST. This sends $Z \otimes Z'$ to $Z \times Z'|_\Delta$.

This gives a pairing:

$$\mathbf{Z}_{\mathrm{tr}}\big((\mathbf{G}_m,1)^n\big) \otimes \mathbf{Z}_{\mathrm{tr}}\Big((\mathbf{G}_m,1)^{n'}\Big) \to \mathbf{Z}_{\mathrm{tr}}\Big((\mathbf{G}_m,1)^{n+n'}\Big)$$

Evaluating this on $X \times \Delta^p$ gives a level-wise product. We can use the Eilenberg-Zilber formula to get a map $\mathbf{Z}(n) \otimes \mathbf{Z}(n') \to \mathbf{Z}(n+n')$: let $A_\bullet$, $B_\bullet$ be simplicial abelian groups (i.e. simplicial objects in the category of abelian group). Then the Eilenberg-Zilber formula gives a quasi-isomorphism between $C(A_\bullet) \otimes C(B_\bullet)$ and $C(A_\bullet \otimes B_\bullet)$. The tensor product on the left is on the category of chain complexes, and the one on the right is on the category of simplicial abelian groups. All of this put together means that the group $\bigoplus_{p,q} H^p(X;\mathbf{Z}(q))$ admits the structure of a bi-graded ring which is graded-commutative with respect to $p$.

Now, we will discuss some special cases. We saw that $H^p(X;\mathbf{Z}(0)) = H^p(X;\mathbf{Z})$ last time. We also have an isomorphism $H^1(X;\mathbf{Z}(1)) \simeq \mathscr{O}_X^\times$ is done in e.g. [11]. In addition, when $X = \operatorname{Spec} F$, we'll see that $H^n(X;\mathbf{Z}(n)) \simeq K_n^M(F)$.

First, for $X = \operatorname{Spec} F$, $H^n(\mathbf{Z}(n)(X)) = H^*(X;\mathbf{Z}(n)) := \mathrm{Cok}\big(d\colon \mathbf{Z}(n)^{n-1}(X) \to \mathbf{Z}(n)^n(X)\big)$. This uses the fact that $X = \operatorname{Spec} F$: topologically, $X$ is a point, so the global sections functor induces an exact equivalence from the category of abelian sheaves on $X$ to the category of abelian groups. Thus, hypercohomology of a complex is the same as the cohomology of its global sections, so $H^p(X;\mathbf{Z}(n))$ is the $p$-th cohomology of the complex $\mathbf{Z}(n)^\bullet(X)$.

We have $d\colon \mathbf{Z}(n)^{n-1}(X) \to \mathbf{Z}(n)^n(X)$ is the same thing as

$$C_1\mathbf{Z}_{\mathrm{tr}}\big((\mathbf{G}_m,1)^n\big)(X) \to C_0\mathbf{Z}_{\mathrm{tr}}\big((\mathbf{G}_m,1)^n\big)(X)$$

An elementary correspondence $Z \subseteq \operatorname{Spec} F \times \mathbf{G}_m^n$ is the same thing as a finite field extension $E/F$ and a map $\operatorname{Spec} E \to \mathbf{G}_m^n$, i.e. an element $x \in E^\times \times \cdots \times E^\times$. In particular, for $a =$

$(a_1, \ldots, a_n) \in (F^\times)^n$, we have a class $[a_1 : \cdots : a_n] \in H^n(\operatorname{Spec} F; \mathbf{Z}(n))$. This equals $[a_1] \cdot [a_2] \cdots [a_n]$ where the product is as defined above.

Now, this gives us a map $K_1^M(F) = F^\times \to H^1(F; \mathbf{Z}(1))$ sending $a$ to $[a]$. We have:

**Lemma 7.3.**

$$[ab] = [a] + [b]$$

*Proof.* Note that $[1] = 0$ because this maps to the kernel of $\mathbf{Z}_{\mathrm{tr}}(\mathbf{G}_m) \to \mathbf{Z}_{\mathrm{tr}}\big((\mathbf{G}_m, 1)\big)$. Now, we claim that $[ab] + [1] = [a] + [b]$.

Note that $\Delta^1 = \operatorname{Spec} F[t_0, t_1]/(t_0 + t_1 = 1) \simeq \operatorname{Spec} F[t] = \mathbf{A}^1_F$, under which the two face maps $\Delta^0 \to \Delta^1$ correspond to the inclusions of the two points $\{0\}$ and $\{1\}$ into $\mathbf{A}^1_F$. We define an interpolation from $[ab] + [1]$ to $[a] + [b]$ parametrized by the $s$-line $\operatorname{Spec} F[s]$. Consider the curve in the $(s, t)$ plane $\operatorname{Spec} F[s, t]$ defined by:

$$\begin{aligned} 0 &= s(t-a)(t-b) + (1-s)(t-ab)(t-1) \\ &= s\left(t^2 - (a+b)t + ab\right) + (1-s)\left(t^2 - (ab+1)t + ab\right) \\ &= t^2 + \big((ab - a - b + 1)s - (ab+1)\big)\, t + ab \end{aligned}$$

This curve is finite over $\operatorname{Spec} F[s] = \mathbf{A}^1_F$, so it defines a correspondence from $\mathbf{A}^1_F$ to $\mathbf{A}^1_F = \operatorname{Spec} F[t]$. Since its intersection with the line $t = 0$ is empty, it actually defines a correspondence to $\mathbf{G}_m$. The restriction of this correspondence to $s = 1$ gives $[a] + [b]$, and the restriction to $s = 0$ gives $[ab] + 1$, so we have the desired homotopy showing $[a] + [b] = [ab] + [1] = [ab]$ in $H^1(F; \mathbf{Z}(1))$. $\square$

This shows that we obtain a well-defined homomorphism of rings

$$T_{\mathbf{Z}} F^\times = \bigoplus_n (F^\times)^{\otimes n} \to \bigoplus_n H^n(F; \mathbf{Z}(n))$$

which sends $a_1 \otimes \cdots \otimes a_n \to [a_1] \cdot [a_2] \cdots [a_n] = [a_1 : \cdots : a_n]$.

Next time, we will see that $[a] \cdot [1 - a] = 0$ if $a \in F^\times \setminus \{1\}$. This will show that the above map descends to a homomorphism $K_*^M(F) \to \bigoplus_n H^n(F; \mathbf{Z}(n))$ which sends $\ell(a)$ to $[a]$. Furthermore, we will see that this is in fact an isomorphism.

# 8 1/26/18

(notes in progress)

# 9 1/29/18

Last time, we defined a map $K_n^M(F) \to H^n(\operatorname{Spec} F; \mathbf{Z}(n))$ denoted $\ell(a_1) \cdot \cdots \cdot \ell(a_n) \mapsto [a_1; \cdots; a_n]$ and proved that this is well-defined (i.e. that it is a homomorphism for $n = 1$ and that the Steinberg relation holds in $H^2(\operatorname{Spec} F; \mathbf{Z}(2))$). We also proved surjectivity for $n = 1$. Proving surjectivity and injectivity for all $n$ requires more work, see [11, Lecture 5] (the proof uses "norm" (transfer) maps $K_n^M(E) \to K_n^M(F)$ for a finite field extension $F \hookrightarrow E$, which are constructed carefully in [7]). Similarly, we may show that $K_n^M(F)/\ell \xrightarrow{\sim} H^n(\operatorname{Spec} F; \mathbf{Z}/\ell(n))$. This result should be regarded as easier than the Milnor conjecture, which asserted an isomorphism between $K_n^M(F)/2$ to $H^n(G_F; \mathbf{Z}/2)$, and boils this down to showing that $H^n(\operatorname{Spec} F; \mathbf{Z}/2(n)) \xrightarrow{\sim} H^n(G_F; \mathbf{Z}/2)$. So far, we have not made any use of "motivic homotopy theory", which will be an ingredient in the proof.

Let's recall the situation in topology. We can consider a category of spaces such as $\mathbf{Top}$ (topological spaces) or $\mathbf{sSets}$ (simplicial sets) with the singular cohomology functor $H^n$ from this category to $\mathbf{Ab}$. This functor admits some interesting factorizations:

$$\begin{tikzcd}
\text{Spaces} \arrow[r, "\Sigma^\infty"] \arrow[dr, "C_*^{\text{sing}}(-;\mathbf{Z})"] \arrow[ddr, "H^n"'] & \text{Ho(Spectra)} \arrow[d, "C_*"] \\
& D(\mathbf{Z}) \arrow[d, "{[-,\Sigma^n \mathbf{Z}]}"] \\
& \mathbf{Ab}
\end{tikzcd}$$

Here $D(\mathbf{Z})$ is the derived category of $\operatorname{Spec} \mathbf{Z}$, defined to be the category of chain complexes of projective $\mathbf{Z}$-modules with morphisms chain homotopy classes of maps of complexes. The map from $D(\mathbf{Z})$ to $\mathbf{Ab}$ which sends a complex to its $n$-th cohomology is representable: $H^n(C^\bullet) = [C^\bullet, \Sigma^n \mathbf{Z}]$ where $\Sigma^n \mathbf{Z}$ is the chain complex with value $\mathbf{Z}$ in grading $n$ and $0$ everywhere else. The category at the top right is the homotopy category of spectra, and the map $X \mapsto \Sigma^\infty(X)$ sending a space to its suspension spectrum could be thought of as taking chains "with values in the sphere spectrum $\mathbf{S}$". Furthermore, cohomology is *representable* in the category of spectra: $H^n(X) = [\Sigma^\infty X, \Sigma^n H\mathbf{Z}]$ in the category of spectra, where $H\mathbf{Z}$ is the Eilenberg-Mac Lane spectrum. Cohomology operations $H^n \to H^{n+k}$ come from maps $H\mathbf{Z} \to \Sigma^k H\mathbf{Z}$ which are "not $\mathbf{Z}$-linear" but only "$S$-linear". (The sphere spectrum is a so-called "$E_\infty$ ring spectrum", which is a generalization of ordinary rings; the existence of non-trivial cohomology operations is an indication that singular chains arise from "base change" along a map of ring spectra $S \to H\mathbf{Z}$. An analogous but simpler fact is that the Bockstein homomorphism $H^*(-;\mathbf{F}_p) \to H^{*+1}(-;\mathbf{F}_p)$ does not come from a $\mathbf{F}_p$ linear map of mod $\mathrm{F}_p$-chains; its existence witnesses that mod $p$ chains $C_*(X;\mathbf{F}_p)$ arises from base change along $\mathbf{Z} \to \mathbf{F}_p$.)

Something similar should work for motivic cohomology, with the category of spaces replaced by $\mathrm{Sm}_F$, the category of smooth $F$-schemes:

$$\begin{tikzcd}
\mathrm{Sm}_F \arrow[r, "\Sigma^\infty"] \arrow[dr] \arrow[ddr, "H^p(-;\mathbf{Z}(q))"'] & \mathrm{SH}(F) \arrow[d] \\
& \mathrm{DM}_-(F) \arrow[d, "{[-,\Sigma^p \mathbf{Z}(q)]}"] \\
& \mathbf{Ab}
\end{tikzcd}$$

We construct the categories appearing on the right-hand side as follows. We may embed $\mathrm{Sm}_F$ into the category of simplicial presheaves on $\mathrm{Sm}_F$, i.e. the category of functors from $\mathrm{Sm}_F^{\mathrm{op}}$ to $\mathbf{sSet}$. We have a notion of weak equivalence on this category: a natural transformation $\eta$ from $F$ to $G$ is a weak equivalence if $\eta_X \colon F(X) \to G(X)$ is a weak equivalence in $\mathbf{sSet}$ for each object $X$. In other words, this category is a *model category*, which means it is suitable for doing homotopy theory.

Alternatively, we may embed the additive category $\mathrm{Corr}_F$ of smooth schemes and correspondences into $\mathrm{sPST}$, the category of additive functors from $\mathrm{Corr}_F^{\mathrm{op}}$ into the category of simplicial abelian groups $\mathbf{sAb}$, which is equivalent to the category of $\mathbf{N}$-graded chain complexes $C_*(\mathbf{Ab})$. This also has a notion of weak equivalences, defined by quasi-isomorphism on the level of chain complexes.

However, these constructions do not have "enough" weak equivalences: they do not capture the "locality" of cohomology on $\mathrm{Sm}_F$, i.e. they do not take into account the presence of a topology on $\mathrm{Sm}_F$. Thus, we want to "localize" further. In the simplicial set version, we invert stalk-wise weak equivalence and maps induced by projection $X \times \mathbf{A}^1 \to X$, and then make the operation of smashing with $(\mathbf{P}^1, \infty)$ invertible to get the category $\mathrm{SH}(F)$ of spectrum objects. In the chain complex version, we invert stalk-wise quasi-isomorphisms and homotopy equivalences to get the category $\mathrm{DM}_-^{\mathrm{eff}}$, and further force the operation of tensoring with $\mathbf{Z}(1)$ to be invertible to get the category $\mathrm{DM}_-$.[11] The chain complex version is easier since we may use the tools of homological algebra rather than model categories.

What do we mean when we say "stalk-wise"? This is with respect to the *Nisnevich topology*. Consider $X = \operatorname{Spec} R$ an affine scheme with a Zariski sheaf $F$ of sets on it. For any point $x \in X$ corresponding to a prime ideal $\mathfrak{p}$, the stalk in the Zariski topology $\varinjlim_{x \in U \subseteq X,\ U \text{ open}} F(U) = F(\operatorname{Spec} R_{(\mathfrak{p})})$. The Nisnevich topology is a Grothendieck topology on the category of $F$-schemes which is finer than the Zariski topology. The stalks here are no longer defined by evaluation at $R_{(\mathfrak{p})}$ but at its *henselization* $R^h_{(\mathfrak{p})}$.

**Definition 9.1.** A local ring $R$ is *henselian* if for every monic polynomial $f(t) \in R[t]$ such that $\overline{f}(t) \in R/\mathfrak{m}[t]$ factors as $\overline{f}(t) = (t - \overline{a_0}) \cdot g(t)$ with $g(\overline{a_0}) \neq 0$, then there exists some $a_0 \in R$ with $f(a_0) = 0$ and $a_0 \mapsto \overline{a_0}$ under the projection $R \to R/\mathfrak{m}$.

Hensel's lemma says that complete local rings are henselian, but the converse is not true. There is a ring $R^h$, the *henselization* of $R$, which is initial for maps from $R$ to henselian local rings. Thus, we have a factorization $R \to R^h \to \widehat{R}$, with $\widehat{R}$ the $\mathfrak{m}$-adic completion. The henselization can be defined as the direct limit of all étale maps $R \to R'$ which induce isomorphisms on the residue field.[12]

# 10 1/31/18

We're heading towards Voevodsky's definition of the category $\mathrm{DM}_-(F)$, the "derived category of motives over $F$". We'll cover roughly §3 of [19] (the treatment in [11] is essentially a verbatim copy).

Let $X$ be a smooth scheme over $F$. We may consider the full sub-category $\mathrm{Et}/X$ of étale $X$-schemes inside the category of $X$-schemes, i.e. the category of arrows $f \colon Y \to X$, where a morphism between arrows is a commuting triangle.

[11] Voevodsky's cancellation theorem, which is a difficult theorem, asserts that this operation is full and faithful. It amounts to demonstrating certain suspension isomorphisms in motivic cohomology. This is a real theorem because we don't have $\mathbf{Z}(1) = \Sigma\mathbf{Z}(0)$.

[12] NB: This is not the same as the *strict henselization*, which gives the stalks in the étale topology. The strict henselization of $R$ is the initial map $R \to R^{\mathrm{sh}}$ such that $R^{\mathrm{sh}}$ is henselian *with separably closed residue field*, and is the limit of all étale maps $R \to R'$.

Let's recall one of several equivalent definitions of an étale map $f\colon Y \to X$:

**Definition 10.1.** We say that $f$ is étale if there is a cover of $Y$ by open affine charts $\operatorname{Spec} R = U \subseteq Y$ and for each $U$, there is an open cover of $f^{-1}(U) \subseteq X$ by open affine charts $W \subseteq \operatorname{Spec}\left(R[x_1, \dots, x_n]/(f_1, \dots, f_n)\right)$ such that $\det\left(\frac{\partial f_i}{\partial f_j}\right) \in R[x_1, \dots, x_n]$ is invertible on $W$.

It's a non-trivial theorem[13] that we may actually take $n = 1$ above. That is, Zariski-locally, any étale map looks like $\operatorname{Spec}\left(R[x]/f(x)[\frac{1}{g}]\right) \to \operatorname{Spec} R$ with $f(x) \in R[x]$ monic and $f'(x)$ a unit in $R[x]/f(x)[\frac{1}{g}]$. These local models are called "standard étale".

For smooth schemes over $\mathbf{C}$, an étale map is locally a codimension $0$ submersion, i.e. a local isomorphism in the analytic topology. In general, we may think of the definition of étale maps as asserting the hypothesis of the inverse function theorem. The idea of replacing the Zariski topology with the étale topology is that we want to consider such things to be local isomorphisms, but this is not actually the case in the Zariski topology.

Now, we may define the appropriate notion of neighborhood for the Nisnevich topology:

**Definition 10.2.** A *Nisnevich neighborhood* of $x \in X$ is a triple $(U, n, h)$ such that the following diagram commutes:

$$\begin{tikzcd} & U \arrow[d, "h"] \\ \operatorname{Spec} k(x) \arrow[ur, "n"] \arrow[r, "x"] & X \end{tikzcd}$$

such that $h$ is étale.

The Nisnevich neighborhoods of a point $x$ form a category, where morphisms are commuting diagrams of the form:

$$\begin{tikzcd} & U \arrow[dd] \arrow[dr] & \\ \operatorname{Spec} k(x) \arrow[ur] \arrow[dr] & & X \\ & U' \arrow[ur] & \end{tikzcd}$$

**Lemma 10.3.** This is a *filtered category*, meaning that: *(i)* for any two Nisnevich neighborhoods $U, U' \to X$ of $x$ (omitting the morphism from $\operatorname{Spec}(k(x))$ from the notation), there exists a Nishnevich neighborhood mapping to both of them, and *(ii)* for any two parallel morphisms $U' \rightrightarrows U$ of Nishnevich neighborhoods of $x$ there exists a morphism $U'' \to U'$ of Nisnevich neighborhoods such that the two compositions $U'' \to U' \rightrightarrows U$ become equal.[14]

Taking colimits over a filtered category (as we shall do in the definition of *stalks*) commutes with arbitrary finite limits.

Now, we may consider a presheaf $\mathscr{F}\colon \left(\mathrm{Et}/X\right)^{\mathrm{op}} \to \mathbf{Set}$ (or $\mathbf{Ab}$, etc.). In particular, this gives a functor on the category of Nisnevich neighborhoods of any $x \in X$.

This lets us define:.

[13] See [15, Tag 02GT]

[14] see e.g. [15, Tag 04GN] for a proof that this is a filtered category. The argument is the same as in the étale case, so if you know a proof there it should apply here too.

**Definition 10.4.** The *Nisnevich stalk* at $x \in X$ is the colimit

$$\varinjlim_{\operatorname{Spec} k(x) \to U \to X} \mathscr{F}(U)$$

where the colimit ranges over the filtered category of Nisnevich neighborhoods of $x$ in $X$.

