1 January 8

We’ll start by discussing Milnor’s construction of what is now called Milnor $K$-theory, done in the 1970 paper [7].

Let $F$ be a field, with unit group $F^\times$. Consider the free associative algebra on $F^\times$, i.e. the tensor algebra: $T(F^\times) = \bigoplus_{n=0}^{\infty} (F^\times)^{\otimes n}$. We define the Milnor $K$-theory (which from now on, we will just refer to as “$K$-theory”) of $F$:

**Definition 1.1.** For a field $F$, the Milnor $K$-theory of $F$ is:

$$K^M_n(F) = T(F^\times) / \langle a \otimes b \mid a, b \in F^\times, a + b = 1 \rangle$$

Since we quotient $T(F^\times)$ by a homogeneous ideal, $K^M_n(F)$ is naturally a graded associative algebra, with graded pieces denoted $K^M_n(F)$. The relation $a \cdot b = 0$ for $a + b = 1$ is called the Steinberg relation.

There is a canonical isomorphism $\ell : F^\times \to K^M_1(F)$ sending $a \in F$ to the class of $a \in T_1(F) = F$. This satisfies:

- $\ell(1) = 0$
- $\ell(ab) = \ell(a) + \ell(b)$
- $\ell(a) \cdot \ell(b) = 0$ whenever $a + b = 1$.

**Lemma 1.2.** If $a + b \in \{0, 1\}$, $\ell(a) \cdot \ell(b) = 0 \in K^M_2(F)$.

**Proof.** If $a + b = 1$, this is one of the properties of $\ell$ mentioned above. If $a + b = 0$, then if $a = 1$, $\ell(a) = 0$, so $\ell(a) \cdot \ell(b) = 0$. If $a \neq 1$, then in $F^\times$ we have:

$$\frac{1 - a}{1 - a^{-1}} = -a \cdot \frac{a - 1}{a - 1} = -a$$

Thus, $\ell(-a) = \ell\left(\frac{1-a}{1-a^{-1}}\right) = \ell(1 - a) - \ell(1 - a^{-1})$, so:

$$\ell(a) \cdot \ell(-a) = \ell(a) \cdot \ell(1 - a) + \ell(a) \cdot \ell(1 - a^{-1})$$

$$= \ell(a) \cdot \ell(1 - a^{-1})$$

$$= -\ell(a^{-1}) \cdot \ell(1 - a^{-1})$$

$$= 0$$

\[\square\]

1 called $\ell$ for “logarithm”
We can proceed similarly to prove:

**Lemma 1.3.** If \( a_1 + \cdots + a_n \in \{0, 1\} \), then \( \ell(a_1) \cdots \ell(a_n) = 0 \in K_n^M(F) \).

The multiplication on \( K_n^M(F) \) is graded commutative:

**Lemma 1.4.** \( \ell(a) \ell(b) = -\ell(b) \ell(a) \) for all \( a, b \in F^\times \)

**Proof.**

\[
\ell(a)\ell(b) + \ell(b)\ell(a) = \ell(a) (\ell(-a) + \ell(b)) + \ell(b) (\ell(a) + \ell(-b)) \\
= \ell(a)\ell(-ab) + \ell(b)\ell(-ab) \\
= (\ell(a) + \ell(b))\ell(-ab) \\
= \ell(ab)\ell(-ab) \\
= 0
\]

By graded commutativity, \( \ell(a) \cdot \ell(a) = -\ell(a) \cdot \ell(a) \), so \( \ell(a) \cdot \ell(a) \) is 2-torsion. We can ask if it is actually 0, i.e. if multiplication is alternating. It turns out that we have:

**Lemma 1.5.** \( \ell(a) \cdot \ell(a) = \ell(-1) \cdot \ell(a) \)

**Proof.** \( \ell(a) \cdot \ell(a) = \ell(a) (\ell(-1) + \ell(-a)) = \ell(-1) \cdot \ell(a) \)

**Example 1.6.** We have a non-trivial ring homomorphism \( K_n^M(R) \to F_2 \) sending \( \ell(a) \) to 0 if \( a > 0 \) and 1 if \( a < 0 \). This sends \( \ell(-1)^n \) to \( 1^n = 1 \), so we get \( \ell(a) \cdot \ell(a) = \ell(a) \) and thus the multiplication is not alternating.

Now, we want to indicate why Milnor \( K \)-theory is a useful invariant of the field \( F \). We recall the state of algebraic \( K \)-theory in 1970. If \( A \) is a ring, there were definitions available for \( K_0(A) \), \( K_1(A) \), \( K_2(A) \), and \( K_3(A) \):

For each \( n \), we may define groups \( \text{St}_n(A) \) called the \( n \)-th **Steinberg group** of \( A \). This has generators \( x_{ij}(\lambda) \) for \( 1 \leq i \neq j \leq n \), \( \lambda \in A \), and relations \( x_{ij}(\lambda)x_{ij}(\mu) = x_{ij}(\lambda + \mu) \) and

\[
[x_{ij}(\lambda), x_{kl}(\mu)] = \begin{cases} 
  x_{il}(\lambda \mu) & j = k \\
  1 & j \neq k 
\end{cases}
\]

There is a homomorphism from \( \text{St}_n(A) \) to \( \text{GL}_n(A) \) sending \( x_{ij}(\lambda) \) to the elementary matrix

\[
e_{ij}(\lambda) = \begin{pmatrix} 
  1 & \lambda & & \\
  & \ddots & & \\
  & & 1 & 
\end{pmatrix}
\]

with the \( \lambda \) in position \( i, j \).

Then, we may define:

**Definition 1.7.** The **Steinberg group** of \( K \) is \( \text{St}(A) = \lim_{\rightarrow n} \text{St}_n(A) \).
The maps $\text{St}_n(A) \rightarrow \text{GL}_n(A)$ patch together to a map $\Phi: \text{St}(A) \rightarrow \text{GL}(A) := \lim_{\rightarrow n} \text{GL}_n(A)$. Then we define:

**Definition 1.8.**
- $K_1(A) = \text{GL}(A)/\Phi(\text{St}(A))$ When $A$ is a field\footnote{this holds for certain other types of rings as well, but is a harder result} there is an isomorphism $K_1(A) = \text{GL}(A)/\Phi(\text{St}(A)) \xrightarrow{\sim} A^\times$ given by the determinant. In particular, $K_1(F) \simeq K_1^M(F)$ for a field $F$.
- $K_2(A) = \ker(\Phi)$.
- $K_3(A) = H_3(\text{St}(A))$.

In 1973, Quillen extended this by defining (Quillen) algebraic $K$-theory $K_*^Q(A) = \pi_*(K(A))$ (i.e. homotopy groups) for a particular topological space $K(A)$. This agrees with the definitions in terms of the Steinberg group for degrees 0 to 3.

Now, assume that $A = F$ is a field. For $\lambda, \mu \in F^\times$, pick elements $D_\mu, D_\lambda \in \text{St}_3(F)$ mapping to $\left(\begin{smallmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & \lambda^{-1} \\ 0 & 1 \end{smallmatrix}\right)$ in $\text{GL}_3(F)$. Then $[D_\lambda, D_\mu] \in \ker(\Phi)$. We denote by $\{\lambda, \mu\}$ the corresponding class in $K_2(F)$.

We have:

**Lemma 1.9.** $\{\lambda, \mu\} = 0$ if $\lambda + \mu = 1$

This is the motivating appearance of the Steinberg relation, and gives a well-defined map $K_2^M(F) \rightarrow K_2(F)$ sending $\ell(a)\ell(b)$ to $\{a, b\}$. Milnor’s definition, which came before Quillen’s definition of higher algebraic $K$-theory, was motivated by the following theorem:

**Theorem 1.10** (Matsumoto). This is an isomorphism from $K_2^M(F)$ to $K_2(F)$.

Another place where the Steinberg relation appears is in the study of Kähler differentials. Recall that for a field $F$, we have the $F$-vector space $\Omega^1_{F/\mathbb{Z}}$ of Kähler differentials of $F$. This is defined by:

$$\Omega^1_{F/\mathbb{Z}} = \bigoplus_{a \in F} F \cdot da/ (d(a + b) = da + db, d(ab) = adb + bda, d(1) = 0)$$

From this, we obtain the algebraic de Rham complex $\Omega^*_{F/\mathbb{Z}} = \bigwedge^* F \Omega^1_{F/\mathbb{Z}}$. This is an alternating graded ring.