If $\mathscr{F}$ is a sheaf of (for example) abelian groups, it does not matter whether we compute the colimit in abelian groups or in sets. The point is that "abelian group" is defined using only finite limits (e.g. the addition $+ : A \times A \to A$ involves the two-fold product of $A$ with itself), so even if we compute the colimit in sets, it automatically inherits an abelian group structure.

**Example 10.5.** The Nisnevich stalk of $\mathscr{O}_{\operatorname{Spec}(R)}$ at the point determined by a prime ideal $\mathfrak{p}$ is precisely the Henselization of the localization $R_{(\mathfrak{p})}$. (The localization itself is the Zariski stalk.)

Note that the Zariski stalk at $x$ is the colimit over the (filtered) sub-category of $U$ such that $U \to X$ is an open immersion (or even the poset consisting of Zariski open subsets containing $x$), so this definition extends that one in some sense. More precisely, it gives a map from the Zariski stalk to the Nisnevich stalk.

Now, let $\mathscr{F} \to \mathscr{G}$ be a morphism of étale presheaves over $X$. We say that it is a *stalkwise isomorphism* if the induced maps of stalks is an isomorphism for all $x \in X$. We want to define Nisnevich sheaves, together with a Nisnevich sheafification functor $\mathscr{F} \to \mathscr{F}^{\sim}_{\mathrm{Nis}}$ which is an initial stalkwise isomorphism to a Nisnevich sheaf. In other words, the sheafification functor should be a left adjoint to the inclusion of the full subcategory of Nisnevich sheaves on $X$ into the category of étale presheaves over $X$. We want this functor to moreover be a left inverse to the inclusion functor, i.e. the sheafification of a sheaf should be itself.

**Remark 10.6.** Why do we define (pre)-sheaves in the Nisnevich topology to be sheaves on the full small étale site of $X$? The problem is that there is no reasonable notion of a "Nisnevich map" $Y \to X$: the Nisnevich condition requires specifying a point $x \in X$. For example, it is not true that if $U \to X$ is a Nisnevich neighborhood of $x$, then there is some refinement of $U$ which is a Nisnevich neighborhood of all points in its image.

Before even asking whether $\mathscr{F}$ is a Nisnevich sheaf, it must be a presheaf on at least the étale maps $U \to X$ (the "small étale site of $X$"). Often, $\mathscr{F}$ will be a functor on all of $\mathrm{Sm}_F$, in which case the sheaf condition is checked after precomposing with $\mathrm{Et}_{/X} \to \mathrm{Sm}_F$, for all $X$.

In order to define Nisnevich sheaves, we need the following definition:

**Definition 10.7.** If $f_i \colon U_i \to X$ are étale maps, we say that $f \colon U := \sqcup_i U_i \to X$ is a *Nisnevich cover* if for all $x \in X$, we may fill in the dotted arrow in the diagram below:

$$\begin{tikzcd} & U \arrow[d] \\ \operatorname{Spec} k(x) \arrow[r, "x"] \arrow[ur, dashed] & X \end{tikzcd}$$

Note that the étale map $U \to X$ is surjective iff for each $x \in X$, there exists some lifting $\operatorname{Spec} k' \to U$ for a finite separable field extension $k'/k(x)$. An étale cover is a surjective étale map. Note that if $U_i \to X$ are open immersions, then this condition says exactly that $X = \cup_i U_i$.

Now, we may define Nisnevich sheaves. For any Nisnevich cover $U \to X$, we may consider the simplicial object

$$X \longleftarrow U \leftleftarrows U \times_X U \Lleftarrow U \times_X U \times_X U \Lleftarrow \cdots$$

in $\mathrm{Et}_{/X}$. If $\mathscr{F}$ is an étale presheaf on $X$, this defines a diagram:

$$\mathscr{F}(X) \longrightarrow \mathscr{F}(U) \rightrightarrows \mathscr{F}(U \times_X U) \Rrightarrow \mathscr{F}(U \times_X U \times_X U) \Rrightarrow \cdots$$

Then we say that $\mathscr{F}$ is a *Nisnevich sheaf* on $X$ iff for every Nisnevich cover $U \to X$, the first map $\mathscr{F}(X) \to \mathscr{F}(U)$ is the equalizer of the two maps $\mathscr{F}(U) \to \mathscr{F}(U \times_X U)$.

Note that when $U = \sqcup U_i$ where $U_i \to X$ are open immersions, $U \times_X U = \sqcup_{i,j} U_i \cap U_j$, $U \times_X U \times_X U = \sqcup_{i,j,k} U_i \cap U_j \cap U_k$, etc., so this definition recovers the ordinary definition of a Zariski sheaf.

Now, we have the category $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Et}/X)$ of Nisnevich sheaves. If $\mathscr{F} \colon \mathrm{Sm}_F^{\mathrm{op}} \to \mathbf{Ab}$ is a functor, since for any $X \in \mathrm{Sm}_F$, we have an inclusion functor $\big(\mathrm{Et}/X\big)^{\mathrm{op}} \hookrightarrow \mathrm{Sm}_F^{\mathrm{op}}$, we may ask whether the composition of $\mathscr{F}$ with this inclusion is a Nisnevich sheaf. If this is the case, we will abuse terminology to say that $\mathscr{F}$ is a Nisnevich sheaf. A similar story works when $\mathscr{F}$ is a functor defined on $\mathrm{Corr}_F$: we say that it is a "Nisnevich sheaf with transfers" if it is a presheaf with transfer which is a sheaf when forgetting the transfers.

**Example 10.8.** For $Y$ a smooth $F$-scheme, $\mathscr{F} = \mathbf{Z}_{\mathrm{tr}}(Y)$ is a Nisnevich sheaf.[15]

We will construct $\mathrm{DM}_-(F)$ as the derived category of the category of chain complexes on the full subcategory of Nisnevich sheaves $\mathscr{F}$ on $\mathrm{Corr}_F$ with homotopy invariant cohomology presheaves.

Next time, we will prove that if $\mathscr{F}$ is a presheaf with transfers, then the Nisnevich sheafification canonically again has transfers. This is one of the places that explicitly uses that we're in the Nisnevich topology.

**Proposition 10.9.** In order to verify that an étale presheaf over $X$ is a Nisnevich sheaf, it suffices to verify the sheaf axioms in the case of covers $\{Y \to X, A \to X\}$ with $A \subset X$ an open subscheme and $Y \to X$ an étale map which restricts to an isomorphism of the reduced schemes determined by the complements $Y \setminus f^{-1}(A)$ to $X \setminus A$.

See the discussion around Definition 12.5 of [11] for a proof.

**Example 10.10.** An example with $F = \mathbf{C}$ would be $X = \mathbf{A}^1 = \operatorname{Spec} \mathbf{C}[t]$, $Y = \mathbf{A}^1 \setminus \{0, z\}$ for some closed point $z \neq 0$, $Y \to X$ induced by $t \mapsto t^2$, and $A = X \setminus \{z^2\}$. The restriction to complementary reduced schemes $Y \setminus f^{-1}(A) = \{-z\} \to \{z^2\} = X \setminus A$ is an isomorphism of schemes (both isomorphic to $\operatorname{Spec} \mathbf{C}$). The resulting cover $\{Y \to X, A \to X\}$ can't be refined to a Zariski cover since the regular function $t$ on $X$ does not have a square root Zariski locally near $z^2 \in X$.

[15]Lemma 6.2 of [11] proves the stronger statement that it is even an étale sheaf. The proof is not long, but uses "faithfully flat descent"; there may be a more elementary proof in the Nisnevich case.

We may consider $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Corr}_F)$ (here, the sheaves are valued in abelian groups) as a subcategory of the category PST of presheaves with transfer. If $\mathscr{F} \in \mathrm{PST}(F)$, we may define a Nisnevich sheafification $\mathscr{F}^{\sim}_{\mathrm{Nis}}$ as a Nisnevich sheaf on the category $\mathrm{Sm}_F$. This will canonically extend to a Nisnevich sheaf on $\mathrm{Corr}_F$.

Now, if $U = \sqcup_i U_i \to X$ is a Nisnevich cover, we obtain a simplicial scheme

$$X \longleftarrow U \leftleftarrows U \times_X U \,\overset{\longleftarrow}{\leftleftarrows}\, U \times_X U \times_X U \cdots$$

This gives a simplicial complex of objects in PST:

$$\mathbf{Z}_{\mathrm{tr}}(X) \longleftarrow \mathbf{Z}_{\mathrm{tr}}(U) \leftleftarrows \mathbf{Z}_{\mathrm{tr}}(U \times_X U) \,\overset{\longleftarrow}{\leftleftarrows}\, \cdots$$

We may consider this as a simplicial presheaf of abelian groups on $\mathrm{Sm}_F$. We have:

**Proposition 10.11.** The complex obtained from this simplicial presheaf by taking Nisnevich stalks at any $x \in V \in \mathrm{Sm}_F$ is exact.

# 11 2/2/18

Last time, we discussed the *Nisnevich topology*. We will start with an example:

**Example 11.1.** Let $X = \operatorname{Spec} \mathbf{R}[t] = \mathbf{A}^1_{\mathbf{R}}$, and consider the point $x = 1\colon \operatorname{Spec} \mathbf{R} \to X$. Then we can say that "$t$ has a square root Nisnevich locally near $1 \in X$". We have an étale map $\operatorname{Spec} \mathbf{R}[\sqrt{t}] \setminus \{0\} \to \operatorname{Spec} \mathbf{R}[t]$, and the point $1\colon \operatorname{Spec} \mathbf{R} \to X$ lifts through a section $\operatorname{Spec} \mathbf{R} \to \operatorname{Spec} \mathbf{R}[\sqrt{t}]$, so this is a Nisnevich neighborhood of $x$ in $X$ (i.e. we map $\operatorname{Spec} \mathbf{R}$ to either one of $\sqrt{t} = \pm 1$: it's okay that there are two choices). Note that $t$ *does not* have a square root Nisnevich locally near $-1$, since there is no lift of $-1\colon \operatorname{Spec} \mathbf{R} \to X$ through $\operatorname{Spec} \mathbf{R}[\sqrt{t}] \setminus \{0\}$, as this requires the field extension $\mathbf{C}/\mathbf{R}$.

Next, we will discuss *Čech theory*. Let $f\colon U = \sqcup_i U_i \to X$ be a Nisnevich cover of $X$. From this, we may obtain an (augmented) *simplicial scheme* with $U_p = U \times_X \times \cdots \times_X U$, with $U$ repeated $p+1$ times. There are $p$ projection maps from this to $U_{p-1}$, defining the structure of the simplicial scheme, i.e. a functor from $\Delta$ to schemes. In fact, we also have $U_{-1} = X$, and this forms a *augmented* simplicial object in schemes: a functor from $(\Delta \cup [-1])^{\mathrm{op}}$ to schemes, with $[-1] = \emptyset$.

Let us briefly discuss the analogue of this construction in $\mathbf{Sets}$. Given a set $X$, we define a simplicial set with $X^{[p]} = X^{p+1}$, for $[p] = \{0, \dots, p\} \in \Delta^{\mathrm{op}}, p > 0$ with the face maps given by projections. Then the homotopy type $|X_\bullet|$ is contractible when $X$ is non-empty and empty when $X$ is empty. The topological realization of this simplicial set is the disjoint union of $X^{p+1} \times \Delta^p$ modulo the usual equivalence relation, and there is an explicit homotopy contracting $|X_\bullet|$, depending on the choice of $x_0 \in X$ (the "straight line homotopy" inside each $\Delta^p$). More generally, we could consider a map $f\colon X \to B$ and form the simplicial topological space $[p] \mapsto X \times_B \cdots \times_B X$. Any section section $x_0\colon B \to X$ of $f$ gives rise to a map $B \to |[p] \to X \times_B \cdots \times_B X|$, and again an explicit "straight line" homotopy can be used to prove that this is a homotopy equivalence.

This phenomenon is a special case of so-called "extra degeneracies" in augmented simplicial objects. Recall that for a simplicial object $[p] \mapsto X_p$ in any category, the *degeneracies* are maps

$s_i\colon X_p \to X_{p+1}$ for $i = 0, \dots, p+1$. An *extra degeneracy* is an additional map $s_{-1}\colon X_{p-1} \to X_p$ satisfying the relations:

$$s_i \circ s_j = (\cdots), \qquad d_i \circ s_j = \begin{cases} (\cdots) & i < j \\ \mathrm{id} & i = j, j+1 \\ s_j \circ d_{i-1} & i > j+1 \end{cases}, \qquad d_i \circ s_{-1} = \begin{cases} \mathrm{id} & i = 0 \\ s_{-1} \circ d_{i-1} & i > 0 \end{cases}$$

In the category of topological spaces, the extra degeneracy gives rise to a homotopy equivalence $|X_\bullet| \xrightarrow{\sim} X_{-1}$. In the category of abelian groups, it gives a chain contraction of the associated chain complex, i.e. a chain homotopy between $\mathrm{id}$ and $0$. Hence, if you have a chain complex that you're trying to prove is acyclic and the chain complex came from an augmented simplicial objectt, then you might try to look for an extra degeneracy in that simplicial object.

Now, with $f\colon U = \sqcup_i U_i \to X$ a Nisnevich cover, we obtain an augmented chain complex of objects of $\mathrm{PST}$ by:

$$0 \longleftarrow \mathbf{Z}_{\mathrm{tr}}(X) \longleftarrow \mathbf{Z}_{\mathrm{tr}}(U) \longleftarrow \mathbf{Z}_{\mathrm{tr}}(U \times_X U) \longleftarrow \cdots$$

Last time, we claimed that this has *exact Nisnevich stalks* (equivalently, vanishing homology sheaves). Namely, if we choose some $v \in V \in \mathrm{Sm}_F$, we obtain a complex of Nisnevich stalks at $v$ given by:

$$\cdots \longleftarrow \varinjlim_{\operatorname{Spec} k(v) \to W \to V} \mathbf{Z}_{\mathrm{tr}}(U_p)(W) \longleftarrow \cdots$$

We may replace $\varinjlim_{\operatorname{Spec} k(v) \to W \to V} \mathbf{Z}_{\mathrm{tr}}(U_p)(W)$ with $\mathbf{Z}_{\mathrm{tr}}(U_p)(\operatorname{Spec} S)$, where $S$ is the henselization of $\mathscr{O}_{V,v}$, a henselian local ring.

The magic property of henselian local rings which ordinary local rings do not satisfy is the following:

**Proposition 11.2.** Given a henselian local ring $S$ and a finite $S$-algebra $S \to A$, there is an isomorphism $A \simeq \prod_i S_i$ with $S_i$ henselian local rings. This lifts the splitting over the residue field of $S$.

*Proof.* This is [15, 04GE]. $\square$

**Example 11.3.** Consider the local ring $S = \mathbf{Z}_{(p)}$ with $p > 2$, i.e. the localization of $\mathbf{Z}$ at the prime ideal $(p)$, and the finite $S$-algebra $A = \mathbf{Z}_{(p)}[t]/(t^{p-1} - 1)$. Over the residue field $\mathbf{F}_p$, we have $\mathbf{F}_p[t]/(t^{p-1} - 1) \simeq \prod_{i=1}^{p-1} \mathbf{F}_p$. However, this splitting does not lift: there is no non-trivial $p-1$-st root of $1$ in $\mathbf{Z}_{(p)}$. On the other hand, Hensel's lemma guarantees that such a lifting works in the henselian local ring $\mathbf{Z}_p$.

Now, consider a correspondence from $W = \operatorname{Spec} S$ to $U_p$. This is given by a finite map $Z \to \operatorname{Spec} S$ and a morphism from $Z$ to $U_p$. We may compose this, as a correspondence, with the canonical map from $U_p$ to $X$; thus, every correspondence from $\operatorname{Spec} S$ to $U_p$ "lives over" a correspondence from $\operatorname{Spec} S$ to $X$. That in turn is given by some linear combination of closed irreducible subsets $T \subseteq \operatorname{Spec} S \times X$ with $T/\operatorname{Spec} S$ finite. The set of irreducible subsets associated to a particular correspondence from $\operatorname{Spec} S$ to $U_p$ may change upon composing with a face map $U_p \to U_{p-1}$ (e.g. there may be some cancellation happening), but only by making it *smaller*, i.e.

passing to a subset. Hence each finite set of irreducible closed subsets of $\operatorname{Spec} S \times X$ gives rise to a subcomplex of the chain complex $(\mathbf{Z}_{\mathrm{tr}}(U_p)(\operatorname{Spec}(S)), \delta)$, and the whole chain complex is the (filtered) colimit of these. To prove that the whole chain complex is acyclic, it therefore suffices to prove that for each finite set $T$ of closed irreducible subsets of $\operatorname{Spec} S \times X$, the corresponding chain complex is acyclic.

Fix $T \subseteq \operatorname{Spec} S \times X$. By Proposition 11.2, $T \simeq \sqcup \operatorname{Spec}(T_i)$ with $T_i$ henselian local rings. Since $U \to X$ is a Nisnevich cover, we may lift $T \to X$ to $T \to U$ (i.e. the Nisnevich condition says we may lift residue fields of each $T_i$, and then because $T_i$ is henselian local we may extend this lift to each $T_i$ and thus to $T$).

Now, this construction gives us lift maps $T \to U_{p-1} \simeq X \times_X U_{p-1}$ to $T \to U_p$, giving the extra degeneracy map $\mathbf{Z}_{\mathrm{tr}}(U_p)(S) \to \mathbf{Z}_{\mathrm{tr}}(U_{p+1})(S)$ and hence the contraction.

**Corollary 11.4.** Let $\mathscr{F} \in \mathrm{PST}(F)$ be a presheaf with transfers. Then the Nisnevich sheafification $\mathscr{F}^{\sim}_{\mathrm{Nis}}$ of the underlying presheaf on $\mathrm{Sm}_F$ gives an object of $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Corr}_F)$, i.e. a Nisnevich sheaf with transfers.

**Corollary 11.5.** The additive category $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Corr}_F)$ is an abelian category.

*Proof.* The only non-obvious part is existence of cokernels. It's easy to see that there are cokernels in $\mathrm{PST}(F)$, namely just the "objectwise cokernels". Sheafifying those gives a sheaf on $\mathrm{Sm}_F$, which by the corollary may be promoted to a sheaf *with transfers*. One then checks that this indeed defines cokernels. $\square$

Let's prove Corollary 11.4.

*Proof.* The corollary is really a "Definition/Corollary", because we have to define functoriality under correspondences. We will do this by "universal example". Fix some $\varphi \in \mathscr{F}^{\sim}_{\mathrm{Nis}}(Y)$ and some correspondence $Z \subseteq X \times Y$ in $\mathrm{Corr}_F(X, Y)$. We want to show that this correspondence sends $\varphi$ to an element $Z^*\varphi$ of $\mathscr{F}^{\sim}_{\mathrm{Nis}}(X)$.

By the construction of the sheafification, the canonical map $\mathscr{F}(Y) \to \mathscr{F}^{\sim}_{\mathrm{Nis}}(Y)$ is typically not surjective, but after passing to a suitable Nisnevich cover $U \to Y$, the element $\varphi$ may be represented by an element $\varphi_0 \in \ker\big(\mathscr{F}(U) \to \mathscr{F}(U \times_Y U)\big)$.

By the Yoneda lemma we may think of this as a map $\varphi_0 \colon \mathbf{Z}_{\mathrm{tr}}(U) \to \mathscr{F}$ which pulls back to $0$ under the map $\mathbf{Z}_{\mathrm{tr}}(U \times_X U) \to \mathbf{Z}_{\mathrm{tr}}(U)$. Now, we know that the following complex has exact Nisnevich stalks, and thus is an exact sequence in the category of Nisnevich sheaves:

$$0 \longleftarrow \mathbf{Z}_{\mathrm{tr}}(Y) \longleftarrow \mathbf{Z}_{\mathrm{tr}}(U) \longleftarrow \mathbf{Z}_{\mathrm{tr}}(U \times_X U) \longleftarrow \cdots$$

Thus, $\mathbf{Z}_{\mathrm{tr}}(Y)$ is the cokernel in this category of Nisnevic sheaves on $\mathrm{Sm}_F$ of the map $\mathbf{Z}_{\mathrm{tr}}(U \times_X U) \to \mathbf{Z}_{\mathrm{tr}}(U)$. Therefore we get that the composition $\varphi_0 \colon \mathbf{Z}_{\mathrm{tr}}(U) \to \mathscr{F} \to \mathscr{F}^{\sim}_{\mathrm{Nis}}$, which is a map in the category of Nisnevich *sheaves* whose composition with $\mathbf{Z}_{\mathrm{tr}}(U \times_X U) \to \mathbf{Z}_{\mathrm{tr}}(U)$ is 0, factors uniquely through a map $\mathbf{Z}_{\mathrm{tr}}(Y) \to \mathscr{F}^{\sim}_{\mathrm{Nis}}(Y)$.

We may evaluate this map at $Z \in \mathbf{Z}_{\mathrm{tr}}(Y)(X)$ to get the desired transfer of $\varphi$. $\square$

# 12 2/5/18

Last time, we discussed the abelian category $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Corr}_F)$ of Nisnevich sheaves with transfer. We can form the bounded derived category $D_-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Corr}_F)$. This includes the objects $= \mathbf{Z}_{\mathrm{tr}}(X)$ and the objects $\mathbf{Z}(i)$.

Inside this category is the full subcategory $\mathrm{DM}_-^{\mathrm{eff}}$ of sheaves with homotopy invariant cohomology sheaves. The cohomology sheaves of a complex $\mathscr{F}^\bullet$ are defined on $X \in \mathrm{Sm}_F$ as the cohomology $H^i(\mathscr{F}^\bullet(X))$ of the complex $\mathscr{F}^\bullet(X)$, taken in the category of Nisnevich sheaves. In other words, this is the kernel of $\mathscr{F}^i(X) \to \mathscr{F}^{i+1}(X)$ mod the image of $\mathscr{F}^{i-1}(X) \to \mathscr{F}^i(X)$), with kernel and image taken in the category of Nisnevich sheaves. This is also equal to the Nisnevich sheafification of the presheaf cohomology of the complex. The condition of a (pre)-sheaf being homotopy invariant means that for any $X$, the projection map $p_1 \colon X \times \mathbf{A}^1 \to X$ induces an isomorphism $(p_1)^* \colon \mathscr{F}(X) \to \mathscr{F}(X \times \mathbf{A}^1)$.

We can "force" a complex of sheaves $\mathscr{F}^\bullet$ to have homotopy invariant cohomology sheaves. That is, we may form the complex $C_*(\mathscr{F}^\bullet) \colon X \mapsto \mathrm{Tot}(\mathscr{F}^\bullet(X \times \Delta^\bullet))$. By construction, the maps $p_1 \colon X \times \mathbf{A}^1 \to X$ induce quasi-isomorphisms $C_*(\mathscr{F}^\bullet)(X) \to C_*(\mathscr{F}^\bullet)(X \times \mathbf{A}^1)$. Thus, the presheaves $X \mapsto H^i(C_*(\mathscr{F}^\bullet)(X))$ are homotopy invariant.

However, the formation of the cohomology sheaves of $C_*(\mathscr{F}^\bullet)$ may destroy the sheaf condition, so we have to sheafify again. One would think that sheafifying then might destroy the homotopy invariance, but the following theorem says that it does not:

**Theorem 12.1.** If $\mathscr{G} \in \mathrm{PSh}_{\mathrm{Nis}}(\mathrm{Corr}_F)$ is homotopy invariant, then its sheafification $\mathscr{G}^\sim$ is also homotopy invariant.

*Proof.* This is in [17].[16]. □

We also have:

**Theorem 12.2.** If $\mathscr{G}$ is a Nisnevich presheaf, then $H^i_{\mathrm{Zar}}(X; \mathscr{G}_{\mathrm{Zar}}) \simeq H^i_{\mathrm{Nis}}(X; \mathscr{G}_{\mathrm{Nis}})$. (here, $\mathscr{G}_{\mathrm{Zar}}$ is the restriction of $\mathscr{G}$ to the small Zariski site of $X$, and $\mathscr{G}_{\mathrm{Nis}}$ is the restriction of $\mathscr{G}$ to the small Nisnevich site of $X$.)

*Proof.* Again, see [17]. □

We have a functor $\underline{C}_* \colon D_-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Corr}_F)) \to \mathrm{DM}_-^{\mathrm{eff}}$ which sends $\mathscr{F}$ to $C_*\mathscr{F}$. Thinking of the former as a triangulated category, we may consider the triangulated category $\mathrm{DM}_-^{\mathrm{eff}}$ as a *localization* of this category with respect to the multiplicative system of morphisms $\mathbf{Z}_{\mathrm{tr}}(X \times \mathbf{A}^1) \to \mathbf{Z}_{\mathrm{tr}}(X)$. (See [15, 05R1] for this construction).

Now, we also have a functor $\mathrm{SM}_F \to \mathrm{DM}_-^{\mathrm{eff}}$ sending $X$ to $M(X) = C_*\mathbf{Z}_{\mathrm{tr}}(X)$. This category contains $M(\mathbf{G}_m^q) = C_*\mathbf{Z}_{\mathrm{tr}}(\mathbf{G}_m^q)$, and $\mathbf{Z}(q)$, which is built out of this. Then, looking at the first "bare hands" construction of motivic cohomology and using the above theorems[17], we have:

**Proposition 12.3.** The motivic cohomology groups $H^p(X; \mathbf{Z}(q))$ are equal to $[M(X); \mathbf{Z}(q)[p]]_{\mathrm{DM}_-^{\mathrm{eff}}}$.

There is also a variant of motivic cohomology called *compactly supported cohomology*. This is defined by:

[16] Voevodsky: cohomological theory of presheaves with transfer.

[17] as well as a comparison of $\mathrm{Ext}^i$ calculated in chain complexes over $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_F)$ versus over $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Corr}_F)$

**Definition 12.4** (compactly supported motivic cohomology)**.**

$$H^p_c(X;\mathbf{Z}(q)) := [M^c(X);\mathbf{Z}(q)[p]]_{\mathrm{DM}^{\mathrm{eff}}_-}$$

What is $M_c(X)$? This should be something like "compactly supported correspondences into $X$". If $X$ is proper, we have $M^c(X) = M(X)$.

In general, we need the following definition:

**Definition 12.5.** The group of *quasi-finite* correspondences[18] $\mathrm{Corr}^{q.f.}_F(X,Y)$ is the set of $\mathbf{Z}$-linear combinations of closed sub-varieties $Z \subseteq X \times Y$ such that the projection to $X$ is *quasi-finite* and *dominant* onto an irreducible component of $X$.

These do not form a category, but $\mathrm{Corr}^{q.f.}_F(X,Y)$ is still functorial in $X \in \mathrm{Corr}_F$. This lets us define:

**Definition 12.6.** $M^c(Y) = C_*(\mathrm{Corr}^{q.f.}_F(-,Y)) \in \mathrm{DM}^{\mathrm{eff}}_-$.

**Proposition 12.7.** If $\overline{Y}$ is a compactification of $Y$, (i.e. $\overline{Y}$ is a proper $F$-scheme and $Y \hookrightarrow \overline{Y}$ is an open immersion), then $M^c(Y)$ is isomorphic to the cone of $M(\overline{Y}\setminus Y) \to M(\overline{Y})$.

The definitions given above work to define $M(X)$, $M^c(X)$ and therefore $H^p(X;\mathbf{Z}(q))$, $H^p_c(X;\mathbf{Z}(q))$ for any scheme $X$ of finite type over $F$ (i.e. not necessarily smooth over $F$). However, some properties of these groups only hold if we assume that resolution of singularities holds over $F$: therefore, they work when $F$ has characteristic $0$ but perhaps they do not in general.

For some purposes, it is convenient to work in a smaller category than $\mathrm{DM}^{\mathrm{eff}}_-$. We define:

**Definition 12.8.** $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$, the "derived category of geometric motives" is the "thick subcategory" of $\mathrm{DM}^{\mathrm{eff}}_-$ generated by $M(X)$ for $X \in \mathrm{Sm}_F$. This is the smallest full subcategory of $\mathrm{DM}^{\mathrm{eff}}_-$ closed under:

- Retracts: i.e. if $\mathscr{F} = \mathscr{F}_0 \oplus \mathscr{F}_1$ and $\mathscr{F} \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$, then $\mathscr{F}_0, \mathscr{F}_1 \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$.
- The shift functor $A \mapsto A[i]$ for $i \in \mathbf{Z}$
- Mapping cones: if $f\colon A \to B$ is a morphism with $A, B \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$, then the mapping cone $C(f)$ is in $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$. i.e. this is some object $C$ such that there is a distinguished triangle

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow A[1] \longrightarrow \cdots$$

**Proposition 12.9.** If $F$ has resolution of singularities, then for any finite type scheme $X$ over $\operatorname{Spec} F$, $M(X)$, $M^c(X)$ are in $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$.

There is a *tensor product* on $\mathrm{DM}_-$ making it into a symmetric monoidal category[19]. We have: $M(X) \otimes M(Y) := M(X \times Y)$. Given $\mathscr{F} \in \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Corr}_F)$, we may canonically resolve $\mathscr{F}$ by representable sheaves as:

$$\mathscr{F} \longleftarrow \bigoplus_{\mathbf{Z}_{\mathrm{tr}}(X)\to\mathscr{F}} \mathbf{Z}_{\mathrm{tr}}(X) \longleftarrow \cdots$$

[18] the notation $\mathrm{Corr}^{q.f.}$ is not standard

[19] and a so-called "tensor triangulated category". The word "triangulated" has to do with taking mapping cones of maps of chain complexes.

This allows us to "linearly extend" the definition of the tensor product from representable objects $M(X) \simeq \mathbf{Z}_{\mathrm{tr}}(X)$ to all sheaves. Given some $A \in \mathrm{DM}^{\mathrm{eff}}_{-}$, the functor $- \otimes A\colon \mathrm{DM}^{\mathrm{eff}}_{-} \to \mathrm{DM}^{\mathrm{eff}}_{-}$ may have an adjoint $\underline{\mathrm{Hom}}(A, -)$. However, for arbitrary $A$, the relevant complex may not be bounded above.

**Proposition 12.10.** $\underline{\mathrm{Hom}}(A, -)$ exists whenever $A \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$.

In particular, we may define $A(1) := A \otimes \mathbf{Z}(1)$ for any $A \in \mathrm{DM}^{\mathrm{eff}}_{-}$.
We have the following crucial theorem:

**Theorem 12.11** (Cancellation theorem)**.** The functor $A \mapsto A(1)$ is full and faithful, i.e. $[A, B]_{\mathrm{DM}^{\mathrm{eff}}_{-}} \xrightarrow{\sim} [A(1), B(1)]_{\mathrm{DM}^{\mathrm{eff}}_{-}}$.

*Proof.* The original proof assumed resolution of singularities, but Voevodsky later found a proof which works in any characteristic. See [20]. □

If there were some object $\mathbf{Z}(-1)$ such that $\mathbf{Z}(1) \otimes \mathbf{Z}(-1) \simeq \mathbf{Z} = \mathbf{Z}(0)$, this theorem would be entirely unsurprising. However, this object does not exist in $\mathrm{DM}^{\mathrm{eff}}_{-}$. We construct the category $\mathrm{DM}_{-}$ by "formally inverting" the functor $- \otimes \mathbf{Z}(1)$, and define inside this the category $\mathrm{DM}_{\mathrm{gm}}$ as before. This will be a "rigid tensor category". (An analogy in topology is the passage from *pointed spaces* to *spectra*, where one formally inverts smash products with $S^1$, after which all finite CW complexes become dualizable. In this case the analogue of the cancellation theorem is far from true: there is a big difference between maps $X \to Y$ and maps $S^1 \wedge X \to S^1 \wedge Y$.)

# 13 2/7/18

Last time, we discussed an embedding of $\mathrm{Corr}_F$ into the category of chain complexes of contravariant functors from $\mathrm{Corr}_F$ to $\mathbf{Ab}$, i.e. the category PST. This embedding is *enriched over* $(\mathbf{Ab}, \otimes)$, meaning that both categories have a natural structure of additive tensor categories, and the embedding respects this structure. We furthermore need to formally invert "weak equivalences" given by the "homotopies" $\mathbf{Z}_{\mathrm{tr}}(X \times \mathbf{A}^1) \to \mathbf{Z}_{\mathrm{tr}}(X)$.

Now, we may carry out a parallel version of this story working over the category $\mathrm{Sm}_F$ of smooth $F$-schemes. We have the Yoneda embedding of this category into the presheaf category $\mathrm{Psh}(\mathrm{Sm}_F)$, as well as the canonical map from $\mathrm{Sm}_F$ to the "simplicialization" $\mathrm{sSm}_F := \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Sm}_F)$. Thus, the Yoneda embedding induces an embedding $\mathrm{sSm}_F \to \mathrm{sFun}(\mathrm{Sm}_F^{\mathrm{op}}, \mathbf{Sets}) = \mathrm{Fun}(\mathrm{Sm}_F^{\mathrm{op}}, \mathbf{sSets})$.

Just as before, we need to "localize" this category by forcing *weak equivalences* to be isomorphisms. We have:

**Definition 13.1.** Given a morphism $f\colon A_\bullet \to B_\bullet$ of simplicial sets, we say that $f$ is a *weak equivalence* if $|A_\bullet| \to |B_\bullet|$ induces isomorphisms on $\pi_0$ and on $\pi_i(-, a)$ for every $i > 0$ and every $a \in A_0$. In other words, a weak equivalence is a homotopy equivalence on the geometric realizations.

In the category $\mathrm{Fun}(\mathrm{Sm}_F^{\mathrm{op}}, \mathbf{sSets})$, we want to invert *stalkwise weak equivalences*. Recall that if $J$ is a filtered category and $j \mapsto A(j)$ is a functor from $J$ to **sSet**, then we have natural isomorphisms:

$$\varinjlim_j \left(\pi_0 A(j)\right) \xrightarrow{\sim} \pi_0 \varinjlim_j A(j), \qquad \varinjlim_j \pi_i \left(A(j), a\right) \xrightarrow{\sim} \pi_i \left(\varinjlim_j A(j), a\right)$$

for any $a$ in the $0$-skeleton of $A$.

Given a functor $\mathscr{F} : \mathrm{Sm}_F^{\mathrm{op}} \to \mathbf{sSets}$ and $x \in X \in \mathrm{Sm}_F$, the Nisnevich stalk at $x$ is the colimit in $\mathbf{sSets}$ of $\mathscr{F}(U)$ over the filtered category of Nisnevich neighborhoods of $X$ $\mathrm{Spec}(k(x)) \to U \to X$. Then we may define:

**Definition 13.2.** A map $\mathscr{F}_\bullet \to \mathscr{G}_\bullet$ of simplicial presheaves on $\mathrm{Sm}_F$ is a *weak equivalence* if the induced map on Nisnevich stalks for any $x \in X \in \mathrm{Sm}_F$ is a weak equivalence in the category of simplicial sets.

**Example 13.3.** Let $\mathscr{F} \in \mathrm{sPSh}(\mathrm{Sm}_F)$ be a simplicial presheaf on $\mathrm{Sm}_F$, sending $[p] \in \Delta$ to $\mathscr{F}_p$. Then the map $\mathscr{F}_p \to (\mathscr{F}_p)^{\sim}_{\mathrm{Nis}}$ between $\mathscr{F}_p$ and its sheafification as a set-valued presheaf is an isomorphism on stalks. Thus, the condition of being a weak equivalence is not sensitive to whether we work with simplicial sheaves or simplicial presheaves.

Morel and Voevodsky start with the category $\mathrm{sSh}_{\mathrm{Nis}}(\mathrm{Sm}_F)$; other authors start with the category of simplicial presheaves $\mathrm{sPSh}_{\mathrm{Nis}}(\mathrm{Sm}_F)$. Both of these categories have the following properties:

- They have a notion of *local equivalence*. This notion satisfies the "2-out-of-3 property" if $f, g$ are composable morphisms, then if $2$ out of $3$ of $f, g, f \circ g$ are local equivalences, so is the third.

- They have *simplicial enrichments*, i.e. the morphism sets $\mathrm{Hom}(\mathscr{F}_\bullet, \mathscr{G}_\bullet)$ are naturally the 0-simplices in a *simplicial set* of morphisms ("mapping spaces"). This is because $\mathrm{Hom}(\mathscr{F}_\bullet, \mathscr{G}_\bullet) \subseteq \prod_X \mathrm{Hom}_{\mathbf{sSets}}(\mathscr{F}(X), \mathscr{G}(X))$, where $\mathrm{Hom}_{\mathbf{sSets}}$ denotes the *simplicial set* of morphisms in the category of simplicial sets[20]. We shall write $\underline{\mathrm{Hom}}(\mathscr{F}, \mathscr{G})$ for the simplicial set of morphisms from $\mathscr{F}$ to $\mathscr{G}$, for emphasis.

However, we have a problem: the $\underline{\mathrm{Hom}}$ simplicial sets are not homotopy invariant. That is, if $\mathscr{F} \xrightarrow{\sim} \mathscr{F}'$ and $\mathscr{G} \xrightarrow{\sim} \mathscr{G}'$ are weak equivalences, it may not be the case that $\underline{Hom}(\mathscr{F}', \mathscr{G}) \to \underline{Hom}(\mathscr{F}, \mathscr{G}')$ is a weak equivalence. This problem is familiar from homological algebra: the formation of total $\mathrm{Hom}$-complexes does not commute with quasi-isomorphisms. (This is the whole reason why higher $\mathrm{Ext}$ groups exist).

To solve this problem, we use the notion of a *model category*. These were introduced by Quillen, and the original paper [13] remains one of the best references. Another reference is [6].

Let $C$ be a category and $W$ a collection of morphisms (thought of as the set of "weak equivalences"). Then, as long as $W$ satisfies some natural conditions, we may construct a *localization* $C[W^{-1}]$, a category with the same objects as $C$, together with a functor $C \to C[W^{-1}]$ which is the identity on objects. Furthermore, this functor is *initial*[21] among functors out of $C$ which send the

[20] the category of simplicial sets is enriched over itself

[21] strictly! (this is reasonable because we're fixing the object set, and all functors are required to be the identity on object sets)

morphisms in $W$ to isomorphisms. The $\mathrm{Hom}$-sets in $C[W^{-1}]$ should be thought of as "$\mathrm{RHom}$" or "derived $\mathrm{Hom}$", but in fact end up naturally being $\pi_0$ of "derived $\mathrm{Hom}$ spaces".