There is a map $\text{dlog}: K_1^M(F) \simeq F^* \rightarrow \Omega^1_{F/\mathbb{Z}}$ given by $a \mapsto a^{-1} da$. If $a + b = 1$ for $a, b \in F^*$, we have $da + db = 0 \in \Omega^1_{F/\mathbb{Z}}$, so:

$$\frac{da}{a} \wedge \frac{db}{b} = a^{-1}b^{-1}(da) \wedge (-da) = 0$$

Thus, $\text{dlog}(a) \wedge \text{dlog}(b) = 0$ for $a + b = 1$, and so we get a (unique) map of rings $K_*^M(F) \rightarrow \Omega^*_{F/\mathbb{Z}}$ sending $\ell(a)$ to $\text{dlog}(a)$.

When $F = \mathbb{Q}(x_1, \ldots, x_n)$ is a purely transcendental extension of $\mathbb{Q}$ of transcendence degree $n$, then $\Omega^1_{F/\mathbb{Z}}$ has basis $dx_1, \ldots, dx_n$. This implies that $\Omega^k_{F/\mathbb{Z}}$ has basis $\{dx_{i_1} \wedge \cdots \wedge dx_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\}$.
\[ i_2 < \cdots < i_k \leq n \} \]. This makes sense when we think of \( F \) as the field of meromorphic functions on \( \mathbb{A}^n_\mathbb{Z} = \text{Spec}(\mathbb{Z}[x_1, \ldots, x_n]) \).

Another fact is that if \( Q \hookrightarrow F \) is a finite extension, then \( K^M_i(F) \simeq \bigoplus_{F \twoheadrightarrow \mathbb{R}} \mathbb{Z}/2 \) for \( i \geq 3 \).

These facts indicate that the study of the higher \( K^M_i(F) \) is interesting mostly for function fields of varieties of sufficiently large dimension relative to \( i \).

## 2 January 10

Last time, we defined Milnor \( K \)-theory of a field \( F \):

\[ K^M_*(F) = \mathbb{Z}\langle \ell(a) \mid a \in F^\times \rangle / (\ell(a) + \ell(b) \mid a + b = 1) \]

We saw the defining Steinberg relation appear in de Rham cohomology and in algebraic \( K \)-theory, and today we will discuss another manifestation of this relation: quadratic forms. Appearing in the title of Milnor’s paper, these are important in Milnor’s original conception of Milnor \( K \)-theory.

**Definition 2.1.** Let \( F \) be a field of characteristic different from \( 2 \). We define a quadratic form \((M, q)\) over \( F \) to be an \( F \)-vector space \( M \) together with a function \( q : M \to F \) which is a homogeneous polynomial of degree two with respect to a basis of \( M \).

Given a quadratic form \((M, q)\), we may define a function

\[ b(x, y) = q(x + y) - q(x) - q(y) \]

this function is bilinear and symmetric, and we may recover \( q \) by the relation \( q(x) = b(x, x)/2 \). Thus, there is a one-to-one correspondence between quadratic forms and symmetric bilinear forms.

Now, assume that the map \( b : M \to M' = \text{Hom}_F(M, F) \) sending \( x \) to \( b(x, \cdot) \) is an isomorphism. In this case, we say that \((M, q)\) is non-degenerate. With respect to a basis, we may write \( b(x, y) = x^t A y \) for a matrix \( A \), and non-degeneracy says exactly that \( \det(A) \neq 0 \).

Two quadratic forms \((M, q), (M', q')\) are equivalent if there exists an \( F \)-linear isomorphism from \( M \) to \( M' \) which pulls back \( q' \) to \( q \). In terms of matrices, this equivalence relation replaces the matrix \( A \) with \( B' A B \) for some \( B \in \text{GL}_n(F) \).

In dimension \( n = 1 \), we may write a quadratic form as \( \langle a \rangle = (F, q(x) = ax^2) \) for \( a \in F^\times \). Then \( \langle a \rangle \simeq \langle b \rangle \) iff \( a/b \in (F^\times)^2 \).

There are binary operations on the set of non-degenerate quadratic forms defined by

\[ (M, q) \oplus (M', q') = (M \oplus M', q + q') \]

where \( (q + q')(m, m') = q(m) + q'(m') \) and

\[ (M, q) \otimes (M', q') = (M \otimes M', qq') \]

Now, using these operations, we may show that the set of equivalence classes of non-degenerate quadratic forms \((M, q)\) over \( F \) is a ring without subtraction. We define:

---

\(^3\) Bass, Tate: *The Milnor ring of a global field*

\(^4\) This is not true when 2 is not invertible!
**Definition 2.2.** The Grothendieck-Witt ring of $F$, denoted $GW(F)$, is the ring obtained by formally adjoining additive inverses to the set of non-degenerate quadratic forms over $F$ with these operations.

We have the following easy proposition:

**Proposition 2.3.** Any $(M, q)$ with $\dim M = n$ may be diagonalized as:

$$(M, q) \simeq \langle a_1 \rangle \oplus \cdots \oplus \langle a_n \rangle$$

for some $a_i \in F$.

This leads to the question of when $\langle a_1 \rangle \oplus \langle a_2 \rangle \simeq \langle a_1' \rangle \oplus \langle a_2' \rangle$.

First, consider $\langle 1 \rangle + \langle -1 \rangle \in GW(F)$, i.e. the (class of the) quadratic form defined by $q(x, y) = x^2 - y^2$. Since $x^2 - y^2 = (x + y)(x - y)$, we may make an invertible change of variables to turn this into the form $q'(x, y) = xy$. This form is called the hyperbolic plane.

Now, we obtain the following diagram:

$$
\begin{array}{c}
GW(F) \\
\downarrow \quad \dim \\
\mathbb{Z} \\
\uparrow (1)+(-1) \quad \downarrow \quad \dim (\mod 2) \\
\mathbb{Z} \\
\end{array}
$$

The image of the diagonal map is an ideal, since $\langle a \rangle \left( \langle 1 \rangle + \langle -1 \rangle \right) = \langle a \rangle + \langle -a \rangle$. This is the class of the form $(x, y) \mapsto ax^2 - ay^2 = (a(x + y))(x - y)$, so by making the change of variables $x' = a(x + y), y' = x - y$, we see that this is isomorphic to the hyperbolic plane.

This allows us to define:

**Definition 2.4.** The Witt ring of $F$, denoted $W(F)$, is the ring $GW(F)/\mathbb{Z} \left( \langle 1 \rangle + \langle -1 \rangle \right)$.

By considering the dimension maps, we obtain a commutative diagram:

$$
\begin{array}{c}
\hat{I} \\
\downarrow \simeq \\
GW(F) \\
\downarrow \dim \\
\mathbb{Z} \\
\end{array} \quad \begin{array}{c}
\rightarrow \\
\downarrow \dim \quad \downarrow \quad \dim (\mod 2) \\
I \\
\mathbb{Z}/2 \\
\end{array}
$$

Here, the vertical rows are exact, i.e. $I, \hat{I}$ are the respective kernels of the dimension maps. We note that $I \cap \mathbb{Z} \left( \langle 1 \rangle + \langle -1 \rangle \right) = 0$.

Now, what does all this have to do with Milnor $K$-theory? We have a map $K_1^M(F) = F^\times \rightarrow I$ given by $a \mapsto \langle a \rangle - \langle 1 \rangle$. Then $a^2 \mapsto \langle a^2 \rangle - \langle 1 \rangle = 0$, since $\langle a^2 \rangle \simeq \langle 1 \rangle$. Since $\ell(a^2) = 2\ell(a)$, we think of this as a map $s: K_1^M(F)/2 \rightarrow I$.

Note that:

$$s(ab) - s(a) - s(b) = \langle ab \rangle - \langle a \rangle - \langle b \rangle + \langle 1 \rangle = (\langle a \rangle - \langle 1 \rangle)(\langle b \rangle - \langle 1 \rangle) \in I^2$$

This gives a corollary:
**Corollary 2.5.** The map $s: K_1^M(F)/2 \longrightarrow I/I^2$ sending $\ell(a)$ to $(\langle a \rangle - \langle 1 \rangle)$ is a homomorphism.

We may tensor this map with itself to get a map $F^\times \otimes F^\times \rightarrow I/I^2 \otimes I/I^2$, and compose this with the multiplication map $I/I^2 \otimes I/I^2 \rightarrow I^2/I^3$. This sends $a \otimes b$ to $(\langle a \rangle - \langle 1 \rangle)(\langle b \rangle - \langle 1 \rangle)$.

**Proposition 2.6.** If $a + b = 1$, $a \otimes b$ maps to 0 under this map.

**Proof.** We may write $\langle ab \rangle + \langle 1 \rangle$ as:

$$\begin{pmatrix} ab \\ 1 \end{pmatrix} = \begin{pmatrix} a(1 - a) \\ 1 \end{pmatrix} = \begin{pmatrix} a - a^2 \\ 1 \end{pmatrix} \sim \begin{pmatrix} a & a \\ a & 1 \end{pmatrix} \sim \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

Thus, we obtain a map $s_2$ from $K_2^M(F) = F^\times \otimes F^\times/(a \otimes b \mid a + b) = 1$ to $I^2/I^3$. By the universal property of the construction of Milnor $K$-theory, we obtain a unique ring map:

$$s: K_*^M(F)/2 \longrightarrow \bigoplus_{n=0}^\infty I^n/I^{n+1} = \text{gr}_I(W(F))$$

sending $\ell(a)$ to $(\langle a \rangle - \langle 1 \rangle) \in I/I^2$. A natural question which appears at this point is:

**Question 1.** Is $\bigcap_{n=0}^\infty I^n = 0$?

This question was settled affirmatively by Arason and Pfister in 1971 [1]. Another fundamental question attached to this story turned out to be much harder:

**Theorem 2.7** (Milnor Conjecture on quadratic forms). The map $s$ defines an isomorphism from $K_*^M(F)/2$ to $\text{gr}_I(W(F))$.

This was proved by Orlov-Vishik-Voevodsky in [5]. Much earlier, Milnor proved that the maps $s_1: K_1^M(F)/2 \longrightarrow I/I^2$ and $s_2: K_2^M(F)/2 \rightarrow I^2/I^3$ are injective. He constructs a splitting using “Stiefel-Whitney classes”: we may think of quadratic forms over $F$ as some sort of vector bundle over $\text{Spec} F$, and $K_*^M(F)$ as some sort of cohomology theory for $\text{Spec} F$, so we might expect there to be a “characteristic class” construction connecting them.

We have a map $w_1: GW(F) \rightarrow F^\times/(F^\times)^2 = K_1^M(F)/2$ sending $\langle a \rangle$ to the class of $\ell(a)$ in dimension 1. More generally, if $A$ is a matrix representing a quadratic form, the class of $A$ maps
to the class of $\ell(\det(A))$. This gives a well-defined map on $GW(F)$ because the determinant of $B^t AB$ differs from the determinant of $A$ by $\det(B)^2$, so they have the same class in $K_1^M(F)/2$.

We want to extend this to a “total Stiefel-Whitney class”:

$$w = 1 + w_1 + \cdots : GW(F) \to \left( \prod_{n=0}^\infty K_n^M(F)/2 \right)^\times$$

sending $\langle a \rangle$ to $1 + w_1(\langle a \rangle) = 1 + \ell(a)$. If $\langle a \rangle \simeq \langle b \rangle$, then $a/b$ is a square, so the classes of $\ell(a)$ and $\ell(b)$ are the same in the codomain.

In order to show this map is really well-defined, it suffices\(^6\) to show that if $\langle a_1 \rangle + \langle a_2 \rangle \simeq \langle b_1 \rangle + \langle b_2 \rangle$ implies that

$$(1 + \ell(a_1))(1 + \ell(a_2)) = (1 + \ell(b_1))(1 + \ell(b_2)) \quad (1)$$

in $K_2^M(F)$. We already proved $w_1$ is well defined, so it remains to consider the component in $K_2^M(F)$.

In other words, we assume that $a_1 x^2 + a_2 y^2 \sim b_1 x^2 + b_2 y^2$. Since $a_1 = a_1(1)^2 + a_2(0)^2$, it is in the image of this quadratic form. Thus (since equivalent forms have the same image), we may write $a_1 = b_1 x^2 + b_2 y^2$ for some $x, y \in F$. Then $1 = \frac{b_1 x^2}{a_1} + \frac{b_2 y^2}{a_1}$. The case $xy = 0$ is easy, so we assume $xy \in F^\times$ and thus, by the Steinberg relation, we have

$$0 = \ell\left( \frac{b_1 x^2}{a_1} \right) \cdot \ell\left( \frac{b_2 y^2}{a_1} \right)$$

In $K_2^M(F)/2$ this is $(\ell(b_1) - \ell(a_1))(\ell(b_2) - \ell(a_1))$, so $\ell(b_1)\ell(b_2) = \ell(a_1)(\ell(b_1) + \ell(b_2) - \ell(a_1))$. We already showed $w_1$ is well defined, so $\ell(a_1) + \ell(a_2) = \ell(b_1) + \ell(b_2) \in K_1^M(F)/2$, so we conclude $\ell(a_1)\ell(a_2) = \ell(b_1)\ell(b_2)$, finishing the proof of (1).

Finally, one may check that $w_1, w_2$ are left inverse to $s_1, s_2$, which shows Milnor’s theorem that $s_1, s_2$ are injective.

### 3 1/12/18

Last time, we discussed the map from $K_1^M(F)/2$ to $gr_I(W(F)) = \oplus I^n/I^{n+1}$. That this is an isomorphism is the content of Milnor’s conjecture on quadratic forms, proved by Orlov-Vishik-Voevodsky. Today, we will discuss another map from $K_1^M/2$, this time with target $H^*(G_F; \mathbb{Z}/2)$, i.e. the Galois cohomology of the trivial Galois module $\mathbb{Z}/2$. Here, $G_F$ is the absolute Galois group of $F$, i.e. the profinite group given by the inverse limit of $\text{Gal}(L/F)$ as $L$ ranges over finite Galois field extensions.

We recall Hilbert’s Theorem 90. Let $F \hookrightarrow L$ be a cyclic Galois field extension of degree $n$, i.e. a Galois field extension such that $\text{Gal}(L/F) \simeq \langle \sigma \rangle \simeq \mathbb{Z}/n\mathbb{Z}$. There is a multiplicative norm map $N_{L/K} : L \to K$ defined by $a \mapsto \det_K(L \xrightarrow{\sigma} L) = \prod_{i=0}^{n-1} \sigma^i(a)$. We think of $L^\times, K^\times$ as $\text{Gal}(L/K)$-modules and write them additively, so $N_{L/K} = 1 + \sigma + \cdots + \sigma^{n-1}$. We have a complex:

$$L^\times \xrightarrow{\sigma^{-1}} L^\times \xrightarrow{N_{L/K}} K^\times$$

\(^6\)see the reference in [7, Lemma 3.1]
The map on the left sends \( b \in L^\times \) to \( \sigma(b)/b \).
We have:

**Theorem 3.1** (Hilbert’s Theorem 90). The above complex is exact in the middle, i.e. the kernel of \( N_{L/K} \) is equal to the image of \( \sigma - 1 \).