We require some extra (but auxiliary) structure on $C$ and $W$ for this all to work. Namely, in addition to $W$, we want two additional classes of morphisms of $C$, namely the set $\mathrm{cof}$ of *cofibrations* and the set $\mathrm{fib}$ of *fibrations*. We require some axioms on $C$ and these collections, in which case we say that $C$ with this extra data is a *model category*:

- $C$ has all limits and colimits.
- $W$ satisfies the "2 out of 3 property", i.e. if two of $f, g, f \circ g$ are in $W$, so is the third.
- The sets $W$, $\mathrm{cof}$, $\mathrm{fib}$ are closed under retracts.
- Given a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

  such that $f \in \mathrm{cof}$, $g \in \mathrm{fib}$, and either $f$ or $g$ is in $W$, then there exists an arrow filling in the dotted arrow and making the diagram commute.

- Given $f\colon X \to Y$, there exists factorizations $f = \pi_1 \circ i_1$ and $f = \pi_2 \circ i_2$ with $\pi_1 \in W \cap \mathrm{fib}$, $i_1 \in \mathrm{cof}$ and $\pi_2 \in \mathrm{fib}$, $i_2 \in W \cap \mathrm{cof}$.

**Example 13.4.** If we let $C = \mathbf{sSets}$, $C$ admits the structure of a model category where the weak equivalences are as defined above, the cofibrations $A_\bullet \to B_\bullet$ are level-wise injections, and the fibrations are "Kan fibrations". From this example, one may develop many others.

**Example 13.5.** If $C$ is any small category, the category $\mathrm{sPSh}(C) = \mathrm{Fun}(C^{\mathrm{op}}, \mathbf{sSets})$ admits two canonical structures with $W$ consisting of object-wise weak equivalences. There is the *injective model structure*, where cofibrations are object-wise cofibrations and fibrations are defined by the lifting property (as in topology). Dually, there is the *projective model structure*, where fibrations are object-wise fibrations and cofibrations are defined by the "co-lifting property" (as in topology).

Next time, we'll briefly discuss the operation of *Bousfield localization*, which takes a model category $(C, W, \mathrm{cof}, \mathrm{fib})$ and a collection $W' \supseteq W$ of morphisms and produces a "universal" model structure on $C$ where $W'$ is contained in the set of weak equivalences, the set of cofibrations is unchanged, and there are fewer fibrations. However, this operation may not always work.

In the case of a simplicially enriched category $C$ (e.g. categories of simplicial sheaves/presheaves), we require a tiny bit more. First, a condition on the simplicial enrichment, that "powers and copowers exist", meaning the following. Let $K$ be a simplicial set and $X \in C$ an object. Then we require that there is an object $K \otimes X \in C$ which represents the functor $\mathbf{sSet}(K, C(X, -))$. Dually, we require that there is an object $X^K \in C$ which represents the functor $\mathbf{sSet}(K, C(-, X))$. Then there is a single compatibility requirement between the simplicial enrichment and the model category structure:

If $L \to K$ is a cofibration of simplicial sets and $X \to Y$ is a cofibration in $C$, then we form the pushout diagram:

$$\begin{tikzcd} L \otimes X \arrow[r] \arrow[d] & L \otimes Y \arrow[d] \\ K \otimes X \arrow[r] & Z \end{tikzcd}$$

Thus, there is a unique map from $Z$ to $K \otimes Y$ which is compatible with the natural maps $K \otimes X \to K \otimes Y$ and $L \otimes Y \to K \otimes Y$. We require that these maps be cofibrations.[22]

# 14 2/9/18

Last time, we discussed how to do "homological algebra" in a setting where one cannot add and subtract maps, so there are no chain complexes and homologies to work with. Quillen's proposal to solve this is to use the structure of a *model category*. This is a piece of additional structure on a category $C$ consisting of three classes of morphisms: the "weak equivalences" $W$, the "cofibrations" $\mathrm{cof}$, and the "fibrations" $\mathrm{fib}$. Various axioms are imposed on $C$ and these classes. We recall some of the most important ones. The first is the lifting and co-lifting axiom. Given a diagram in $C$:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

The lifting axiom says that if $g$ is a fibration and $f$ is a cofibration and at least one of $g, f$ is a weak equivalence, then there exists a morphism filling in the dotted arrow.

Another important axiom is the factorization axiom: any map $f\colon X \to Y$ can be factored as a cofibration followed by a fibration in two ways such that either the fibration or the cofibration is additionally a weak equivalence. Some authors require these factorizations to be functorial in $f$.

The structure of a model category gives rise to some natural classes of objects: an object $X$ is called *fibrant* if the map to the terminal object is a fibration, and it is called *cofibrant* if the map from the initial object is a cofibration. ($C$ is assumed to have all limits and colimits).

These concepts also make sense in the setting where $C$ is a simplicially enriched category, i.e. when the Hom-sets $\mathrm{Hom}(X, Y)$ have a functorial structure of simplicial sets. We require some additional axioms on $(C, W, \mathrm{cof}, \mathrm{fib})$ to make the model structure compatible with this enrichment, as discussed in the last lecture.

A particular example of the factorization axioms shows that objects admit *fibrant* and *cofibrant* approximations. A cofibrant approximation to an object $Y$ is a factorization of the initial map $\emptyset \to Y$ as a cofibration $\emptyset \to Y^c$ followed by a fibration which is a weak equivalence $Y^c \twoheadrightarrow^{\sim} Y$. Thus, this "approximates" $Y$ by a cofibrant object $Y^c$. Dually, a fibrant approximation to an object $X$ is a factorization of the terminal map $X \to \{*\}$ as a cofibration which is a weak equivalence $X \hookrightarrow^{\sim} X^f$ followed by a fibration $X^f \twoheadrightarrow \{*\}$. Thus, this "approximates" $X$ by a fibrant object $X^f$.

[22] Some "dual" properties about fibrations then follow from various adjunctions, cf e.g. Proposition 4.12 in [6]

In the category of simplicial sets, the condition of an object being cofibrant is the vacuous condition, and the condition of being fibrant says that the object is a "Kan Complex", i.e. this says exactly that the terminal map is a Kan fibration.

In this setting, we may define a *derived* $\mathrm{Hom}$ functor $C^{\mathrm{op}} \times C \to \mathbf{sSet}$. This sends $(X, Y)$ to the simplicial $\mathrm{Hom}$-set $\underline{\mathrm{Hom}}(X^c, Y^f)$. This is like defining the $\mathrm{Ext}$ groups in an abelian category via injective and projective resolutions. The axioms imply that this derived $\mathrm{Hom}$ functor respects weak equivalences (so it sends weak equivalences in $C$ to weak equivalences in $\mathbf{sSet}$), which is *not true* for the ordinary $\mathrm{Hom}$ functor.

Let us now briefly discuss Bousfield localization of simplicial model categories. Given a further set of morphisms $S \subseteq \mathrm{mor}(C)$, we say that an object $Y$ is *$S$-local* if $Y$ is fibrant and for any $f\colon X \to X'$ in $S$, the map $\mathrm{Hom}(f, Y)\colon \mathrm{Hom}(X', Y) \to \mathrm{Hom}(X, Y)$ is a weak equivalence.

If both $X, X'$ are cofibrant, we say that $f$ is a *$S$-local equivalence* if for any $S$-local object $Y$, $\mathrm{Hom}(f, Y)$ is a weak equivalence. Thus, if we perform a universal operation which forces the morphisms in $S$ to become weak equivalences, we must also do so for the class of $S$-local equivalences.

There is an operation to do this, i.e. a universal way to change the model structure on $C$ to make the morphisms in $S$ weak equivalences. This is called *Bousfield localization*, and exists under certain hypotheses (some of which are set-theoretic in nature to avoid proper class problems). This keeps the class of cofibrations unchanged, but shrinks the class of fibrations to the class of *$S$-local fibrations*. We define these to be the fibrations $f\colon X \twoheadrightarrow Y$ which satisfy the lifting property for any cofibration which is also an $S$-local equivalence. This consists, for example, of all cofibrations in $S$, so this is a more stringent condition on $f$ than the ordinary lifting axiom.

Now, we return to the category $\mathrm{sPSh}(\mathrm{Sm}_F)$, the category of contravariant morphisms from the category $\mathrm{Sm}_F$ to the category of simplicial sets. We recall the "official" definition of $\mathrm{Sm}_F$: this is the full category of the category of $F$-schemes whose objects are the finite-type smooth maps $X \to \mathrm{Spec}\, F$. This is an *essentially small category*, i.e. there is a *set* of isomorphism classes of objects.

We have the notion of "global" weak equivalences: if $\mathscr{F}, \mathscr{G}$ are simplicial presheaves, we say that a morphism $\mathscr{F} \to \mathscr{G}$ is a global weak equivalence iff for every object $X \in \mathrm{Sm}_F$, $\mathscr{F}(X) \to \mathscr{G}(X)$ is a weak equivalence of simplicial sets. This extends to a model structure, where the cofibrations consist exactly of monomorphisms, and the fibrations are defined as those morphisms satisfying the lifting axiom.[23]

We can perform a Bousfield localization on this category to *force* Nisnevich-local weak equivalences - i.e. morphisms such that the induced map on every Nisnevich stalk is a weak equivalence - to be weak equivalences. This leads to the notion of *Nisnevich descent*:

**Definition 14.1.** We say that a sheaf $\mathscr{F}$ satisfies *Nisnevich descent* if for any open immersion $U \hookrightarrow X$ and any étale map $f\colon V \to X$ which is an isomorphism over $X \setminus U$, then if we apply $\mathscr{F}$ to the "elementary distinguished square":

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \arrow[d, "f"] \\ U \arrow[r] & X \end{tikzcd}$$

[23] this uses only that $\mathrm{Sm}_F$ is essentially small: for any small category $C$ it was proved in [13] that one can construct a model category structure on functors $C \to \mathrm{sSets}$ in this way.

we get a *homotopy cartesian square*[24]:

$$\begin{tikzcd} \mathscr{F}(X) \arrow[r] \arrow[d] & \mathscr{F}(V) \arrow[d] \\ \mathscr{F}(U) \arrow[r] & \mathscr{F}(U \times_X V) \end{tikzcd}$$

We should think of these as approximately being the fibrant sheaves after we Bousfield-localize the Nisnevich-local weak equivalences. This isn't exactly true, but we have the following lemma, found in [?mv]:

**Lemma 14.2.** If $\mathscr{F}$ has Nisnevich descent, then for any object $X$, the fibrant replacement $\mathscr{F} \to \mathscr{F}^f$ induces maps

$$\mathscr{F}(X) \xrightarrow{\sim} \mathscr{F}^f(X)$$

which are weak equivalences in $\mathbf{sSet}$. (In other words, $\mathscr{F} \to \mathscr{F}^f$ is a global equivalence.)

As a final step, we need to localize this category even further, so our sheaves are "$\mathbf{A}^1$-homotopy invariant". In other words, we Bousfield localize at the set $S$ of projection maps $X \times \mathbf{A}^1 \to X$ for every $X \in \mathrm{Sm}_F$ (thought of as representable presheaves). After this, for any *fibrant* $\mathscr{F}$, the map $\mathscr{F}(X) \to \mathscr{F}(X \times \mathbf{A}^1)$ will be a weak equivalence of simplicial sets for any $X$.

This construction gives us the *unstable category of motivic spaces* $\mathrm{Spc}_F^{\mathbf{A}^1}$. This is a model category whose underlying category is the category of simplicial presheaves on $\mathrm{Sm}_F$, the weak equivalences are the Nisnevich-local $\mathbf{A}^1$-homotopy equivalences, the cofibrations are the monomorphisms, and the class of fibrations is induced by the rest of the data.

The terminal object, thought of as the "one-point set" is the representable presheaf associated to $\mathrm{Spec}\, F$. The "basepointed" version of $\mathrm{Spc}_F^{\mathbf{A}^1}$ is just the "under category" for the terminal object, or in other words functors from $\mathrm{Sm}_F$ to pointed simplicial sets (a.k.a. simplicial sets with a chosen 0-simplex). Let us denote the basepointed category by $\mathrm{Spc}_*^{\mathbf{A}^1}$. There are two notions of "circle" in this category. The first is the "simplicial circle" $S_s^1$, represented by $S^1 = \Delta[1]/\delta\Delta[1]$, where $\Delta[1]$ is the simplicial set sending $[p] \in \Delta$ to $\Delta([p], 1)$. (More precisely, it is the constant presheaf which sends any smooth scheme to that simplicial set.) The basepoint in the simplicial circle (i.e. the collapsed $\delta\Delta[1]$) gives a map from the terminal object to $S_s^1$, making it an object in the pointed category. It turns out that this is equivalent to the simplicial presheaf represented by the "node" $\mathbf{A}^1/0, 1$, i.e. the pushout of the diagram:

$$\begin{tikzcd} \{0,1\} \arrow[r] \arrow[d] & \mathbf{A}^1 \simeq \{*\} \arrow[d] \\ \{*\} \arrow[r] & {} \end{tikzcd}$$

There is also the "Tate circle", which si the representable presheaf represented by $\mathbf{G}_m = \mathrm{Spec}\, F[t^{\pm 1}]$. This is also "pointed", with distinguished point given by $1 \in \mathbf{G}_m$.

Next, we want to "calculate" the homotopy type of $\mathbf{P}^1$. This is $\mathbf{A}^1$ "glued to" $\mathbf{A}^1$ along $\mathbf{G}_m$, with one map $\mathbf{G}_m \to \mathbf{A}^1$ given by $t \mapsto t$ and the other map $\mathbf{G}_m \to \mathbf{A}^1$ given by $t \mapsto t^{-1}$. $\mathbf{A}^1$ is

[24]this means essentially that the induced map of homotopy fibers is a homotopy equivalence

homotopy equivalent to a point, and $\mathbf{G}_m$ is the pointed object $S^1_t$. We may define *smash products* of pointed objects $(\mathscr{F}, x\colon \operatorname{Spec} F \to \mathscr{F})$ and $(\mathscr{G}, y\colon \operatorname{Spec} \mathscr{F} \to \mathscr{G})$, simply as object-wise smash product of pointed simplicial sets, and it turns out that we always have a homotopy pushout diagram:

$$\begin{array}{ccc} \mathscr{F} & \longrightarrow & \{*\} \\ \downarrow & & \downarrow \\ \{*\} & \longrightarrow & S^1_s \wedge \mathscr{F}. \end{array}$$

Thus, we have:

$$\mathbf{P}^1 \simeq S^1_s \wedge S^1_t.$$

# 15 2/12/18

We will continue discussing the category $C = \mathrm{sPSh}(\mathrm{Sm}_F)$ of simplicial presheaves on the category of smooth $F$-schemes, together with its model structure. The class of weak equivalences, denoted $W$, consists of the $\mathbf{A}^1$-local weak equivalences, and with associated fibrations and cofibrations. An important idea to keep in mind is that the main "homotopy theoretic" data is specified by $(C, W)$, and the fibrations and cofibrations are more of an auxiliary technical tool: thus, any meaningful construction should only really depend on $(C, W)$, at least up to weak equivalence.

For example, the derived Hom is given by $R\operatorname{Hom}(\mathscr{F}, \mathscr{G}) = \operatorname{Hom}(\mathscr{F}^c, \mathscr{G}^f) \in \mathbf{sSet}$, so it appears to depend crucially on which objects are fibrant and cofibrant. However, there is another[25] way of formulating this derived Hom which agrees with this up to weak equivalence and which is constructed only depending on $(C, W)$.

One may discuss pushouts in the category of presheaves. These are given by object-wise pushouts. There are also *homotopy pushouts*, which map to the object-wise pushouts. Given a diagram of the form:

$$\begin{array}{ccc} \mathscr{F}_{01} & \longrightarrow & \mathscr{F}_1 \\ \downarrow & & \\ \mathscr{F}_0 & & \end{array}$$

We may obtain the homotopy pushout by replacing $\mathscr{F}_{01} \to \mathscr{F}_1$ by a weakly equivalent cofibration, and then taking the object-wise pushout.[26] The map from the homotopy pushout to the pushout is a weak equivalence if one of the legs is a cofibration.

**Example 15.1.** Consider a Zariski open subset $U \hookrightarrow X$ and an étale map $V \to X$ which is an isomorphism over $X \setminus U$. Then we get a fiber diagram[27]:

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

[25] see e.g. Dwyer–Kan: *Calculating simplicial localizations*

[26] For a general model category, we need to replace both maps by cofibrations, but under a minor technical condition which is satisfied in our setting, only one is necessary. (The technical condition is that the model category is *left proper*.)

[27] this kind of square is often called an "elementary distinguished square"

Both horizontal arrows are Zariski open immersions. Thus, the associated maps of simplicial presheaves are monomorphisms and therefore cofibrations. The canonical map from the homotopy pushout to the actual pushout in (simplicial) presheaves is therefore an equivalence. The actual pushout is not representable, but the induced map to the presheaf represented by $X$ is an isomorphism on stalks (this is a statement about $S$-valued points, for $S$ a Henselian local ring). Hence the diagram is "homotopy cartesian" (which is just a name for the induced map from homotopy pushout to $X$ being a weak equivalence).

We may also construct *Thom spaces*. First, we recall the situation in topology. Let $V \to X$ be a vector bundle. We may define the *Thom space* $\mathrm{Th}(V) = DV/S(V)$, where $DV$ and $S(V)$ are the associated disk and sphere bundles respectively (this depends on the choice of a metric on $V$). Note that the inclusion $D(V) \hookrightarrow V$ is a homotopy equivalence over $X$, as is the inclusion $SV \hookrightarrow V - X$. One way of rephrasing the definition of $\mathrm{Th}(V)$ is that it is a pushout of the following diagram[28]:

$$\begin{tikzcd} S(V) \arrow[r] \arrow[d] & D(V) \arrow[d] \\ \mathrm{pt} \arrow[r] & \mathrm{Th}(V) \end{tikzcd}$$

By applying the homotopy equivalences $S(V) \xrightarrow{\sim} V - X$ and $DV \xrightarrow{\sim} V$, we see that this is the homotopy pushout of:

$$\begin{tikzcd} V - X \arrow[r] \arrow[d] & V \arrow[d] \\ \mathrm{pt} \arrow[r] & \mathrm{Th}(V) \end{tikzcd}$$

(In topology, the map $V - X \hookrightarrow V$ is not a cofibration, and the actual pushout here would not be well-behaved, e.g. most likely not Hausdorff.)

Now, to mimic this construction in algebraic geometry, we let $X \in \mathrm{Sm}_F$ be a smooth scheme over $F$ and let $V \to X$ be a vector bundle over $X$, thought of as the total space. If $X = \operatorname{Spec} R$ is affine and $V$ is the vector bundle associated to the locally free $R$-module $M$, $V = \operatorname{Spec}(\mathrm{Sym}_R(M^\vee))$.

Then, the Thom space $\mathrm{Th}_X(V)$ is the pushout in the category $\mathrm{sPSh}(\mathrm{Sm}_F)$ of the diagram:

$$\begin{tikzcd} V - X \arrow[r] \arrow[d] & V \arrow[d] \\ \{*\} \arrow[r] & \mathrm{Th}_X(V) \end{tikzcd}$$

Since $V - X \to V$ is the inclusion of a Zariski open, the top horizontal map is an injection on represented presheaves, hence a cofibration, so the Thom space is equivalent to the homotopy pushout.

One of the reasons Thom spaces are useful in topology is the *Pontryagin-Thom construction*. We let $M^d \hookrightarrow \mathbf{R}^{n+d}$ be a compact smooth $d$-dimensional submanifold. Then we may embed the *normal bundle* $\nu M$ into $\mathbf{R}^{n+d}$ as a tubular neighborhood.

[28]this is the actual pushout, but the top horizontal map is a cofibration so it is equivalent to the homotopy pushout

Then we have a map $\varphi\colon S^{n+d} \simeq \mathbf{R}^{n+d} \cup \{\infty\}$ to $\mathrm{Th}(\nu M) = D(\nu M)/S(\nu M)$, the "Pontryagin–Thom collapse map". This may be regarded as a universal fundamental class for $M$, in the following sense. If we consider $[\pi] \in \pi_{n+d}\mathrm{Th}(\nu M)$, we may apply the Hurewicz map to send this to a class in relative homology $H_{n+d}(\mathrm{Th}(\nu M), *)$, where $*$ denotes the basepoint in the Thom space. We have a Thom isomorphism $H_{n+d}(\mathrm{Th}(\nu M), *) \simeq H_d(M; \widetilde{\mathbf{Z}})$, where $\widetilde{\mathbf{Z}}$ denotes the local coefficient system associated to the orientation character $w_1(\nu M) : \pi_1(M) \to \mathbf{Z}^\times$ for the normal bundle, which in general may be non-orientable. The resulting homology class is a fundamental class $[M]$, so in this sense the class $[\pi]$, or even the map $\pi$ may be regarded as a homotopical enhancement of the fundamental class. It is quite canonical, for example the embedding $M^d \hookrightarrow \mathbf{R}^{n+d}$ exists and is unique up to isotopy, provided $n \gg d$.

Of course, most of those words (tubular neighborhood, etc) don't make sense in algebraic geometry, but nevertheless we have the following remarkable replacement.

**Theorem 15.2** (Morel-Voevodsky)**.** If $Z \subseteq X \in \mathrm{Sm}_F$ is a closed subvariety, then we have an isomorphism:

$$\mathrm{Th}(N_{X,Z}) \simeq X/(X \setminus Z)$$

This isomorphism takes place in the $\mathbf{A}^1$-local homotopy category, but comes from a particular "zig-zag" of $\mathbf{A}^1$-local weak equivalences in $C = \mathrm{Spd}_*^{\mathbf{A}^1}$.

*Proof.* We recall the construction of the normal bundle in algebraic geometry. The closed embedding $Z \hookrightarrow X$ is specified by an ideal sheaf $\mathscr{I} \hookrightarrow \mathscr{O}_X$, and the normal bundle is the sheaf on $Z$ given by $(\mathscr{I}/\mathscr{I}^2)^\vee|_Z$, which is locally free under our assumptions.

Of course, there is nothing like a tubular neighborhood theorem in algebraic geometry, but we may geometrically realize the normal bundle via a blowup construction called "deformation to the normal cone".[29] We consider the blowup $\mathrm{Bl}_{Z\times\{0\}}(X \times \mathbf{A}^1)$. We have a pullback diagram:

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \mathrm{Bl}_{Z\times\{0\}}(X \times \mathbf{A}^1) \arrow[d] \\ Z \times \{0\} \arrow[r] & X \times \mathbf{A}^1 \end{tikzcd}$$

There is an isomorphism $\mathbf{P}(N_{X,Z} \oplus \mathscr{O}_Z) \simeq E$. There is an open immersion $N_{X,Z} \hookrightarrow \mathbf{P}(N_{X,Z} \oplus \mathscr{O}_Z)$ determined by $[x] \mapsto [(x, 1)]$. There is a complementary closed embedding $\mathbf{P}(N_{X,Z}) \hookrightarrow \mathbf{P}(N_{X,Z} \oplus \mathscr{O}_Z)$ defined by $[x] \mapsto [(x, 0)]$.

Now, the map $Z \times \mathbf{A}^1 \hookrightarrow X \times \mathbf{A}^1$ canonically factors through $\mathrm{Bl}_{Z\times\{0\}}(X \times \mathbf{A}^1)$ (this follows from the universal property of the blowup, since $Z \times \{0\}$ is an effective Cartier divisor in $Z \times \mathbf{A}^1$). This restricts to a canonical inclusion of $Z \times \{0\}$ into $E \subseteq \mathrm{Bl}_{Z\times\{0\}}(X \times \mathbf{A}^1)$.

Now define $D = \mathrm{Bl}_{Z\times\{0\}}(X \times \mathbf{A}^1) \setminus \mathrm{Bl}_Z(X)$ and let $t\colon D \to \mathbf{A}^1$ be the projection onto the $\mathbf{A}^1$ factor. We have $t^{-1}(0) \simeq N_{X,Z}$, and $Z \times \{0\}$ maps into this as the 0-section. We also have $t^{-1}(1) \simeq X$, and $Z \times \{1\}$ maps into $Z$ under this identification.

Now, both $\mathrm{Th}_Z(N_{X,Z}) = N_{X,Z}/(N_{X,Z} - Z)$ and $X/(X \setminus Z)$ map into $D/(D - (Z \times \mathbf{A}^1))$. We claim that these are both weak equivalences, which is proved in three steps.

The first step is to consider the case that $X = Z \times \mathbf{A}^d$, and $Z \hookrightarrow X$ is given by the 0-section $Z \times \{0\} \hookrightarrow Z \times \mathbf{A}^d$. Then it is easy to see the isomorphism $\mathrm{Th}(N_{X,Z}) \simeq X/(X - Z)$, since

[29] See §25 in [?ravi] for a more thorough discussion, and [5] for a comprehensive treatment.

we may identify $N_{X,Z}$ with $Z \times \mathbf{A}^d \simeq X$, and one may moreover verify[30] that both maps to $D/(D-(Z \times \mathbf{A}^1))$ are weak equivalences.

The second step, which is the most interesting step, is to prove the claim under the assumption that there exists a pullback diagram of the form:

$$\begin{tikzcd} Z \arrow[r, "q|_Z"] \arrow[d] & \mathbf{A}^{n-c} \times \{0\} \arrow[d] \\ X \arrow[r, "q"] & \mathbf{A}^n \end{tikzcd}$$

With $q$, and hence $q|_Z$, étale. From this, we get a diagram:

$$\begin{tikzcd} & X \times_{\mathbf{A}^n} (Z \times \mathbf{A}^c) \arrow[r] \arrow[d] & Z \times \mathbf{A}^c \arrow[d, "q|_Z \times \mathrm{id}"] \\ Z \times_{\mathbf{A}^{n-c}} Z \arrow[ur] \arrow[dr] & X \arrow[r, "q"] & \mathbf{A}^n = \mathbf{A}^{n-c} \times \mathbf{A}^c \\ & Z \arrow[u] \arrow[r, "q|_Z"] & \mathbf{A}^{n-c} \times \{0\} \arrow[u] \end{tikzcd}$$

Here, the vertical maps in the top square are both étale, and the two squares are Cartesian. The diagonal $\Delta \subset Z \times_{\mathbf{A}^{n-c}} Z$ is of course closed, but since $q_{|Z}$ is étale it is also open. Hence $Z \times_{\mathbf{A}^{n-c}} Z \simeq \Delta \sqcup U$ with $\Delta$ the diagonal and $U$ its complement. Then we have closed subspaces $U \subset Z \times_{\mathbf{A}^{n-c}} Z \subset X \times_{\mathbf{A}^n} (Z \times \mathbf{A}^C)$, so the complement $V = X \times_{\mathbf{A}^n} (Z \times \mathbf{A}^C) - U$ is (Zariski) open in $X \times_{\mathbf{A}^n} (Z \times \mathbf{A}^C)$. After removing it, the complement $\Delta$ maps to $Z$ by an isomorphism, so we have constructed an *elementary distinguished square*:

$$\begin{tikzcd} W \arrow[r] \arrow[d] & V = X \times_{\mathbf{A}^n} (Z \times \mathbf{A}^C) - U \arrow[d] \\ X - Z \arrow[r] & X. \end{tikzcd}$$

Since we saw that elementary distinguishes squares are homotopy pushouts, we have weak equivalences:

$$V/(V-W) \xrightarrow{\sim} X/(X-Z)$$

and

$$V/(V-W) \xrightarrow{\sim} Z \times \mathbf{A}^c/(Z \times (\mathbf{A}^c \setminus \{0\}))$$

Now, we may apply the first step to this last quotient to get the desired zig-zag of weak equivalences.

The third step is to use that Zariski-locally, any smooth pair $Z \hookrightarrow X$ admits a pullback diagram with $(q, q_{|Z})$, and use a gluing argument. $\square$

[30]we skipped this in class though

# 16 2/14/18

Last time, we discussed Thom spaces. We proved that if $Z \subseteq X$ is a closed immersion with $Z, X \in \mathrm{Sm}_F$, then $X/(X - Z)$ and $\mathrm{Th}(N_{X,Z}) = N_{X,Z}/(N_{X,Z} - Z)$ are weakly equivalent. That allows us to define a "Pongtryagin-Thom construction", i.e. a "collapse map" $X \to X/(X - Z) \simeq \mathrm{Th}(N_{X,Z})$. To prove this weak equivalence, we proceeded as follows:

1. Define the "zig-zag" of morphisms via the deformation to the normal cone.
2. Settle the case $X = Z \times \mathbf{A}^d$.
3. Settle the case where $X$ is étale over $\mathbf{A}^d$ and $Z$ is the pullback of $X$ to $\mathbf{A}^{d-c} \times \{0\}$.
4. Use the following fact: if $X, Z$ are smooth varieties and $x \in X$, then there is a Zariski open neighborhood $U \ni x$ with $(U, U \cap Z)$ as in the previous step.

Next, we will discuss Eilenberg-Mac Lane spaces $K(\mathbf{Z}(i), j) \in \mathrm{Spc}^{\mathbf{A}^1}$. These are simplicial presheaves such that for any $X \in \mathrm{Sm}_F$, we have:

$$[X, K(\mathbf{Z}(i), j)] := \pi_0 \mathrm{RHom}(X, K(\mathbf{Z}(i), j)) \simeq H^j(X; \mathbf{Z}(i))$$

Here, this isomorphism takes place in the homotopy category of simplicial presheaves with $\mathbf{A}^1$-local weak equivalences.

Now, what is the relationship between simplicial presheaves with Nisnevich local equivalence and sheaf cohomology? For any $X, Y$, we have:

$$[X, Y] = \pi_0 \underline{\mathrm{Hom}}(X, Y^f)$$

This is because every object is already cofibrant (since the map from the initial object is always a monomorphism).

We say that $\mathscr{F}_\bullet \in \mathrm{sPSh}(\mathrm{Sm}_F)$ has *Nisnevich descent* if for any elementary distinguished square

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

then the following square is also homotopy Cartesian:

$$\begin{tikzcd} \mathscr{F}(X) \arrow[r] \arrow[d] & \mathscr{F}(V) \arrow[d] \\ \mathscr{F}(U) \arrow[r] & \mathscr{F}(U \times_X V) \end{tikzcd}$$

If $\mathscr{F}$ satisfies this property, then $\mathscr{F}(X) \to \mathscr{F}^f(X)$ is a weak equivalence for any $X$. This means that $\underline{\mathrm{Hom}}(X, \mathscr{F}) \simeq \underline{\mathrm{Hom}}(X, \mathscr{F}^f) =: \mathrm{RHom}(X, \mathscr{F})$.

Now, let $\mathscr{A} : \mathrm{Sm}_F^{\mathrm{op}} \to \mathbf{Ab}$ be a Nisnevich sheaf. We may compose $\mathscr{A}$ with the forgetful map $\mathbf{Ab} \to \mathbf{Set}$ and the embedding $\mathbf{Set} \to \mathbf{sSet}$ to get a simplicial presheaf. Because $\mathscr{A}$ is a Nisnevich sheaf, it automatically has Nisnevich descent. Thus, we have:

$$[X, \mathscr{A}] \simeq \mathscr{A}(X) = H^0(X, \mathscr{A})$$

Here, the last group is Zariski (or Nisnevich) sheaf cohomology on $X$. This gives the first relationship between homotopy theory of simplicial presheaves with Nisnevich local equivalences and sheaf cohomology.

Now, let $A$ be an abelian group and $n \in \mathbf{N}$. We will define $K(A, n) \simeq B \cdots BA = B^n A = \mathrm{DK}(A[n])$. Here, $B$ is the "classifying space" functor, which takes topological abelian groups to topological abelian groups, and hence may be iterated. $\mathrm{DK}$ is the ***Dold-Kan functor***, giving an equivalence of categories[31] between $\mathbf{N}$-graded chain complexes $\mathrm{Ch}(\mathbf{Ab})$ and $\mathbf{sAb}$.

We may apply this construction object-wise to get the simplicial presheaf $\underline{B\mathscr{A}} \colon \mathrm{Sm}_F^{\mathrm{op}} \to \mathrm{sSets}$. This satisfies

$$\pi_i(B\mathscr{A}(X)) = \begin{cases} \mathscr{A}(X) & i = 1 \\ 0 & \text{else} \end{cases}$$

and similarly for $\underline{B\mathscr{A}}$.

However, this simplicial presheaf will not satisfy descent in general. Consider the following homotopy Cartesian square:

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

This gives a diagram:

$$\begin{tikzcd} B\mathscr{A}(X) \arrow[r] \arrow[d] & B\mathscr{A}(V) \arrow[d] \\ B\mathscr{A}(U) \arrow[r] & B\mathscr{A}(U \times_X V) \end{tikzcd}$$

The condition of this square being homotopy Cartesian is (at least morally) equivalent to the existence of a long exact "Mayer-Vietoris" sequence:

$$\pi_k(B\mathscr{A})(X) \longrightarrow \pi_k(B\mathscr{A})(U) \oplus \pi_k(B\mathscr{A})(V) \longrightarrow \pi_k B\mathscr{A}(U \times_X V) \longrightarrow \pi_{k-1}(B\mathscr{A})(X) \longrightarrow \cdots$$

This would give us a short exact sequence:

$$0 \longrightarrow \mathscr{A}(X) \longrightarrow \mathscr{A}(U) \oplus \mathscr{A}(V) \longrightarrow \mathscr{A}(U \times_X V) \longrightarrow 0$$

However, this sequence need not be right exact, since higher cohomology $H^1(X, \mathscr{A})$ might intervene. However, if the fibrant replacement $B\mathscr{A} \to (B\mathscr{A})^f$ would happen to have $\pi_0(B\mathscr{A})^f(X)$ equal to $H^1(X; \mathscr{A})$, this might fix the problems if the long-exact sequence in homotopy groups became replaced with something continuing as the Mayer–Vietoris sequence. Hence we guess that the more generally the map

$$(B^n\mathscr{A})(X) \to (B^n\mathscr{A})^f(X)$$

should defines an isomorphism on $\pi_{\geq n}$, and have $\pi_{n-i}(B^n\mathscr{A})^f(X) \simeq H^i(X; \mathscr{A})$. This is indeed the case, as we shall now explain by constructing an explicit model.

We pick an injective resolution $\mathscr{A} \to \mathscr{I}^\bullet$. Passing to Nisnevich stalks gives a quasi-isomorphism from $A$ concentrated in degree $0$ to the complex:

$$0 \longrightarrow \mathscr{I}^0 \longrightarrow \mathscr{I}^1 \longrightarrow \mathscr{I}^2 \longrightarrow \cdots \longrightarrow \mathscr{I}^{n-1} \longrightarrow \ker(\mathscr{I}^n \to \mathscr{I}^{n+1}) \longrightarrow 0$$

[31] on the nose! Not just "homotopically"

The Dold-Kan functor takes this to a Nisnevich local equivalence of simplicial presheaves from $B^n\mathscr{A}$ to $\underline{B}^n\mathscr{A}$, defined to be the Dold-Kan image of the truncated injective resolution.

Since homotopy groups of the Dold–Kan construction is homology of the chain complexes, we get $\pi_{n-i}(\underline{B}^n\mathscr{A})(X)) = H^i_{\mathrm{Nis}}(X;\mathscr{A})$ by the definition of sheaf cohomology in terms of injective resolutions.

**Lemma 16.1.** $\underline{B}^n\mathscr{A}$ has Nisnevich descent.

We may prove this using the Mayer-Vietoris long exact sequence.

Since $\underline{B}^n\mathscr{A}$ has Nisnevich descent, $\underline{B}^n\mathscr{A} \to (\underline{B}^n\mathscr{A})^f$ is an object-wise weak equivalence. This means that:

$$\begin{aligned}
[X, \underline{B}^n\mathscr{A}] &:= \pi_0 \mathrm{RHom}(X, \underline{B}^n\mathscr{A}) \\
&= \pi_0 \underline{\mathrm{Hom}}(X; (\underline{B}^n\mathscr{A})^f) \\
&= \pi_0 \underline{\mathrm{Hom}}(X; \underline{B}^n\mathscr{A}) \\
&= \pi_0(\underline{B}^n\mathscr{A})(X) = H^n(X;\mathscr{A}),
\end{aligned}$$

so we have in fact constructed a "representing object" for sheaf cohomology (at least for sheaves defined on the all of $\mathrm{Sm}_F$).

The *same* idea works for $\mathscr{A}_\bullet$, a chain complex of simplicial presheaves, with hypercohomology replacing cohomology. One performs the same construction level-wise and then takes a total complex.

Now, we have:

$$H^i(X : \mathbf{Z}(j)) = H^i_{\mathrm{Nis}}(X; C_*\mathbf{Z}_{\mathrm{tr}}(\mathbf{G}_m, 1)^j[-j]) = [X; \underline{\mathbf{Z}(j)[i]}]$$

Here, $\mathbf{Z}(j)[i] = \underline{B}^i\mathbf{Z}(j)$. This follows from the above discussion: however, the last set refers to homotopy classes of maps of simplicial presheaves after inverting just Nisnevich local weak equivalences. We have said nothing at all about $\mathbf{A}^1$-homotopy!

However, the same (difficult) argument that shows that $H^i(-;\mathbf{Z}(j))$ is homotopy invariant shows that the formation of $\mathbf{Z}(j)[i]$ is $\mathbf{A}^1$-local. This uses the fact that $\mathbf{Z}(j)$ is actually a presheaf with transfers.

# 17 2/16/18

Last time we saw that Nisnevich cohomology of a smooth scheme $X$ with coefficients in a sheaf $\mathscr{A}$ is representable by a (fibrant) simplicial presheaf $\underline{B}^n\mathscr{A}$. As a corollary we can define, for any $X \in \mathrm{sPSh}(\mathrm{Sm}_F)$ (not necessarily representable!)

$$H^n(X;\mathscr{A}) = [X, \underline{B}^n\mathscr{A}]_{\mathrm{Ho}\left(\mathrm{sPSh}(\mathrm{Sm}_F)\right)} \simeq [X \sqcup \{*\}, \underline{B}^n\mathscr{A}]_{\mathrm{Ho}\left(\mathrm{sPSh}_*(\mathrm{Sm}_F)\right)}$$

It is sometimes convenient to work with the category of pointed simplicial presheaves $\mathrm{sPSh}_*(\mathrm{Sm}_F)$. There is a natural forgetful functor $U$ from this category to the category of simplicial presheaves, and a functor from $\mathrm{sPSh}(\mathrm{Sm}_F)$ to $\mathrm{sPSh}_*(\mathrm{Sm}_F)$ sending a simplicial presheaf $X$ to $X \sqcup \{*\}$.