This complex is reminiscent of the resolution used to calculate group cohomology for the cyclic group \( \text{Gal}(L/K) \), i.e. the complex:

\[
\begin{array}{ccccccc}
L^\times & \xrightarrow{\sigma^{-1}} & L^\times & \xrightarrow{1+\sigma+\cdots+\sigma^{n-1}} & L^\times & \xrightarrow{\sigma^{-1}} & L^\times \rightarrow \\
\end{array}
\]

This shows us that Hilbert’s Theorem 90 is equivalent to the following statement, which is true for any finite Galois field extensions:

**Theorem 3.2** (Hilbert’s Theorem 90, version 2). For any finite Galois field extension \( L/F \), \( H^1(\text{Gal}(L/F), L^\times) = 0 \).

We can patch together the Galois cohomology of the finite extensions of \( F \) to get absolute Galois cohomology of \( F \): Pick a separable closure \( F \rightarrow F_s \) and take colimits over all finite Galois sub-extensions \( F \rightarrow L \subseteq F_s \).
We have:

**Definition 3.3.** \( H^n(G_F; F_s^\times) := \lim_{\rightarrow L} H^n(\text{Gal}(L/F); L^\times) \)

We may also define Galois cohomology intrinsically, without passing to finite sub-extensions, in the category of discrete \( G_F \)-modules by using continuous cochains. We get another version of Hilbert’s theorem 90:

**Theorem 3.4** (Hilbert’s Theorem 90, version 3). \( H^1(G_F; F_s^\times) = 0 \)

Assume that \( \text{char}(F) \neq 2 \). We have an action of \( G_F \) on \( F_s^\times \), leading to the short exact Kummer sequence of \( G_F \)-modules:

\[
\begin{array}{ccccccc}
1 & \longrightarrow & \mu_2(F_s^\times) = \{ \pm 1 \} & \longrightarrow & F_s^\times & \xrightarrow{x^2} & F_s^\times \\
& & & & & & 1 \\
\end{array}
\]

There is a similar sequence with \( \mu_n \) and \( n \)-th powers when \( \text{char}(F) \nmid n \).

This gives a long exact sequence:

\[
\cdots \longrightarrow H^0(G_F; F_s^\times) \longrightarrow H^0(G_F; F_s^\times) \xrightarrow{s} H^1(G_F; \mathbb{Z}/2) \longrightarrow H^1(G_F; F_s^\times) = 0
\]

We may rewrite this as:

\[
F^\times \xrightarrow{x^2} F^\times \longrightarrow H^1(G_F; \mathbb{Z}/2) \longrightarrow 0
\]

Thus, we have an isomorphism \( s: K_1^M(F)/2 = F^\times/(F^\times)^2 \xrightarrow{\sim} H^1(G_F; \mathbb{Z}/2) \). Now, there are cup products in Galois cohomology, giving a map \( \cup: H^1(G_F; \mathbb{Z}/2) \otimes H^1(G_F; \mathbb{Z}/2) \rightarrow H^2(G_F; \mathbb{Z}/2) \). This gives us another manifestation of the Steinberg relation:
\textbf{Proposition 3.5} (Bass-Tate). If \( a, b \in F^\times \) satisfy \( a + b = 1 \), then \( \delta(a) \sim \delta(b) = 0 \in H^2(G_F; \mathbb{Z}/2) \).

This gives the immediate corollary.

\textbf{Corollary 3.6.} The Kummer map \( s \) determines a well-defined map of rings \( s : K^M_*(F)/2 \to H^*(G_F; \mathbb{Z}/2) \).

Now, we prove the proposition:

\textit{Proof.} Assume that \( a, b \notin (F^\times)^2 \), \( b = 1 - a \). Pick \( \alpha \in F_s \) with \( \alpha^2 = a \). Let \( E = F[\alpha] \simeq F[X]/(X^2 - \alpha) \) be the field extension generated by \( \alpha \). This gives an embedding map \( \pi : G_E \hookrightarrow G_F \) realizing \( G_E \) as an index-two subgroup of \( G_F \).

This determines a commutative diagram, with \( \pi^* \) the restriction map and \( \pi_* \) the co-restriction or transfer map.

\[
\begin{array}{ccc}
E^\times & \xrightarrow{\delta} & H^1(G_E; \mathbb{Z}/2) \\
| & & | \\
\downarrow{\pi_*} & \downarrow{\pi_*} & \downarrow{\pi_*} \\
F^\times & \xrightarrow{\delta} & H^1(G_F; \mathbb{Z}/2)
\end{array}
\]

We can think of this transfer map topologically. Let \( f : X \to Y \) be a finite covering space. Then there is a map \( f_* : H^*(X; \mathbb{Z}/2) \to H^*(Y; \mathbb{Z}/2) \) (more generally, this works for any constant coefficient module or sheaves when there is a map of sheaves compatible with \( f \), defined at the level of cochains by summing over lifts of chains in \( Y \) to chains in \( X \).

This gives the transfer map in group cohomology because (for \( M \) a constant coefficient group) \( H^*(G; M) = H^*(BG; M) \), and when \( H \subseteq G \) is a finite-index subgroup, we obtain a finite covering map \( BH = EG/H \to BG = EG/G \).

Note that the transfer map is not multiplicative, but it is a homomorphism of modules over the ring \( H^*(G_F; \mathbb{Z}/2) \), which acts on \( H^*(G_E; \mathbb{Z}/2) \) via \( \pi^* \). Topologically, we see this because \( \pi_*(x \sim \pi^*(y)) = \pi_*(x) \sim y \).

Now, we may compute that \( N_{E/F}(\alpha) = \alpha(1 - \alpha) = -a, N_{E/F}(1 - \alpha) = \pi_*(x \sim \pi^*(y)) = \pi_*(x) \sim y \).

Then, the commutative diagram gives us:

\[
\delta_F(a) \sim \delta_F(1 - a) = \delta_F(a) \sim \delta_F(N_{E/F}(1 - \alpha))
= \delta_F(a) \sim \pi_*(\delta_E(1 - \alpha))
= \pi_*(\pi^*(\delta_E(a)) \sim \delta_E(1 - \alpha))
= \pi_*(\delta_E(1 - \alpha))
= \pi_*(\delta_E(\alpha^2) \sim \delta_E(1 - \alpha))
= 0
\]

\footnote{Note that for the purposes of group cohomology, \( EH \) is any contractible space with a free action of \( H \); by restriction the \( G \)-action on \( EG \) to \( H \), we may identify \( EG \) with \( EH \) as topological spaces.}
This construction works more generally, proceeding via the degree $n$ Kummer sequence:

$$1 \rightarrow \mu_n(F_s^\times) \rightarrow F_s^\times \rightarrow F_s^\times \rightarrow 1$$

This gives a map $K_M^M(F)/n \sim H^1(G_F; \mu(n))$. By an analogous result to the one above, this extends to a map of rings:

$$K_M^M(F)/n \rightarrow \bigoplus_i H^i(G_F; \mu_{\otimes Z/n^i})$$

See [9] for details.

This leads to:

**Conjecture 1** (Milnor Conjecture). The map $K_M^M(F)/2) \rightarrow H^*(G_F; \mathbb{Z}/2)$ is an isomorphism.

We have the incredible result:

**Theorem 3.7** (“Norm Residue Theorem”: Voevodsky, Root). For any field $F$ with $\text{char}(F) \nmid n$, the map $K_M^M(F)/n \rightarrow \bigoplus_i H^i(G_F; \mu_{\otimes Z/n^i})$ is an isomorphism.

Prior to its proof, this was known as the Bloch-Kato conjecture (not to be confused with the Bloch-Kato conjecture on special values of $L$-functions, which is wide open).