For $X \in \mathrm{sPSh}_*$, we may define reduced cohomology (i.e. "cohomology relative to the base-point")

$$\widetilde{H}^n(X;\mathscr{A}) = [X, \underline{B}^n\mathscr{A}]_{\mathrm{Ho(sPSh_*)}} = [S^1_s \wedge X, \underline{B}^{n+1}\mathscr{A}] \simeq \widetilde{H}^{n+1}(X \wedge S^1_s;\mathscr{A})$$

The sequence of sheaves $\underline{B}^n\mathscr{A}$ has the structure of a "$\Omega$-spectrum", as there is a canonical Nisnevich local weak equivalence $\underline{B}^n\mathscr{A} \xrightarrow{\sim} \Omega^1_s\underline{B}^{n+1}\mathscr{A}$, where $\Omega^1_s$ denotes the loop space with respect to the simplicial circle $S^1_s$ (this is just a fancy way to express taking object-wise loop spaces). This is the structure which we really used to define cohomology with coefficients in $\mathscr{A}$, and it works at this level of generality:

Given any sequence $E$ consisting of fibrant $E_n$ in $\mathrm{sPSh}_*(\mathrm{Sm}_F)$ and weak equivalences $E_n \xrightarrow{\sim} \Omega_s E_{n+1}$, we could define $H^n(X;E) = [X, E_n]_{\mathrm{Ho(sPSh_*)}}$. The condition of being an "$\Omega$ spectrum" in this sense gives this set of homotopy classes the structure of an abelian group.

Now, so far, we have not really done anything "motivic", since we have not dealt with $\mathbf{A}^1$-homotopy invariance. To force the $H^n(X;\mathscr{A})$ to satisfy Nisnevich descent, we were able to use the fibrant replacement construction. We want a similar construction which forces $\mathbf{A}^1$-homotopy invariance. The natural candidate for this is to replace a simplicial presheaf $\mathscr{F}$ with $\mathrm{Sing}(\mathscr{F})$, the simplicial presheaf defined by:

$$X \mapsto ([p] \mapsto \mathscr{F}_p(X \times \Delta^p))$$

We called this construction $C_*\mathscr{A}$ when we were talking about chain complexes of abelian sheaves $\mathscr{A}$.

Unfortunately, the constructions of replacing $\mathscr{F}$ by $\mathscr{F}^f$ and of replacing $\mathscr{F}$ by $\mathrm{Sing}(\mathscr{F})$ are not compatible, in the sense that $\mathrm{Sing}(\mathscr{F})^f$ may not be $\mathbf{A}^1$-homotopy invariant and that $\mathrm{Sing}(\mathscr{F}^f)$ may not satisfy Nisnevich descent. We can try to force this to happen by taking the sequence:

$$\mathscr{F}_\bullet \to \mathscr{F}^f_\bullet \to \mathrm{Sing}(\mathscr{F}^f_\bullet) \to \Big(\mathrm{Sing}(\mathscr{F}^f)\Big)^f \to \cdots$$

Every other object in this sequence satisfies Nisnevich descent, and the remaining objects (apart from the original $\mathscr{F}$) are $\mathbf{A}^1$-homotopy invariant. It turns out that this means the (filtered) colimit is an $\mathbf{A}^1$-homotopy invariant simplicial presheaf with Nisnevich descent.

This uses a particular property of the Nisnevich topology which is not true for arbitrary Grothendieck topologies, namely that the covering axiom can be checked via finite diagrams - namely, elementary distinguished squares. Then for a particular elementary distinguished square of objects of $\mathrm{Sm}_F$, the condition of a presheaf $\mathscr{F}$ satisfying Nisnevich descent for this square is preserved under filtered colimit.

If you are familiar with constructing fibrant replacements in model categories, this construction might not seem so baroque as it otherwise might: often, one has to repeat a construction some big infinite cardinal number of times in order to make things work in this setting.

Now, the notation $[X,\mathscr{F}]$ meaning homotopy classes of maps from $X$ to $\mathscr{F}$, depends on what the weak equivalences are in the category: there is a canonical map $[X,\mathscr{F}]_{\mathrm{Nis}} \to [X,\mathscr{F}]_{\mathrm{Nis}+\mathbf{A}^1}$, but there is no particular reason to expect it to be an isomorphism.

However, for motivic cohomology, something amazing happens:

**Theorem 17.1.** If $\mathscr{F}$ is a homotopy invariant presheaf with transfers, then $\mathscr{F}^{\sim}_{\mathrm{Nis}}$ and $H^i_{\mathrm{Nis}}(-,\mathscr{F}^{\sim}_{\mathrm{Nis}})$ are automatically $\mathbf{A}^1$-homotopy invariant.

If $\mathscr{F}_\bullet$ is a chain complex in the category $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Corr}_F)$ of Nisnevich sheaves with transfers, then $C_*\mathscr{F}$ has $\mathbf{A}^1$-homotopy invariant cohomology presheaves $X \mapsto H_i((C_*\mathscr{F})(X))^{\sim}_{\mathrm{Nis}}$. A spectral sequence argument shows that this implies that $X \mapsto H^i(X; C_*\mathscr{F})$ is also $\mathbf{A}^1$-homotopy invariant.

# 18 2/21/18

Let us say a bit more about the "Eilenberg-Mac Lane spaces" representing motivic cohomology in the model category $\mathrm{Spc}_*^{\mathbf{A}^1}$, which has as its underlying category the category $\mathrm{sPSh}(\mathrm{Sm}_F)$ of simplicial presheaves on $\mathrm{Sm}_F$ and as its weak equivalences the $\mathbf{A}^1$-Nisnevich-local weak equivalences. The associated homotopy category is called the *motivic homotopy category*. We've developed a general recipe for $\mathbf{A}^1$-localization, by giving an "explicit" fibrant replacement involving an infinite process.

Previously, we developed a parallel story in the abelian world. We included $\mathrm{Sm}_F$ into the additive category $\mathrm{Corr}_F$, and then looked at the category $\mathrm{Fun}(\mathrm{Corr}_F^{\mathrm{op}}, \mathrm{Ch})$. Then by localizing the Nisnevich-local quasi-isomorphisms, we obtained the derived category $D(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Corr}_F))$. From this, we built $\mathrm{DM}_-^{\mathrm{eff}}$, the full subcategory with homotopy invariant homology presheaves. There is a functor $C_*$ from $D(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Corr}_F))$ to $\mathrm{DM}_-^{\mathrm{eff}}$, and we used a non-trivial result to show that this preserves the Nisnevich sheaf property, so we did not need an infinite process.

We can relate these two constructions via the functor[32] $\Gamma$ sending a Nisnevich sheaf of chain complexes $\mathscr{A}_*$ to $\Gamma\mathscr{A}_* \in \mathrm{sPSh}(\mathrm{Sm}_F)$. This functor $\Gamma$ is defined by truncating $\mathscr{A}_*$ at $0$ and then using the Dold-Kan correspondence (giving an equivalence of categories between $\mathbf{N}$-graded chain complexes and simplicial objects in an abelian category). This construction is compatible with the weak equivalences, so it gives a functor on homotopy categories:

$$\Gamma\colon D(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Corr}_F)) \to \mathrm{Ho}(\mathrm{sPSh}(\mathrm{Sm}_F))$$

In particular, this functor restricts to a functor on the subcategory $\mathrm{DM}_-^{\mathrm{eff}}$.

We have:

$$\underline{H}^0_{\mathrm{Nis}}(X; \mathscr{A}_\bullet) = [X, \Gamma\mathscr{A}_\bullet]_{\mathrm{Ho}(\mathrm{Spc}_*^{\mathbf{A}^1})} = \pi_0 \mathrm{RHom}(X, \Gamma\mathscr{A}_\bullet)$$

More generally, we have:

$$\pi_i \mathrm{RHom}(X, \Gamma\mathscr{A}_\bullet) = \underline{H}^{-i}_{\mathrm{Nis}}(X; \mathscr{A}_\bullet)$$

When $\mathscr{A}_\bullet \in \mathrm{DM}_-^{\mathrm{eff}}$, so it has homotopy-invariant homology presheaves (and hence homology sheaves, by [17]), we get an isomorphism $\underline{H}^n(X; \mathscr{A}_\bullet) \xrightarrow{\sim} \underline{H}^n(X \times \mathbf{A}^1; \mathscr{A}_\bullet)$, and the above discussion shows that this means that $(\Gamma\mathscr{A}_\bullet)^f \in \mathrm{Spc}^{\mathbf{A}^1}$ is $\mathbf{A}^1$-local.

Letting $M(X)$ be the object in $\mathrm{DM}_-^{\mathrm{eff}}$ associated to the representable sheaf on $\mathrm{Corr}_F$ determined by $X$, we have:

$$\underline{H}^0_{\mathrm{Nis}}(X; \mathscr{A}_\bullet) = [M(X), \mathscr{A}_\bullet]_{\mathrm{DM}_-^{\mathrm{eff}}} \simeq [X, (\Gamma\mathscr{A}_\bullet)]_{\mathrm{Ho}(\mathrm{Spc}^{\mathbf{A}^1})}$$

This means that $X \mapsto [M(X), A]_{\mathrm{DM}_-^{\mathrm{eff}}}$ is also representable in $\mathrm{Ho}(\mathrm{Spc}^{\mathbf{A}^1})$.

Recall the definition of $\mathbf{Z}(i) = \Sigma^{-i} C_* \mathbf{Z}_{\mathrm{tr}}((\mathbf{G}_m, 1)^{\wedge i}) \in \mathrm{DM}_-^{\mathrm{eff}}$. The above discussion shows that $\Sigma^j \mathbf{Z}(i)$ represents $H^j(-; \mathbf{Z}(i))$. Thus, in $\mathrm{sPSh}(\mathrm{Sm}_F)$, we have:

$$K(\mathbf{Z}(i), j) = \Gamma\Big(\Sigma^{j-i} \mathbf{Z}_{\mathrm{tr}}(\mathbf{G}_m, 1)^{\wedge i}\Big)$$

[32]we didn't use this notation last time, but the relationship is $\underline{B}^n \mathscr{A} = \Gamma(\mathscr{A}[n])$

As before, the fact that cohomology is representable allows us to define cohomology for any $X \in \mathrm{sPSh}(\mathrm{Sm}_F)$ as:

$$H^i(X; \mathbf{Z}(j)) = [X, K(\mathbf{Z}(j), i)]_{\mathrm{Ho}(\mathrm{Spc}^{\mathbf{A}^1})}$$

In particular, we can do this for $X = K(\mathbf{Z}(i), j)$ itself. We see that:

$$H^{i'}(K(\mathbf{Z}(j), i), \mathbf{Z}(j')) = [K(\mathbf{Z}(j), i), K(\mathbf{Z}(j'), i')]_{\mathrm{Ho}(\mathrm{Spc}^{\mathbf{A}^1})}$$

Since $K(\mathbf{Z}(j), i)$ represents the cohomology functor, this is also the same thing as the set of natural transformations from the functor $H^i(-, \mathbf{Z}(j))$ to $H^{i'}(-, \mathbf{Z}(j'))$ as functors from $\mathrm{Spc}^{\mathbf{A}^1}$ to $\mathbf{Set}$.

So far, the discussion has been fairly formal. Now, we will turn to some more serious results. Let $A, B \in \mathrm{DM}^{\mathrm{eff}}_{-}$. Then we have a functor $A \mapsto A \otimes \mathbf{Z}(1)$. We have:

**Theorem 18.1** (Cancellation Theorem)**.** For any $A, B \in \mathrm{DM}^{\mathrm{eff}}_{-}$, the functor $- \otimes \mathbf{Z}(1)$ induces an isomorphism:

$$[A, B]_{\mathrm{DM}^{\mathrm{eff}}_{-}} \xrightarrow{\sim} [A \otimes \mathbf{Z}(1), B \otimes \mathbf{Z}(1)]_{\mathrm{DM}^{\mathrm{eff}}_{-}}$$

The shift $\Sigma\mathbf{Z}(1)$ is given by $C_*\mathbf{Z}_{\mathrm{tr}}(\mathbf{G}_m, 1) = M(\mathbf{G}_m, 1) = \mathrm{cok}(M(\mathrm{pt}) \to M(\mathbf{G}_m))$. This $(\mathbf{G}_m, 1)$ gives an object $S^1_t \in \mathrm{Spc}^{\mathbf{A}^1}_*$. More generally, given a smooth variety $X$ with a basepoint $x$, we may define $M(X, x)$ similarly as the cokernel of the map $M(\mathrm{pt}) \to M(X)$ defined by $x$.

Clearly the shift functor is fully faithful as well, so tensoring with $\Sigma\mathbf{Z}(1) = M(S^1_t)$ is also fully faithful.

Then the cancellation theorem leads to suspension isomorphisms in motivic cohomology:

$$\widetilde{H}^i(X; \mathbf{Z}(j)) = [M(X, x), \Sigma^i\mathbf{Z}(j)]_{\mathrm{DM}^{\mathrm{eff}}_{-}} \xrightarrow{\sim} \widetilde{H}^{i+1}(S^1_t \wedge X; \mathbf{Z}(j+1))$$

We also have a suspension isomorphism in the "simplicial direction", using the simplicial circle $S^1_s = \Delta[1]/(\partial\Delta[1])$, i.e. the constant simplicial presheaf with value given by the simplicial set $\Delta[1]/(\partial\Delta[1])$. We get:

$$\widetilde{H}^i(X; \mathbf{Z}(j)) \xrightarrow{\sim} \widetilde{H}^{i+1}(S^1_s \wedge X; \mathbf{Z}(j))$$

On the level of representing objects, we may state these as weak equivalences

$$K(\mathbf{Z}(j), i) \xrightarrow{\sim} \Omega^1_s K(\mathbf{Z}(j), i+1)$$

and

$$K(\mathbf{Z}(j), i) \xrightarrow{\sim} \Omega^1_t K(\mathbf{Z}(j+1), i+1)$$

in the category $\mathrm{Spc}^{\mathbf{A}^1}_*$ of pointed simplicial presheaves, with $\mathbf{A}^1$ Nisnevich local weak equivalences. Here, $\Omega^1_s$ denotes the loop space with respect to $S^1_s$, in other words just the object-wise loop space of simplicial presheaves, and $\Omega^1_t$ the pointed mapping space from $S^1_t$[133]

This allows us to define the *motivic stable category*, which is supposed to be like a category of spectra. This consists of collections of fibrant objects $E_n \in \mathrm{Spc}^{\mathbf{A}^1}_{\bullet}$ for $n \in \mathbf{Z}$ equipped with maps:

$$E_n \xrightarrow{\sim} \Omega^1_s\Omega^1_t E_{n+1}$$

Note that since $S^1_s \wedge S^1_t \simeq (\mathbf{P}^1, \infty)$, the right hand side is pointed maps from $(\mathbf{P}^1, \infty)$ to $E_{n+1}$.

[33]by definition, this is right adjoint (in the actual categories, not just in homotopy categories) to the smash product with $S^1_t$. Strictly speaking we should have written $K(\mathbf{Z}(j), i+1)^f$ before taking these loop spaces, but we shall tacitly redefine the notation $K(\mathbf{Z}(j), i)$ to mean something fibrant, weakly equivalent to what we defined before

# 19 2/23/18

We will introduce the notation $H^{p,q}(X;A)$ for the motivic cohomology group $H^p(X;\mathbf{Z}(q)\otimes_{\mathbf{Z}} A) = H^p(X;A(q))$. Last time, we discussed that motivic cohomology is representable:

$$H^p(X;A(q)) = [X_+, K(A(q),p)]_{\mathrm{Ho}(\mathrm{Spc}_*^{\mathbf{A}^1})}$$

Here, $X_+ = X \sqcup \{+\}$, the pointed scheme formed by adding a disjoint basepoint to $X$. Also, if $X \in \mathrm{Spc}_*^{\mathbf{A}^1}$, we can extend the definition of motivic cohomology by letting $\widetilde{H}^{p,q}(X,A) = [X, K(A(q),p)]$. (The tilde denotes that this is "cohomology relative to the basepoint": if $X$ comes from a smooth scheme with a basepoint map $\mathrm{Spec}(F) \to X$, then $H^{p,q}(X) \cong H^{p,q}(\mathrm{Spec}(F)) \oplus \widetilde{H}^{p,q}(X)$.)

We also discussed the *suspension isomorphisms*, which follow from the cancellation theorem. We have isomorphisms:

$$H^{p,q}(X;A) \xrightarrow{\sim} H^{p+1,q}(S^1_s \wedge X;A), \qquad H^{p,q}(X;A) \xrightarrow{\sim} H^{p+1,q+1}(S^1_t \wedge X;A)$$

Here, $S^1_t = (\mathbf{G}_m, 1) = \Sigma\mathbf{Z}(1)$ is the "Tate circle" and $S^1_s$ is the "simplicial circle".

In topology, there is only one circle $S^1$, and only the first index is meaningful, so the suspension isomorphism takes the form:

$$\widetilde{H}^p(X;\mathbf{F}_\ell) \xrightarrow{\sim} \widetilde{H}^{p+1}(S^1 \wedge X;\mathbf{F}_\ell)$$

Here, $\ell$ is a prime number.

In topology, at least when $\ell$ is an odd prime, we have *reduced power* operations denoted:

$$\mathcal{P}^a\colon H^n(X;\mathbf{F}_\ell) \to H^{n+2a(\ell-1)}(X;\mathscr{F}_\ell)$$

These commute with the suspension isomorphisms, so they are "stable" cohomology operations.

Another stable cohomology operation is the Bockstein homomorphism $\beta\colon H^n(X;\mathscr{F}_\ell) \to H^{n+1}(X;\mathscr{F}_\ell)$. These operations generate the *Steenrod algebra* $\mathscr{A}_\ell$ of all stable cohomology operations. This is an algebra over the ring $H^*(\mathrm{pt};\mathscr{F}_\ell) = \mathbf{F}_\ell$.

We have an $\mathbf{F}_\ell$ basis given by:

$$\mathcal{P}^I = \beta^{\epsilon_0}\mathcal{P}^{i_1}\cdots\beta^{\epsilon_{n-1}}\mathcal{P}^{i_n}\beta^{\epsilon_n}$$

for $\epsilon_i \in \{0,1\}$, $i_a \in \mathbf{N}$, and $i_a \geq \ell i_{a+1} + \epsilon_a$. We include $\mathcal{P}^0 = \mathrm{id}$ in this set. We can describe the multiplication in $\mathscr{A}_\ell$ via the *Adem relations*:

$$\mathcal{P}^a\mathcal{P}^b = \sum(\cdots)\mathcal{P}^i\mathcal{P}^j$$

whenever $a < \ell \cdot b$. Here, all $i,j$ considered will satisfy $i \geq \ell j$. There is a corresponding relation giving a formula for $\mathcal{P}^a\beta\mathcal{P}^b$ when $a \leq \ell b$.

Now, we may mimic the construction of $\mathscr{A}_\ell$ to define the *motivic Steenrod algebra* $\mathscr{A}_\ell^{\mathrm{mot}}$ for $\ell \neq \mathrm{char}(F)$ as the algebra of cohomology operations $\theta_{p,q}\colon H^{p,q}(-;\mathbf{F}_\ell) \to H^{p+a,q+b}(-;\mathbf{F}_\ell)$ which commute with both suspension isomorphisms. Such operations are called *bi-stable*. It turns out that the structure of this algebra as an algebra over the ring $H^{*,*}(\mathrm{Spec}(F);\mathbf{F}_\ell)$ is essentially the same as in topology. However, this ring is much more complicated than $\mathbf{F}_\ell$: for example, we have seen that $H^{p,p}(\mathrm{Spec}(F), \mathbf{F}_\ell) = K_p^M(F)/\ell$.