Already in degree 2, this is a hard result:

**Theorem 3.8** (Merkurjev). $K_M^M(F)/2 \rightarrow H^2(G_F; \mathbb{Z}/2)$ is an isomorphism.

**Theorem 3.9** (Merkurjev-Souslin). $K_M^M(F)/n \rightarrow H^2(G_F; \mu_{\otimes 2})$ is an isomorphism.

One reason to care about the result in degree 2 is that $H^2(G_F; \mathbb{Z}/2)$ is isomorphic to the 2-torsion part of the Brauer group of $F$, denoted $\text{Br}(F)$.

This group is constructed from the set of isomorphism classes of central simple $F$-algebras $A$, i.e. associative $F$-algebras with center $F$ such that $A \otimes_F E \simeq M_n(E)$ (the $n$-dimensional matrix algebra over $E$) for some $n$ and some finite extension $E/F$. We mod out this set by the equivalence relation $A \sim M_n(A)$. This has a group structure with multiplication given by tensor product over $F$ and unit element given by the trivial central simple $F$-algebra $F$. The inverse of $A$ is $A^{\text{op}}$, the opposite algebra of $A$ (i.e. it has the same underlying $F$-vector space as $A$, but with multiplication given by $a \cdot A^{\text{op}} b = b \cdot A a$). Thus, the two-torsion elements are exactly the central simple algebras $A$ with $A \simeq A^{\text{op}}$.

By composing the Kummer map $K_M^M(F)/2 \rightarrow H^2(G_F; \mathbb{Z}/2)$ and the isomorphism of the latter group with $\text{Br}(F)$, we may verify that $\ell(a)\ell(b)$ maps to the class of the central simple algebra $F\langle x, y \rangle/(x^2 - a, y^2 - b, xy + yx)$. The surjectivity part of Merkurjev’s theorem implies that these so-called “quaternion algebras” generate the 2-torsion of the Brauer group.

### 3.1 Cheat sheet by SG on group cohomology

**3.1.1 As a special case of $\text{Ext}$**

If $G$ is a (discrete) group, and $M$ is a module over the group ring $\mathbb{Z}[G]$, then group cohomology may be defined as $H^i(G; M) = \text{Ext}_{\mathbb{Z}[G]}^i(\mathbb{Z}, M)$. 

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3.1.2 Via singular homology

For any group $G$ there exists a connected CW complex $BG$ with basepoint $\ast \in BG$, an isomorphism $\phi : \pi_1(BG, \ast) \cong G$, such that the universal cover of $BG$ is contractible. The triple $(BG, \ast, \phi)$ is unique up to homotopy equivalence in a suitable sense (and the homotopy equivalence is unique up to homotopy). If we pick such a triple for $G$, we get one for any subgroup. Indeed, if $EG \to BG$ denotes the universal cover then $G$ acts on $EG$ by deck transformations and if $H < G$ is a subgroup, then the quotient map $EG \to EG/H$ is a covering space. Hence we may take $BH = EG/H$. In this model, $BH \to BG$ is a covering space; it is a finite covering space if $|G : H| < \infty$.

Then we may define $H^i(G; \mathbb{Z}) = H^i(BG)$, so group cohomology is a special case of singular cohomology. If $\hat{M}$ has non-trivial action, group cohomology becomes a special case of “cohomology with local coefficients” (see e.g. Hatcher’s book).

3.1.3 Via an explicit cochain complex

Finally, one may define $C^n(G; M)$ as the set of all functions $f : G \times \cdots \times G \to M$, where the product has $n$ factors. There is a coboundary map $\delta : C^{n-1}(G; M) \to C^n(G; M)$ given by

$$\delta f(g_1, \ldots, g_n) = g_1.f(g_2, \ldots, g_n) + \sum_{i=1}^{n-1} (-1)^i f(g_1, \ldots, g_i g_{i+1}, \ldots, g_n) + (-1)^n f(g_1, \ldots, g_{n-1}),$$

where the first term involves the action of $g_1$ on $f(g_2, \ldots, g_n) \in M$.

$n = 1$ is particularly interesting: $f : G \to M$ is a cocycle if and only if it satisfies $f(g_1 g_2) = g_1.f(g_2) + f(g_1)$. If the action of $G$ on $M$ is trivial, this says precisely that $f$ is a homomorphism and in this case $H^1(G; M)$ is just $\text{Hom}(G, M)$.

3.1.4 Transfer

For any group homomorphism $\phi : H \to G$ there is an induced $f^* : H^*(G; M) \to H^*(H; M)$, where $\hat{H}$ acts on $M$ via $\phi$. If $H \subset G$ has finite index, there is a “transfer” (also called “corestriction”) map $f_* : H^*(G; M) \to H^*(H; M)$ in the other direction. The most important properties to know about this construction is that it is a homomorphism of $H^*(G; M)$-modules, and that the composition $f_* \circ f^*$ is multiplication by the index of $H$ in $G$.

I find the topological description (lifting simplices in a model for $BH \to BG$ which is a finite covering space) the most intuitive, but the transfer can of course be described in any of the three equivalent models of group cohomology given above. For example, in the explicit cochain complex, one chooses a set-theoretic section $s : G/H \to G$ of the quotient map (i.e. picks a representative of each coset), defines $\phi_* : C^*(H; M) \to C^*(G; M)$ by

$$\phi_*(f)(g_1, \ldots, g_n) = \sum_{x \in G/H} (sx)^{-1}.f((sx)g_1(sxg_1)^{-1}, \ldots, (sx)g_n(sxg_n)^{-1}),$$

and checks that this is a map of cochain complexes and has the desired properties.
3.1.5 Profinite topological groups

For certain kinds of topological groups $G$, there is a way to take the topology into account when defining group cohomology. If you’re familiar with cohomology of discrete groups and pretend that we just take cohomology of the underlying discrete group, your intuition might not be too far off. (If on the other hand you are familiar with taking “classifying spaces” of topological groups, then you should know that this is not the correct thing to do here.)

For much more on this, see e.g. Serre’s *Galois cohomology*. For even more, see Neukirch–Schmidt–Wingberg: *Cohomology of number fields*.

If $G$ is a topological group we may consider open normal subgroups $H \subset G$ of finite index. For such $H$ the quotient group $G/H$ is a finite group and the quotient topology is the discrete one. There is a canonical continuous homomorphism $G \to G/H$, and hence a continuous map

$$G \to \lim_{\leftarrow H} G/H,$$

where the inverse limit runs over open normal subgroups of finite index. This inverse limit inherits a topology from the product of the discrete $G/H$, and the resulting topological group is called the *profinite completion* of $G$. The topological group $G$ is called *profinite* if the canonical continuous homomorphism from $G$ is a homeomorphism.

If $M$ is an abelian group with an action $G \times M \to M$ which is continuous in the discrete topology of $M$, then we can define the continuous cochain complex $C^\ast_{\text{cont}}(G; M)$ using only continuous functions. One can show that the cohomology of this cochain complex is canonically isomorphic to the direct limit

$$\lim_{\to H} H^\ast(G/H; M^H),$$

where $H$ runs through open normal subgroups of $G$ and $M^H \subset M$ is the subgroup fixed by $H$. Either can be taken as the definition of (continuous) cohomology of the profinite group $G$ with coefficients in $M$. The colimit description is useful if you are already familiar with cohomology of discrete groups and want to transfer some of your knowledge into continuous cohomology of profinite groups.

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The goal for the next few lectures will be to develop *motivic cohomology*. This was conjectured to exist in the 1980’s by Lichtenbaum ([4]) and Beilinson ([2]). Their idea is to associate cohomology groups $H^{p,q}(X)$ to a scheme $X$ satisfying certain properties, such as the Euler characteristic being related to special values of $\zeta$-functions. Voevodsky’s motivation came more from the Atiyah-Hirzebruch spectral sequence, which relates topological $K$-theory to singular cohomology: motivic cohomology is supposed to have a similar relationship with Quillen’s algebraic $K$-theory.