The reduced powers are operations:

$$\mathcal{P}^a\colon H^{p,q}(X, \mathbf{F}_\ell) \to H^{p+2a(\ell-1),q+a(\ell-1)}(X; \mathbf{F}_\ell)$$

and the Bockstein homomorphisms are operations:

$$\beta\colon H^{p,q} \to H^{p+1,q}$$

We also define $B^a := \beta\mathcal{P}^a$. Now, we have:

**Proposition 19.1.** If $\ell \neq \mathrm{char}(F)$ and $\ell \neq 2$, then the same Adem relations hold in the motivic Steenrod algebra as in the ordinary Steenrod algebra. When $\ell = 2$, some small modification is needed.

In topology, when $\ell = 2$, the reduced powers are often denoted differently: we have $\mathcal{P}^a = \mathrm{Sq}^{2a}$ and $\beta\mathcal{P}^a = \mathrm{Sq}^{2a+1}$. In general, the constructions of the reduced powers come from the fact that the cup product is graded commutative on $C^*(X; \mathscr{F}_\ell)$ up to homotopy, but not on the nose. Voevodsky's construction ([18]) in the motivic setting follows a similar principle. We will discuss a construction of these operations in topology which is formally very analogous to Voevodsky's.

In a particular case, the *total power operation* $\mathcal{P} = \sum_{a=0}^\infty \mathcal{P}^a$ will be an element of the cohomology group $H^{*,*}(K(\mathbf{F}_\ell(p), 2p); \mathbf{F}_\ell)$, giving a natural transformation $H^{2p,p}(-; \mathbf{F}_\ell) \to H^{*,*}$. Voevodsky uses a particular model for the Eilenberg-Mac Lane space $K(\mathbf{F}_\ell(p), 2p)$, and constructs an "$\ell$-th power map" $K(\mathbf{F}_\ell(p), 2p) \times \mathrm{B}S_\ell \to K(\mathbf{F}_\ell(p\ell), 2p\ell)$.

Now, $K(\mathbf{Z}(i), j)$ corresponds under the Dold-Kan functor $\Gamma$ to the object

$$\mathbf{Z}_{\mathrm{tr}}((\mathbf{G}_m, 1)^{\wedge i})[j-i] = M(\mathbf{G}_m, 1)[j-i]$$

Recalling that $(\mathbf{G}_m, 1) = S_t^1$ and that $\mathbf{P}^1 \simeq S_s^1 \wedge S_t^1 \simeq \mathbf{A}^1/(\mathbf{A}^1 \setminus \{0\})$. (The first of these weak equivalences is derived from $\mathbf{P}^1$ being the pushout of $\mathbf{A}^1 \leftarrow \mathbf{G}_m \to \mathbf{A}^1$.) Using this, we may see that:

$$K(\mathbf{Z}(p), 2p) \simeq C_*(\mathbf{Z}_{\mathrm{tr}}(\mathbf{A}^p)/\mathbf{Z}_{\mathrm{tr}}(\mathbf{A}^p \setminus \{0\}))$$

This is analogous to the construction of the *Dold-Thom* model for $K(\mathbf{Z}, n)$ in topology. This is the free topological abelian group generated by $S^n = \mathbf{R}^n \cup \{\infty\}$. We think of a map $f\colon X \to K(\mathbf{Z}, n)$ as sending $f(x)$ to a $\mathbf{Z}$-linear combination of points in $\mathbf{R}^n$, so this is like a correspondence from $X$ to $\mathbf{R}^n$. Thus, we may think of $K(\mathbf{Z}, n)$ as a "labeled configuration space": this consists of finite sets of points in $\mathbf{R}^n$, each of which are labeled by integers, and we require that when points collide, we add their labels. This works similarly if the $\mathbf{Z}$ coefficients are replaced by $\mathbf{F}_\ell$.

The cup product gives a map $K(\mathbf{Z}, n) \wedge K(\mathbf{Z}, m) \to K(\mathbf{Z}, n+m)$: given labeled points $\{a_i \cdot x_i\}$ with $x_i \in \mathbf{R}^n$ and $\{b_i \cdot y_i\}$ with $y_i \in \mathbf{R}^m$, the product is the set $\{a_i b_i \cdot (x_i, y_i)\}$ with $(x_i, y_i) \in \mathbf{R}^{n+m} = \mathbf{R}^n \times \mathbf{R}^m$. However, this operation is not commutative, since the first factor determines the first $n$ coordinates and the second factor determines the last $m$ coordinates. On the other hand, there is a natural action of the symmetric group $S_{n+m}$ on $K(\mathbf{Z}, n+m)$, and the cup product is commutative after taking a quotient by this action.

To get the $\ell$-th power maps, we take the diagonal embedding $K(\mathbf{F}_\ell, 2n) \to K(\mathbf{F}_\ell, 2n) \wedge \cdots \wedge K(\mathbf{F}_\ell, 2n)$, with $\ell$ factors on the right hand side. This sends $\sum_i a_i x_i$ to $\sum_{i_1,\ldots,i_\ell} (a_{i_1} \cdots a_{i_\ell})(x_{i_1}, \ldots, x_{i_\ell})$. Since $S_\ell$ acts on the right-hand side, we get a map $K(\mathbf{F}_\ell, 2n) \times \mathrm{B}S_\ell \to K(\mathbf{F}_\ell, 2\ell n)$.

Now, we have the following fact in group cohomology: if $G_\ell \subseteq G$ is a Sylow $\ell$-subgroup, then the restriction map $H^*(G;\mathbf{F}_\ell) \to H^*(G_\ell;\mathbf{F}_\ell)$ is injective. (This is because the composition of this map with the transfer or co-restriction map is multiplication by $(G : G_\ell)$, which is prime to $\ell$). Thus, we have:

$$H^*(\mathrm{B}S_\ell;\mathbf{F}_\ell) \hookrightarrow H^*(\mathrm{B}C_\ell;\mathbf{F}_\ell)^{\mathrm{Aut}(C_\ell)} = \mathbf{F}_\ell[u,v]$$

Here, $v = \beta u$, and $C_\ell$ is the cyclic group of order $\ell$. This follows from the factorization:

$$\begin{tikzcd} & \mathbf{F}_\ell^\times \ltimes \mathbf{F}_\ell \arrow[d] \\ \mathbf{F}_\ell = C_\ell \arrow[ur] \arrow[r] & S_\ell \end{tikzcd}$$

# 20 2/26/18

Let $\ell$ be an odd prime number, $S_\ell$ the symmetric group on $\ell$ letters, and $\mathrm{B}S_\ell$ its classifying space. When defining Steenrod operations, one essentially must compute $H^*(S_\ell;\mathbf{F}_\ell) = H^*(\mathrm{B}S_\ell;\mathbf{F}_\ell)$. We may consider the cyclic group of order $\ell$ as $(\mathbf{F}_\ell,+)$ embedded as the subgroup $\{\left(\begin{smallmatrix}1&*\\&1\end{smallmatrix}\right) \subseteq \mathrm{GL}_2(\mathbf{F}_\ell)$. This sits inside the larger subgroup $\{\left(\begin{smallmatrix}*&*\\&1\end{smallmatrix}\right)\}$, which is isomorphic to $\mathbf{F}_\ell^\times \ltimes \mathbf{F}_\ell$ and acts on $\mathbf{F}_\ell \times \{1\}$. This defines an inclusion $\mathbf{F}_\ell \subseteq \mathbf{F}_\ell^\times \ltimes \mathbf{F}_\ell \subseteq S_\ell$, where $\mathbf{F}_\ell$ is the Sylow $\ell$-subgroup of $S_\ell$ and $\mathbf{F}_\ell^\times \ltimes \mathbf{F}_\ell$ is its normalizer.

Now, we have a restriction map $i^* \colon H^*(\mathrm{B}S_\ell;\mathbf{F}_\ell) \to H^*(\mathrm{B}\mathbf{F}_\ell;\mathbf{F}_\ell)$ and a transfer/corestriction map $\mathrm{tr}\colon H^*(\mathrm{B}\mathbf{F}_\ell;\mathbf{F}_\ell) \to H^*(\mathrm{B}S_\ell;\mathbf{F}_\ell)$. Their composition is multiplication by $(S_\ell : \mathbf{F}_\ell) = (\ell-1)!$, which may be thought of as an element in $\mathbf{F}_\ell^\times$. Thus, the restriction map is injective.

We may identify the cohomology $H^*(\mathrm{B}\mathbf{F}_\ell;\mathbf{F}_\ell)$ with $\mathbf{F}_\ell[u,\beta u]$ with the element $u \in H^1(\mathrm{B}\mathbf{F}_\ell;\mathbf{F}_\ell) \simeq \mathrm{Hom}(\mathbf{F}_\ell,\mathbf{F}_\ell)$ corresponding to the identity map. The Bockstein homomorphism $\beta$ takes $H^1(\mathrm{B}\mathbf{F}_\ell,\mathbf{F}_\ell)$ to $H^2(\mathrm{B}\mathbf{F}_\ell,\mathbf{F}_\ell)$. We may identify $\beta u$ as the Euler class $e(V)$ for the vector bundle $V \to \mathrm{B}\mathbf{F}_\ell$ corresponding to the map $\mathbf{F}_\ell \to \mathrm{GL}_1(\mathbf{C}) \subseteq \mathrm{GL}_2(\mathbf{R})$ sending $1$ to $e^{2\pi i/\ell}$. We can see that there is a basis for $H^i(\mathrm{B}\mathbf{F}_\ell;\mathbf{F}_\ell)$ consisting of $(\beta u)^i$ and $u(\beta u)^{i-1}$.

The Serre spectral sequence associated to the fibration $\mathrm{B}\mathbf{F}_\ell \to \mathrm{B}(\mathbf{F}_\ell^\times \ltimes \mathbf{F}_\ell) \to \mathrm{B}\mathbf{F}_\ell^\times$ leads to an additive isomorphism $H^*(\mathrm{B}(\mathbf{F}_\ell^\times \ltimes \mathbf{F}_\ell);\mathbf{F}_\ell) \simeq H^*(\mathrm{B}\mathbf{F}_\ell;\mathbf{F}_\ell)^{\mathbf{F}_\ell^\times}$. We can compute the action of $\mathbf{F}_\ell^\times$ on $H^*(\mathrm{B}\mathbf{F}_\ell;\mathbf{F}_\ell)$ explicitly: $\zeta \in \mathbf{F}_\ell^\times$ acts as $\zeta^i$ on $(\beta u)^i$ and $u(\beta u)^{i=1}$. Now, since $\zeta^i = 1$ for all $\zeta \in \mathbf{F}_\ell^\times$ iff $(\ell-1) \mid i$, the fixed points are spanned by $(\beta u)^{(\ell-1)i}$ and $u(\beta u)^{(\ell-1)i-1}$. Note that $\beta$ takes the latter to the former.

Now, there is a general theorem in group cohomology that says if the Sylow $\ell$-subgroup $G_\ell$ of a group $G$ is abelian, then $H^*(G;\mathbf{F}_\ell) \xrightarrow{\sim} H^*(N_G(G_\ell);\mathbf{F}_\ell)$. [34] Thus, we have an isomorphism $H^*(\mathrm{B}(\mathbf{F}_\ell^\times \ltimes \mathbf{F}_\ell);\mathbf{F}_\ell) = \mathbf{F}_\ell[v,(\beta v)]$, with $v = u(\beta u)^{\ell-2} \in H^{2\ell-3}(\mathbf{F}_\ell^\times \ltimes \mathbf{F}_\ell);\mathbf{F}_\ell)$.

Now, we may identify the class $\beta v$ as the Euler class $e(\overline{\rho})$, where $\rho$ is the vector bundle on $\mathrm{B}S_\ell$ associated to the reduced regular representation: i.e. the quotient of the permutation representation $\rho\colon S_\ell \to \mathrm{GL}_\ell(\mathbf{C})$ by the span of $(1,1,\cdots,1)$, i.e. the trivial sub-representation. Note that $\overline{\zeta}|_{\mathbf{F}_\ell} = L_1 \oplus \cdots \oplus L_{\ell-1}$ is the sum of all of the different characters of the additive group of $\mathbf{F}_\ell$.

Now, all this gives us:

**Corollary 20.1.** For any topological space $X$, $H^*(X \times \mathrm{B}S_\ell;\mathbf{F}_\ell) = H^*(X)[v,\beta v]$.

[34] Sometimes called "Swan's theorem", see e.g. [1, Exercise 2.6] or [4, Corollary 5.7.4]

We return to the discussion of Eilenberg-Mac Lane spaces. Recall that we thought of $K(\mathbf{F}_\ell, 2n)$ as the free topological $\mathbf{F}_\ell$-vector space on $S^{2n}$ via the Dold-Thom theorem. We can think of this as a labeled configuration space of points $\sum_i a_i x_i$ with $a_i \in \mathbf{F}_\ell$ and $x_i \in S^{2n} = \mathbf{R}^{2n} \cup \{\infty\}$. We realized the cup product map $K(\mathbf{F}_\ell, 2n) \times K(\mathbf{F}_\ell, 2m) \to K(\mathbf{F}_\ell, 2(n+m))$ defined by $(\sum_i a_i x_i, \sum_j b_j y_j) \mapsto \sum_{i,j} (a_i b_j)(x_i, y_j)$.

Now, we have a natural transformation in cohomology from $H^{2n}(-; \mathbf{F}_\ell)$ to $H^{2n}(X^\ell) \times \cdots \times H^{2n}(X^\ell)$ with $\ell$ copies by sending $x$ to

$$(x \otimes 1 \otimes \cdots \otimes 1, 1 \otimes x \otimes 1 \otimes \cdots \otimes 1, \cdots, 1 \otimes \cdots \otimes 1 \otimes x)$$

Multiplying gives $x \otimes \cdots \otimes x \in H^{2\ell n}(X^\ell)$ and pulling back to the diagonal gives $x^\ell \in H^{2n\ell}(X; \mathbf{F}_\ell)$.

We may think of $x$ as a map $X \to K(\mathbf{F}_\ell, 2n)$. We have a diagram:

$$\begin{tikzcd}
X^\ell \arrow[r, "{(x,x,\cdots,x)}"] & K(\mathbf{F}_\ell, 2n) \times \cdots \times K(\mathbf{F}_\ell, 2n) \arrow[d, "\smile"] \\
X \arrow[u, "\Delta"] \arrow[r, "x^\ell"] & K(\mathbf{F}_\ell, 2n\ell)
\end{tikzcd}$$

This diagram commutes "on the nose". Now, both spaces on the top row have an $S_\ell$ action defined by permuting the factors, and the space $K(\mathbf{F}_\ell, 2n\ell)$ has an action of $S_\ell$ induced from the action on $S^{2n\ell}$, thought of as $(\mathbf{C}_n \oplus \cdots \oplus \mathbf{C}^n) \cup \{\infty\}$. The maps are equivariant "on the nose", but the action on $K(\mathbf{F}_\ell, 2n\ell)$ is non-trivial on the point-set level. We will now apply a homotopy quotient. This is done by the *Borel construction*: to take a homotopy quotient of a space $Y$ by the action of a group $G$, we take the product $Y \times \mathrm{E}G$ and consider the diagonal action of $G$. This is a free action, so we may take a well-behaved quotient. Here, $\mathrm{E}G$ is the universal cover of $\mathrm{B}G$ and is a contractible space with a free action of $G$.

We have a diagram:

$$\begin{tikzcd}
 & (\mathrm{E}S_\ell \times X^\ell \times K(\mathbf{F}_\ell, 2n\ell))/S_\ell \arrow[d, shift left] & \\
\mathrm{B}S_\ell \times X \arrow[ur] \arrow[r] & (\mathrm{E}S_\ell \times X^\ell)/S_\ell \arrow[u, bend left, "s"] \arrow[r] & \mathrm{B}S_\ell
\end{tikzcd}$$

Here, the map $s$ is like a "twisted cocycle": a *section* of a bundle of pointed spaces with fibers $K(\mathbf{F}_\ell, 2n\ell)$. (The point being that a section of a trivial(ized) bundle is just a map, so this is "like a map" to $K(\mathbf{F}_\ell, 2n\ell)$.) The bundle is pulled back from a bundle over $BS_\ell$. The diagonal map is just obtained by composing the left-most horizontal map (coming from the diagonal of $X$) with $s$. The section $s$ itself comes from the map $X^\ell \to K(\mathbf{F}_\ell, 2n\ell)$ in the previous diagram.

In general, if $E \to B$ is a fiber bundle with fiber $K(\mathbf{F}_\ell, m)$ such that $\pi_1(B)$ acts on $K(\mathbf{F}_\ell, m)$ via a map $\pi_1(B) \to \mathrm{GL}_m(\mathbf{R})$ and the action of $\mathrm{GL}_m(\mathbf{R})$ on $S^m$, then it is trivial whenever $\pi_1(B)$ acts by orientation preserving maps (for some choice of basepoint $b \in B$) on the associated vector bundle. This is the case when $m = 2n$ is even and the action comes from $\mathrm{GL}_n(\mathbf{C}) \subseteq \mathrm{GL}_m(\mathbf{R})$.

The trivialization comes from a map $f\colon E \to K(\mathbf{F}_\ell, m)$ corresponding to an element of $H^m(E; \mathbf{F}_\ell)$. The Serre spectral sequence gives a map from $H^*(B; H^m(K(\mathbf{F}_\ell, m); \mathbf{F}_\ell)) \to H^{*+m}(E; \mathbf{F}_\ell)$, and we can look at the image of the Thom class. This map $f$ induces an isomorphism on $\pi^{-1}(b) \simeq K(\mathbf{F}_\ell, m)$, so we get the desired trivialization $(f, \pi)\colon E \xrightarrow{\sim} K(\mathbf{F}_\ell, m) \times B$. With respect to this trivialization, the section $s$ becomes just a map $BS_\ell \times X \to K(\mathbf{F}_\ell, 2n\ell)$, so it represents a cohomology class.

In the motivic setting, the role of $\mathrm{B}\mathbf{F}_\ell$ is played by $\mathrm{B}\mu_\ell$, with $\mu_\ell$ the group *scheme* of $\ell$-th roots of unity. The difference does not matter when the field $F$ contains all the $\ell$-th roots of unity (since then $\mu_\ell \simeq \mathbf{F}_\ell$). Otherwise, many of the arguments are similar, except for the last bit with the Thom class.

# 21 2/28/18

Last time we discussed a construction of the "total power operation" $K(\mathbf{F}_\ell, 2n) \times BS_\ell \to K(\mathbf{F}_\ell, 2n\ell)$ in topology. At a crucial step, it used Thom isomorphism for vector bundles to trivialize a certain fibration. Let us briefly discuss how Thom isomorphism works in the motivic setting.