We will briefly discuss (complex) *topological $K$-theory*. Let $X$ be a finite CW complex; to $X$ we may associate the group $K^0(X)$, defined as the Grothendieck group of the category of (complex) vector bundles on $X$. It turns out that this functor is representable: $K^0(X) \simeq [X, \mathbb{Z} \times BU] = \pi_0(\text{Maps}(X, \mathbb{Z} \times BU))$. Here $BU = \lim_{\to U(n) \times U(n)}$. Here $BU = \lim_{\to U(n) \times U(n)}$.

These groups are extended to negative degrees by defining $K^{-n}(X) = K^0(S^n X)$, and the resulting groups are also representable. If we write $ku_0 = \mathbb{Z} \times BU$, we have $K^{-n}(X) = [X, \Omega^n ku_0]$,
where $\Omega^n$ is the $n$-th fold loop space. The space $\mathbb{Z} \times BU$ admits deloopings, i.e. $ku_0 \simeq \Omega ku_1 \simeq \Omega^2 ku_2$, and so on. The $ku_i$ form a spectrum. Thus, we have homotopy equivalences:

$$\text{Maps}(X, ku_0) \simeq \Omega \text{Maps}(X, ku_1) \simeq \Omega^2 \text{Maps}(X, ku_2)$$

We have $K^{-n}(X) = \pi_n \text{Maps}(X, ku_0) = \pi_{n+k} \text{Maps}(X, ku_k)$ for any $k$. Thus, we may define $K^*(X)$ in positive degree as well by defining $K^n(X) = \pi_{k-n} \text{Maps}(X, ku_k)$ for sufficiently large $k$.

The Atiyah-Hirzebruch spectral sequence comes from the Postnikov tower of $ku_0 = \mathbb{Z} \times BU$. This gives a diagram:

Here, $\tau \leq n(ku_0)$ has $\pi_k = 0$ for all $k > n$, and the vertical maps are homotopy fibrations. The homotopy fiber is an Eilenberg-Mac Lane space $K(\pi_n(ku_0), n)$. These Eilenberg-Mac Lane spaces have the property that they represent cohomology: $[X, K(A, n)] \simeq H^n(X; A)$ for any abelian group $A$.

The Postnikov tower gives a tower of maps from $X$:

The homotopy fibers are $\text{Maps}(X, K(\pi_n(ku_0), n))$. This tower gives a spectral sequence for homotopy groups (with bi-degrees labeled as in the cohomological Serre spectral sequence):

$$H^p(X, \pi_{-q}(ku_0)) \implies \pi_{-p-q} \text{Maps}(X, ku_0) = K^{p+q}(X)$$

This is a “fourth quadrant” cohomological spectral sequence: i.e. it’s supported when $p > 0, q < 0$ with differentials increasing $p$ and decreasing $q$.

Now, we know the homotopy groups of $ku_0 = \mathbb{Z} \times BU$, as a result of the following theorem:
**Theorem 4.1** (Bott Periodicity).

\[ \pi_*(\mathbb{Z} \times BU) = \begin{cases} \mathbb{Z} & * \geq 0 \text{ even} \\ 0 & \text{else} \end{cases} \]

Thus, the Atiyah-Hirzebruch spectral sequence takes the form

\[ E_2^{p,q} = \begin{cases} H^p(X; \mathbb{Z}) & \text{for } q \leq 0 \text{ even} \\ 0 & \text{otherwise} \end{cases} \implies K^{p+q}(X). \]

The first possible differentials are \( d_3 : E_3^{p,-2j} \to E_2^{p+3,-2j-2} \). It turns out that they can be identified with the composition

\[ H^p(X; \mathbb{Z}) \to H^p(X; \mathbb{Z}/2) \xrightarrow{Sq^2} H^{p+2}(X; \mathbb{Z}/2) \xrightarrow{\beta} H^{p+3}(X; \mathbb{Z}), \]

where \( Sq^2 \) denotes the “Steenrod square” cohomology operation (cf. e.g. Section 4.L in Hatcher’s textbook, or similar accounts in other textbooks), and \( \beta \) denotes the “Bockstein homomorphism” (i.e. the connecting homomorphism in the long exact sequence associated to \( 0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2 \to 0 \)). In particular the \( d_3 \) differential satisfies \( 2d_3 = 0 \), and the higher differentials can also be shown to be torsion. As an aside, the whole spectral sequence may be equipped with “Adams operations” converging to the usual Adams operations \( \psi^k : K^*(X) \to K^*(X) \), in which \( \psi^k \) acts as \( k^j \) on the row \( E_2^{*, -2j} \).

In Quillen K-theory, we associate to a scheme \( X \) a topological space \( K(X) \) with \( K^Q_0(X) := \pi_0(K(X)) \). As before, this space admits arbitrarily many de-loopings. There’s supposed to be a so-called “motivic spectral sequence” whose \( E_2 \)-page is given by a cohomology groups which we can denote (in a grading convention which makes it look analogous to the AHSS, but may not be the usual grading of the motivic spectral sequence)

\[ E_2^{p,q} = \begin{cases} H^p(X; \mathbb{Z}(-q/2)) & \text{for } q \leq 0 \text{ even} \\ 0 & \text{otherwise} \end{cases} \implies \pi_{-p-q}(K(X)), \]

where \( H^p(X; \mathbb{Z}(-q/2)) \) is for now just notation for a functor indexed by two integers \((p, -q/2)\) that we haven’t defined yet. (In the actual definition \( \mathbb{Z}(-q/2) \) will be a cochain complex of sheaves of abelian groups.)

Beilinson and Lichtenbaum were able to predict some properties which these groups should have, at least when \( X \) is smooth over a field:

- \( H^*(X; \mathbb{Z}(0)) \) should be (Zariski) sheaf cohomology with coefficients in \( \mathbb{Z} \). (This is 0 in higher degrees when \( X \) is smooth.)

- \( H^p(X; \mathbb{Z}(1)) \) should be \( H^{p-1}(X; \mathcal{O}_X^\times) \), again in Zariski sheaf cohomology.

- \( H^{2q}(X; \mathbb{Z}(q)) \) should be the Chow groups \( \text{CH}^q(X) \) which measure algebraic cycles of dimension \( q \) up to rational equivalence.\(^8\)

\(^8\)Note that this means that the Chow groups of \( X \) should, after accounting for some differentials, give a filtration of \( K_0(X) \). Indeed, the construction of Chern characters gives an isomorphism from \( K_0(X) \otimes \mathbb{Q} \) to \( \bigoplus_q \text{CH}^q(X) \otimes \mathbb{Q} \).
In the case $X = \text{Spec } F$, the above properties tell us that we should have $H^1(\text{Spec } F; \mathbb{Z}(1)) = H^0(\text{Spec } F; F^\times) = F^\times = K^M_1(F)$. We should also have $H^0(\text{Spec } F; \mathbb{Z}(0)) = \mathbb{Z} = K^M_0(F)$. More generally, we should have $H^p(\text{Spec } F; \mathbb{Z}(p)) = K^M_p(F)$. Thus, this spectral sequence should give a relationship between Milnor and Quillen $K$-theory for $F$. Because Quillen $K$-theory satisfies the Steinberg relation and agrees with Milnor $K$-theory in degrees up to 2, there is a canonical map of graded rings $K^M_*(F) \to K^Q_*(F)$, and we hope for this to be an edge map in the motivic spectral sequence.

Beilinson also proposed how to find these motivic cohomology groups. $H^*(X; \mathbb{Z}(0))$ and $H^*(X; \mathbb{Z}(1))$ are both sheaf cohomology for the Zariski topology. The former is just sheaf cohomology of the constant sheaf $\mathbb{Z}$. The latter is, up to a degree shift by 1, sheaf cohomology of the sheaf $G_m = \mathcal{O}_X^\times$. Thus, $H^*(X; \mathbb{Z}(1))$ is cohomology with coefficients in the object $G_m[-1]$. Beilinson's proposal, which Voevodsky accomplished, is to realize the $\mathbb{Z}(q)$ as chain complexes of sheaves.

How do we associate cohomology groups to complexes of sheaves? If we are given a cochain complex $F^\bullet$ of sheaves of abelian groups, we may pick injective resolutions $F^p \to I^p\to$ in such a way that we get the following diagram:

\[
\begin{array}{ccccccc}
F^0 & \to & F^1 & \to & \cdots \\
\uparrow & & \uparrow & & \\
F^{0,0} & \to & F^{0,1} & \to & \cdots \\
\uparrow & & \uparrow & & \\
F^{1,0} & \to & F^{1,1} & \to & \cdots \\
\downarrow & & \downarrow & & \\
\vdots & & \vdots & &
\end{array}
\]

Taking global sections gives us a bi-complex, and we can define hypercohomology $H^*(X; F^\bullet) = H^*(\text{Tot}(I^{\bullet,\bullet}))$, i.e. the cohomology groups of the total complex associated to this double complex. This is the complex with groups $\text{Tot}(I^{\bullet,\bullet})^n = \bigoplus_{p+q=n} I^{p,q}$ with differential given (up to sign) on each $I^{p,q}$ as the sum of the horizontal and vertical differentials.\(^9\)

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In order to construct motivic cohomology, we introduce a category $\text{SH}(F)$, the Morel-Voevodsky stable homotopy category of $F$, which is supposed to be a “stable homotopy theory of smooth varieties”. We expect cohomological functors to factor through a canonical functor from $\text{Sm}_F$, the category of smooth schemes over $F$, to $\text{SH}(F)$.

In particular, we expect to have the following diagrams of functors:

\[
\begin{array}{ccc}
\text{Sm}_F & \to & \text{SH}(F) \\
\downarrow & & \downarrow \\
H^p(-, \mathbb{Z}(q)) \ ; K^Q_p(-) & \to & \text{Ab}
\end{array}
\]

\(^9\)note: there is a natural quasi-isomorphism from $F^\bullet$ to $\text{Tot}(I^{\bullet,\bullet})$, so the hypercohoomology of $F^\bullet$ is also $H^i(RF)$ with $RF$ the derived functor of $F$, defined on the derived category of sheaves. See the Stacks Project, Tag 0133
We also want the vertical functors to be representable, as is the case with topological $K$-theory and singular homology. One construction of the motivic spectral sequence uses Voevodsky’s “slice filtration” (analogous to the Postnikov filtration in usual stable homotopy theory). Just as singular cohomology of a space may be defined without first defining the stable homotopy category and Eilenberg–MacLane spectra, so may motivic cohomology be defined in a direct and “elementary” way (using only chain complexes, no fancier stable homotopy theory). Today, we will discuss the “direct” definition of $H^p(X; \mathbb{Z}(q))$ for $X$ smooth over a field $F$. Some references are the summary in Section 2.1 of Voevodsky’s ’96 preprint “The Milnor Conjecture”, [8], and the textbook [6].

First, we discuss cycles on a smooth $F$-scheme $X$: compare the theory developed in [3].

**Definition 5.1.** A cycle on a smooth $F$-scheme $X$ is a finite linear combination $\sum n_V[V]$ with $n_V \in \mathbb{Z}$ and $V \subseteq X$ a closed subvariety (i.e. a closed irreducible topological subspace or equivalently a closed irreducible reduced subscheme, but not necessarily smooth).

If $f : X \to Y$ is a proper map, then the topological image $W = f(V) \subseteq Y$ is a closed and irreducible subvariety of $Y$. We define:

**Definition 5.2.** For $f : X \to Y$ a proper map between smooth $F$-schemes, the proper pushforward $f_*$ is defined by:

$$f_*([V]) = \begin{cases} 0 & \text{dim } W < \text{dim } V \\ (K(V) : K(W))[W] & \text{dim } W = \text{dim } V \end{cases}$$

Here $(K(V) : K(W))$ is the degree of the function field extension $K(W) \hookrightarrow K(V)$, which is finite when $\text{dim } W = \text{dim } V$.

Cycles allow the development of intersection theory:

**Definition 5.3.** If $V, W \subseteq X$ are subvarieties of the smooth $F$-scheme $X$, then we say the intersection $V \cap W$ is proper if each irreducible component $T \subseteq V \cap W$ satisfies

$$\text{codim}(T) = \text{codim}(V) + \text{codim}(W)$$

This is something like transversality in topology, but it is much weaker. For example, the line $Y = 0$ and the parabola $Y = X^2$ in the affine plane $\mathbb{A}^2_F = \text{Spec } F[X, Y]$ have proper intersection but are not transverse since they are tangent at their intersection point $(0, 0)$.

If $V, W$ have proper intersection, we want to define

$$[V] \cdot [W] = \sum_{T \subseteq V \cap W} n_T[T]$$

for “multiplicities” $n_T$. To define this, let $\mathcal{O}_{X,T}$ be the local ring of $X$ along $T$, i.e. the stalk of $\mathcal{O}_X$ at the generic point $\text{Spec } K(T) \hookrightarrow T$. (This ring consists exactly of those regular functions defined on open subsets $U \subseteq X$ such that $U \cap T \neq \emptyset$.) Let $\mathcal{I}, \mathcal{J}$ be the ideals of $\mathcal{O}_{X,T}$ corresponding to the closed subvarieties $V, W$ respectively. Now, we define:

**Definition 5.4** (Serre’s formula). If $V, W$ are two closed subvarieties of a smooth $F$-scheme $X$ with proper intersection, then for each irreducible component $T \subseteq V \cap W$, we have:

$$n_T = \sum_{i=0}^{\infty} (-1)^i \text{length}_{\mathcal{O}_{X,T}} \text{Tor}^i_{\mathcal{O}_{X,T}}(\mathcal{O}_{X,T}/\mathcal{I}, \mathcal{O}_{X,T}/\mathcal{J})$$

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Note that this formula specializes to \( n_T = \text{length}_{\mathcal{O}_{X,T}} \mathcal{O}_{X,T}/(J + F) \) if we are in a setting where the higher Tor groups vanish. Note that \( J + F \) is the ideal of \( \mathcal{O}_{X,T} \) corresponding to the scheme-theoretic intersection of \( V \) and \( W \) at \( T \).

**Example 5.5.** Let \( X = \mathbb{A}^2_F = \text{Spec } F[X, Y] \), \( V \) the line \( Y = 0 \) so \( V = \text{Spec } F[X, Y]/(Y) \), and \( W \) the parabola \( Y = X^2 \), so \( W = \text{Spec } F[X, Y]/(Y - X^2) \). Then the topological intersection of \( V \) and \( W \) consists of the point \((0, 0)\), so we have \( T = F[X, Y]/(X, Y) \) and \( \mathcal{O}_{X,T} = F[X, Y]_{(x,y)} \). Then \( \mathcal{O}_{X,T}/(J + F) = F[X, Y]_{(x,y)}/(Y, X^2) \simeq F[X]/(X^2) \). This has length 2 over \( \mathcal{O}_{X,T} \), and \( n_T = 2 \). Thus, we have:

\[
[V] \cdot [W] = 2[(0, 0)]
\]

Next, we will define the category of correspondences over \( F \). The objects of this category will be smooth \( F \)-schemes, but there will be additional morphisms. There is an analogue to this construction in topology:

Let \( X \) be a space. Then we can define the \( n \)-th symmetric power of \( X \) \( \text{SP}^n(X) = X^n/S_n \) (where \( S_n \) is the symmetric group acting on \( X^n \) by permuting the factors). Then we have a diagram:

\[
\begin{array}{ccc}
X & \longrightarrow & \bigsqcup_{n=0}^\infty \text{SP}^n(X) \\
& & \downarrow \\
& & \text{SP}(X)
\end{array}
\]

Here, \( \bigsqcup_{n=0}^\infty \text{SP}^n(X) \) is the free topological abelian monoid generated by \( X \), and \( \text{SP}(X) \) is the free topological abelian group generated by \( X \). Points of \( \text{SP}(X) \) consist of \( \mathbb{Z} \)-linear combination of points of \( X \).