In the motivic setting, there is a *projective bundle formula*. If $V \to X$ is a $d$-dimensional vector bundle and $\mathbf{P}V \to X$ is its projectivization, we have:

$$M(\mathbf{P}(V)) \simeq M(X) \oplus M(X)(1) \oplus \cdots \oplus M(X)(d-1)$$

This means that $H^{*,*}$ is a free module over $H^{*,*}(X)$ on generators $1, c, \ldots, c^{d-1}$ for $c = [\theta(1)] \in H^{2,1}(\mathbf{P}(V))$. In the case $V$ is a trivial bundle, this shows that $H^{*,*}(\mathbf{P}^n)$ is a free module over $H^{*,*}(\operatorname{Spec} F)$. This lets us define the *Thom* class $\lambda_V$ via the sequence:

$$\mathbf{P}(V) \to \mathbf{P}(V \oplus \mathscr{O}) \to \mathrm{Th}(V)$$

We define $\lambda_V$ to be the class which maps to $1c^d + \sum_{i=0}^{d-1} a_i c^i$, and note that this uniquely defines $a_i \in H^{*,*}(X)$. Indeed, the map in motivic cohomology induced by $\mathbf{P}(V) \to \mathbf{P}(V \oplus \mathscr{O})$ sends $1, c, \ldots, c^{d-1} \in H^{*,*}(\mathbf{P}(V \oplus \mathscr{O}))$ to a basis for $H^{*,*}(\mathbf{P}(V))$ as a module over $H^{*,*}(X)$. Hence the last basis vector $c^d$ may be adjusted to a basis vector $\lambda_V$ for the kernel which is then free of rank one over $H^{*,*}(X)$.

What about classifying spaces? Given a group scheme $G$, we naturally get a functor of points $\mathrm{Sm}_F^{\mathrm{op}} \to \mathrm{Groups}$, and we can define $\mathrm{B}G$ to be its "sheafification" (or fibrant replacement) in the Nisnevich or étale site.

Some specific group schemes appearing include $\mathbf{G}_m = \mathrm{GL}_1$. This is $\mathrm{Spec}(F[t, t^{-1}])$ with multiplication given by $t \mapsto t \otimes t$. This admits a map from $G = \mu_\ell$. This is given by $\mathrm{Spec}(F[t]/(t^\ell - 1))$, and is the kernel of the $\ell$-th power map. The other group scheme appearing is $G = S_\ell = \sqcup_{S_\ell} \mathrm{Spec}(F)$, which is a constant group scheme.

There is also a geometric construction of classifying spaces. Assume that $\rho \colon G \hookrightarrow \mathrm{GL}_n$ is a faithful representation $\rho \colon G \hookrightarrow \mathrm{GL}_n$. For each $N$, we may consider the embedding of $\mathrm{GL}_n$, and hence of $G$, into $\mathrm{GL}_{nN}$, which sends a matrix $g$ to the block diagonal matrix consisting of $N$ copies of $g$. This defines a linear action of $G$ on $\mathbf{A}^{nN}$. This action is not free, but we may consider the maximal Zariski open subset $V_N$ on which $G$ acts freely. It is a theorem that $\varinjlim_{N\to\infty} V_N$ is $\mathbf{A}^1$-contractible. At least when $G$ is sufficiently nice, the quotient $V_N \to V_N/G$ exists and is a principal $G$-bundle. Thus, it makes sense to define:

$$\mathrm{B}_{\mathrm{gm}}(G) = \varinjlim_{N\to\infty} (V_N/G)$$

This is isomorphic to the étale sheafification of the functor of points given by $BG$. Up to $\mathbf{A}^1$-homotopy equivalence, this construction is independent of $\rho$.

**Example 21.1.** Let $G = \mathbf{G}_m$ and $\rho$ the identity map $\mathbf{G}_m \to \mathbf{G}_m = \mathrm{GL}_1$. This acts diagonally on $\mathbf{A}^N$, and the action is free exactly on $\mathbf{A}^N \setminus \{0\}$, with quotient $\mathbf{P}^{N-1}$. This gives us:

$$\mathrm{B_{gm}}\mathbf{G}_m = \mathbf{P}^\infty_F := \varinjlim_N \mathbf{P}^N_F$$

This shows us that $H^{*,*}(\mathrm{B}\mathbf{G}_m) \simeq H^{*,*}(F)[[c]]$ with $c \in H^{2,1}$.

We may also take the one-dimensional representation of $\mu_\ell$ determined by the canonical inclusion into $\mathbf{G}_m$. This acts diagonally and freely on $\mathbf{A}^n \setminus \{0\}$. There is a fibration of $\mathbf{A}^n \setminus \{0\}/\mu_\ell \to \mathbf{P}^{N-1}$. This is isomorphic to the complement of the $0$ section $\mathscr{O}(-\ell) \setminus \mathbf{P}^{N-1} \to \mathbf{P}^{N-1}$. Now, if $V \to X$ is any vector bundle, it is actually a $\mathbf{A}^1$-homotopy equivalence, because locally it is the product of $X$ with a contractible affine space.

The cofiber of the map $\mathscr{O}(-\ell) \setminus \mathbf{P}^{N-1} \to \mathbf{P}^{N-1}$ is the Thom space $\mathrm{Th}_{\mathbf{P}^{N-1}}(\mathscr{O}(-\ell))$. The Thom isomorphism says that cupping with the Thom class $\lambda_{\mathscr{O}(-\ell)}$ gives an isomorphism from $H^{*,*}(\mathbf{P}^{N-1})$ to $\widetilde{H}^{*,*}(\mathrm{Th}(\mathscr{O}(-\ell)))$. The inclusion of $\mathbf{P}^{N-1}$ as the zero section gives a pullback map $\widetilde{H}^{*,*}(\mathrm{Th}(\mathscr{O}(-\ell)))$, and just as in topology we can *define* the Euler class $e(\mathscr{O}(-\ell)) \in H^{*,*}(\mathbf{P}^{N-1})$ to be the class determined by the pullback of the Thom class. This is $\pm\ell \cdot c$ where $c = [\mathscr{O}(1)] \in H^{2,1}(\mathbf{P}^{N-1})$ is the class generating the cohomology. This class is thus trivial in $H^{*,*}(-; \mathbf{Z}/\ell)$.

Thus, we have an exact sequence:

$$0 \to H^{*,*}(\operatorname{Spec} F; \mathbf{Z}/\ell)[[c]] \to H^{*,*}(\mathrm{B}\mu_\ell; \mathbf{Z}/\ell) \to H^{*,*}(\operatorname{Spec} F; \mathbf{Z}/\ell)[[c]] \to 0$$

The addive structure on cohomology may be deduced from this, and with a bit more work Voevodsky also determines the multiplicative structure:

$$H^{*,*}(X \times \mathrm{B}\mu_\ell; \mathbf{Z}/\ell) = H^{*,*}(X; \mathbf{Z}/\ell)[[u, v]]/\sim$$

where $v$ is the Euler class of $\mu_\ell \subseteq \mathbf{G}_m$ and $u$ is a certain class with $v = \beta u$. Here, $\sim$ means that $u^2 = 0$ if $\ell$ is odd and $u^2 = \tau v + \rho u$ when $\ell = 2$. We define $\rho$ as the class corresponding to $\ell(-1) \in K_1^M(\operatorname{Spec} F)/2 = H^{1,1}(\operatorname{Spec}(F); \mathbf{Z}/2)$, and $\tau \in H^{0,1}(\operatorname{Spec}(F); \mathbf{Z}/2) = \mu_2(F)$ corresponds to $-1 \in \mu_2(F)$. Here, we require that $\ell \neq \mathrm{char}(F)$, as usual. These classes $\tau$ and $\rho$ appear in many formulas in motivic cohomology at the prime $\ell = 2$.

For $S_\ell$, we use the reduced regular representation $\overline{\rho}$. We consider $e(\overline{\rho}) \in H^{2\ell-2,\ell-1}(\mathrm{B}S_\ell; \mathbf{Z})$ and take its image $d$ under the map to $H^{2\ell-2,\ell-1}(\mathrm{B}S_\ell; \mathbf{Z}/\ell)$. THen, there is a unique $c \in H^{2\ell-3,\ell-1}(\mathrm{B}S_\ell; \mathbf{Z}/\ell)$ with $d = \beta c$. Then we have a map $H^{*,*}(X; \mathbf{Z}/\ell)[[c, d]] \to H^{*,*}(X \times \mathrm{B}S_\ell; \mathbf{Z}/\ell)$.

# 22 3/2/18

(Executive summary by SG.)

(Discussed an isomorphism $H^{*,*}(X; \mathbf{Z}/\ell)[[c, d]]/\sim \to H^{*,*}(X \times \mathrm{B}S_\ell; \mathbf{Z}/\ell)$ and how the construction of the total power operation carries over to the motivic setting. At a certain point it is important to use the Thom isomorphism to untwist a "twisted cocycle", similarly to the topological case.)

(After this, I discussed some constructions and results from Milnor's paper "On the Steenrod algebra and its dual" [9].)

# 23 3/5/18

Last time, we discussed the Steenrod algebra $\mathscr{A}$, which can be written as $H^{*,*}(\mathrm{pt};\mathbf{F}_\ell)\langle\beta^{\epsilon_0}P^{s_1}\beta^{\epsilon_1}\cdots P^{s_k}\beta^{\epsilon_k}\rangle$ with $s_i \geq \ell\cdot s_{i+1}+\epsilon_i$. We call these generators "admissible", and they form a basis as a module over $H^{*,*}(\mathrm{pt};\mathbf{F}_\ell)$. The products of $P$ and $\beta$ in other orders can be written as linear combinations of the admissible generators, just as in topology. (For $\ell > 2$ it's exactly as in topology, for $\ell = 2$ there are correction terms involving the special elements $\rho$ and $\tau$ from the presentation of the cohomology ring of $B\mu_\ell$.)

There is a coproduct $\psi\colon \mathscr{A}\to\mathscr{A}\otimes_{H^{*,*}(\mathrm{pt})}\mathscr{A}$, and this encodes a formula for $\theta(uv)$ with $\theta\in\mathscr{A}$ in terms of Steenrod operations on $u, v$. We define particular classes $\xi_i,\tau_i\in\mathrm{Hom}_{H^{*,*}(\mathrm{pt})}(\mathscr{A},H^{*,*}(\mathrm{pt}))=\mathscr{A}^\vee$. From the action of $\mathscr{A}$ on $H^{*,*}(\mathrm{B}\mu_\ell;\mathbf{Z}/\ell)\simeq H^{*,*}(\mathrm{pt})\langle v^i, uv^i \mid i\geq 0\rangle$. Here, $u\in H^{1,1}$, and $v=\beta u\in H^{2,1}$. We define $\xi_i$ and $\tau_i$ by requiring that for $\theta\in\mathscr{A}$, we have:

$$\theta(v)=\sum_{i=0}^{\infty}\langle\xi_i,\theta\rangle v^{\ell^i},\qquad \theta(u)=\langle\xi_0,\theta\rangle+\sum_{i=0}^{\infty}\langle\tau_i,\theta\rangle v^{\ell^i}$$

For $\ell>2$, $\mathscr{A}^\vee$ is a free graded commutative $H^{*,*}(\mathrm{pt})$-algebra on $\tau_0,\tau_1,\ldots,\xi_1,\xi_2,\ldots$. We have $\xi_i\in\mathscr{A}^\vee_{2(\ell^i-1),\ell^i-1}$ and $\tau_i\in\mathscr{A}^\vee_{2\ell^i-1,\ell^i-1}$.

At $\ell=2$, a similar statement is true except for the fact that we have the relation $\tau_i^2=\tau\xi_{i+1}+\rho(\tau_{i+1}+\tau_0\xi_{i+1})$. Here, $\ell(-1)=\rho\in H^{1,1}(\mathrm{pt})=K^1_M(F)$. However, this is still free as a module on the monomials.

The proofs of this freeness result for the dual is surprisingly easy. In the topological case, it is proved by Milnor in [9], and the motivic proof is analogous. We first verify that the number of candidate basis elements for the dual gives the correct dimension in each bi-degree. The somewhat strange-looking requirement for a monomial in the $\beta, P$ to be admissible can be rewritten by defining $r_i=s_i-\ell\cdot s_{i+1}+\epsilon_i$, and then the requirement says that $r_i\geq 0$. We have $s_n=\sum_{i=n}^{\infty}(\epsilon_i+r_i)\ell^{i-n}$, so the $\epsilon_i, r_i$ determine the $s_i$. We calculate that the degree of a monomial $\beta^{\epsilon_0}P^{s_1}\beta^{\epsilon_1}\cdots P^{s_k}\beta^{\epsilon_k}$ is:

$$\epsilon_0+\sum_j(\epsilon_j+2(\ell-1)s_j)=\epsilon_0+\sum_{i=1}^{\infty}(\epsilon_i(2\ell^i-1)+r_i(2\ell^i-2))$$

Here, $\epsilon_i\in\{0,1\}$ and $r_i\in\mathbf{N}$. Now, for $I=(\epsilon_0,r_1,\epsilon_1,\cdots,r_k,\epsilon_k)$ a tuple of indices with $\epsilon_i\in\{0,1\}$ and $r_i\in\mathbf{N}$, we define $\theta(I)$ as the monomial in the $\beta, P$ with these indices. This is a basis vector in $\mathscr{A}^{*,*}$. We also define $\omega(I)=\prod_{i=0}^{\infty}\tau_i^{\epsilon_i}\prod_{i=1}^{\infty}\xi_i^{r_i}\in\mathscr{A}^\vee$. Our degree calculation shows that the bi-degree of $\omega(I)$ is equal to the bi-degree of $\theta(I)$. Then we can calculate that:

$$\langle\theta(I),\omega(J)\rangle=\begin{cases}\pm1 & I=J\\ 0 & I<J\\ ? & I>J\end{cases}$$

Here, " $>$ " means with respect to the lexicographic order. This means that the change of basis matrix in $\mathscr{A}^\vee$ from the dual basis of the $\theta(I)$ to the basis consisting of the $\omega(J)$ is upper triangular with $\pm1$ on the diagonal, so it is invertible and therefore the $\omega(I)\in\mathscr{A}^\vee$ form a basis over $H^{*,*}(\mathrm{pt})$.

The existence of this basis for the dual gives an alternative basis for $\mathscr{A}$ as a (left) $H^{*,*}(\mathrm{pt})$-module consisting of the dual basis to $\omega(I)$. In particular, we have elements $Q_k$ called "Milnor primitives" which are dual to the $\tau_k$, and live in $\mathscr{A}^{2\ell^k-1,\ell^k-1}$. For $E = (\epsilon_0, \ldots, \epsilon_k)$ and $R = (r_1, \ldots, r_k)$, we define $\rho(E, R) \in \mathscr{A}$.

In terms of this basis, we may compute the coproduct on $\mathscr{A}^\vee$ by:

$$\xi_k \mapsto \sum_{i=0}^{k} \xi_{k-i}^{\ell^i} \otimes \xi_i, \qquad \tau_k \mapsto \tau_k \otimes 1 + \sum_{i=0}^{k} \xi_{k-i}^{\ell^i} \otimes \tau_i$$

This implies that $Q_k^2 = 0$ in $\mathscr{A}$, and they generate an exterior sub-algebra of $\mathscr{A}$. We call them primitive because $Q_k(xy) = (Q_k x) \cdot y \pm x(Q_k y)$ - this is true for any $\ell$ in topology and for $\ell > 2$ in the motivic theory. We also have $Q_0 = \beta\colon H^{*,*} \to H^{*+1,*}$.

This leads to something called "Margolis homology", which we shall explain shortly. Let us first discuss a special case called "Bockstein homology", associated to $Q_0 = \beta$, and start with the situation in topology. This is defined by

$$\beta H^n(X; \mathbf{Z}/\ell) := \ker(\beta\colon H^n(X) \to H^{n+1}) / \operatorname{im}(\beta\colon H^{n-1} \to H^n)$$

This construction may look a little bit artificial, as it does not look like a typical cohomology theory. (Unlike for example singular cohomology or topological $K$-theory, it is not representable by a space.) It arises as the $E_2$ page of the Bockstein spectral sequence. This arises from an attempt to understand $H^*(X; \mathbf{Z}_\ell)$ in terms of $H^*(X; \mathbf{Z}/\ell)$ together with the endomorphism $\beta$. Here, $H^*(X; \mathbf{Z}_\ell)$ will be of the form $\bigoplus \mathbf{Z}_\ell \oplus \bigoplus \mathbf{Z}_\ell/\ell^{n_i}$. The $\mathbf{Z}_\ell$ summands show up as single $\mathbf{Z}/\ell$ summands in $H^*(X; \mathbf{Z}/\ell)$, while each $\mathbf{Z}_\ell/\ell^n$ summands show up as a pair of $\mathbf{Z}/\ell$ summands in $H^*(X; \mathbf{Z}/\ell)$. For $n = 1$ these are connected by a non-zero Bockstein homomorphism, and for $n > 1$ they are connected by "higher Bocksteins". A better way to say this is via the Bockstein spectral sequence. We have an exact couple given by:

$$\begin{tikzcd}
H^*(X; \mathbf{Z}_\ell) \arrow[rr, "\ell"] & & H^*(X; \mathbf{Z}_\ell) \arrow[dl, "\rho"] \\
& H^*(X; \mathbf{Z}/\ell) \arrow[ul, "\beta"] &
\end{tikzcd}$$

This leads to a spectral sequence in the standard way, and we see that $\mathbf{Z}/\ell^n$ torsion in $H^*(X; \mathbf{Z}_\ell)$ comes from non-zero $d_n$ differentials in the Bockstein spectral sequence. We may think of this construction as coming from the two maps $\mathbf{Z}_\ell \to \mathbf{F}_\ell$ and $\mathbf{Z}_\ell \to \mathbf{Q}_\ell$, plus the action of $\ell$ on $\mathbf{Z}_\ell$.

In topology, there is an analogous construction with non-trivial grading. This comes from the ring $\mathbf{F}_\ell[v_n]$ with $|v_n| = 2(\ell^n - 1)$ together with the projection map to $\mathbf{F}_\ell$ and the localization map to $\mathbf{F}_\ell[v_n, v_n^{-1}]$. We have $\mathbf{F}_\ell[v_n] = \pi_* k(n)$ with $k(n)$ the ring spectrum of "connective Morava $k$-theory", which has an endomorphism given by multiplication by $v_n$. The ring $\mathbf{F}_\ell[v_n, v_n^{-1}]$ is $\pi_* K(n)$, with $K(n)$ the ring spectrum for (periodic) Morava $K$-theory. (For some purposes this behaves a bit like the complete DVR $\mathbf{Z}_\ell$: the "residue field" is still $\mathbf{F}_\ell$, the "fraction field" is $K(n)$, and $v_n$ is a "uniformizer".) There is again an exact couple:

$$\begin{tikzcd}
k(n)^*(X) \arrow[rr, "v_n"] & & k(n)^*(X) \arrow[dl, "d_1 = Q_k"] \\
& H^*(X; \mathbf{Z}/\ell) \arrow[ul] &
\end{tikzcd}$$

This gives us a spectral sequence analogous to the Bockstein spectral sequence, whose $E_2$ page is the Margolis homology $MH^*(X) := \ker(Q_k)/\operatorname{im}(Q_k)$, converging (probably under some finiteness assumptions) to Morava $K$-theory of $X$. Notice that it is much easier to define Margolis homology on its own (one only needs to construct $Q_k$ and prove $Q_k^2 = 0$) than to construct Morava $K$-theory and the spectral sequence.

Apparently an early preprint version of Voevodsky's paper used a "motivic Morava $K$-theory", but later versions got away with only defining "motivic Margolis homology".

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