A map \( f: X \to \bigsqcup_{n=0}^\infty \text{SP}^n(Y) \) is like a “multi-valued function” from \( X \) to \( Y \), i.e. each point of \( X \) maps to some finite set of points of \( Y \), possibly with multiplicity. A map \( f: X \to \text{SP}(Y) \) sends each point of \( X \) to some \( \mathbb{Z} \)-linear combination of points in \( Y \).

Given maps \( f: X \to \text{SP}(Y) \) and \( g: Y \to \text{SP}(Z) \), we obtain a map \( X \to \text{SP}(Z) \):

\[
X \xrightarrow{f} \text{SP}(Y) \xrightarrow{\text{SP}(g)} \text{SP}(\text{SP}(Z)) \longrightarrow \text{SP}(Z)
\]

Here, the final map \( \text{SP}(\text{SP}(Z)) \) is defined by “expanding” linear combinations of linear combinations of points of \( Z \). Thus, we have a category whose objects are topological spaces and whose morphisms from \( X \) to \( Y \) consist of maps from \( X \) to \( \text{SP}(Y) \). We want to do something similar in algebraic geometry.

**Definition 5.6.** An *elementary correspondence* \( f: X \rightsquigarrow Y \) is a subvariety \( V \subseteq X \times Y \) such that the map \( V \to X \) defined by restricting the first projection \( p_1: X \times Y \longrightarrow X \) to \( V \) is a finite morphism which is surjective onto a component of \( X \).

The intuition is that such \( V \) are the graphs of “multi-valued functions” from \( X \) to \( Y \).

This lets us define:

**Definition 5.7.** \( \text{Corr}_F(X, Y) \) is the set of \( \mathbb{Z} \)-linear combinations of elementary correspondences \( X \rightsquigarrow Y \). This is a subset of the set of cycles on \( X \times Y \).
We give some examples of correspondences:

**Example 5.8.** Given a morphism \( f : X \to Y \), the graph \( \Gamma_f \hookrightarrow X \times Y \) is a correspondence from \( X \) to \( Y \), which is elementary when \( X \) is connected. (Otherwise it is the formal sum of one elementary correspondence for each component of \( X \).)

**Example 5.9.** Consider the multi-valued function \( C \to C \) defined by \( z \mapsto \pm \sqrt{z} \). We can realize this as a correspondence from \( X = A^1_C \) to \( X = A^1_C \). Define \( V \subseteq X \times Y = A^2_C \) by \( V = \text{Spec} \ C[X,Y]/(X - Y^2) \). The projection to \( X \) is a surjective finite morphism of degree 2, so this is an elementary correspondence.

We may define a bilinear composition law on the groups of correspondences, making the category of smooth \( F \)-schemes and correspondences into an additive category:

**Definition 5.10.** For \( X,Y,Z \) smooth \( F \)-schemes, the composition law on \( \text{Corr}_F \) is the bilinear map \( \text{Corr}_F(X,Y) \times \text{Corr}_F(Y,Z) \to \text{Corr}_F(X,Z) \) defined by sending \((V,W)\) with \( V \subseteq X \times Y, W \subseteq Y \times Z \) elementary correspondences to \( p^*(V \times Z \cdot [X \times W]) \), with \( p: X \times Y \times Z \to X \times Z \) the projection map and \( \cdot \) the intersection formula defined above.

**Lemma 5.11.** The above definitions give a well-defined associative composition law, making \( \text{Corr}_F \) into a category. Moreover, there is a faithful embedding \( \text{Sm}_F \hookrightarrow \text{Corr}_F \) which is the identity on objects and sends \( f: X \to Y \) to \( \Gamma_f \subseteq X \times Y \).

6 1/22/2018

We continue to develop motivic cohomology. Last time, we considered the category \( \text{Sm}_F \) of smooth varieties over the field \( F \), and we defined a functor from this to a category called \( \text{Corr}_F \), the “correspondence category”. This is an additive category, with \( X \sqcup Y \) the categorical co-product and product in this category. However, this category is not abelian, i.e. there are not usually kernels and cokernels.

**Remark 6.1.** Suppose a category has the properties that finite products and coproducts exist, that the canonical map from the initial object to the terminal object is an isomorphism, and that the canonical map \( X \sqcup Y \to X \times Y \) from the coproduct to the product is an isomorphism for all objects \( X \) and \( Y \). Then all morphism sets canonically inherit the structure of commutative monoids: for \( f,g: X \to Y \), we define \( f + g: X \to Y \) by \( \text{codiag} \circ (f \times g) \), with \( \text{codiag} : Y \times Y \to Y \sqcup Y \) given by \( \text{id} \sqcup \text{id} \). The category is additive if these abelian monoids are groups, in which case composition is automatically bilinear. Notice that being additive is entirely a property of a category (not extra data!).

There is a fully faithful embedding from \( \text{Corr}_F \) to the category PST of “presheaves with transfer” on \( \text{Sm}_F \). This is the category of additive functors from \( \text{Corr}_F^{\text{op}} \) to \( \text{Ab} \). These functors are specified by associating an abelian group \( S(X) \) to each smooth \( F \)-scheme \( X \), together with homomorphisms \( S(X) \otimes \mathbb{Z} \text{Corr}_F(X,Y) \to S(Y) \) which are compatible with the composition on \( \text{Corr}_F \).

We denote this embedding by \( X \mapsto \mathbb{Z}_{\text{id}} (X) = \text{Corr}_F(-,X) \). The reason for the terminology is that we obtain “transfer” maps \( f_* \) for finite surjective maps \( f: X \to Y \). This is because for such an
We can’t form anything like an actual smash product in the category of schemes, but in homotopy invariant?

And here, \( X \) is an isomorphism for all \( G \).

Consider \( A^1 \setminus \{0\} = \text{Spec} \, F[t, t^{-1}] \), also called \( G_m \). Consider the maps \( pt = \text{Spec} \, F \to G_m \to pt \) where the first map is the point 1 and the second is the terminal map. We obtain a diagram:

\[
\begin{array}{c}
Z_{tr}(pt) \to Z_{tr}(A^1 \setminus \{0\}) \to Z_{tr}(pt)
\end{array}
\]

Since the composition of the two maps is the identity, we get a canonical splitting:

\[
Z_{tr}(A^1 \setminus \{0\}) \simeq Z_{tr}(pt) \oplus \text{Cok} \left( Z_{tr}(pt) \to Z_{tr}(A^1 \setminus \{0\}) \right) =: Z_{tr}(pt) \oplus Z_{tr}(G_m, 1)
\]

We may play a similar game and get a diagram of split exact sequences:

\[
\begin{array}{c}
Z_{tr}(pt \times pt) \to Z_{tr}(pt \times G_m) \to Z_{tr}(G_m, 1) \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
Z_{tr}(G_m \times pt) \to Z_{tr}(G_m \times G_m) \to \text{Cok} \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
Z_{tr}(G_m, 1) \to \text{Cok} \to Z_{tr}(G_m, 1)^\wedge 2
\end{array}
\]

This gives us:

\[
Z_{tr}(G_m \times G_m) \simeq Z_{tr}(pt) \oplus 2Z_{tr}(G_m, 1) \oplus Z_{tr}((G_m, 1)^2)
\]

More generally, we have:

\[
Z_{tr}(G_m^n) \simeq \bigoplus_{i=0}^{n} \binom{n}{i} Z_{tr}((G_m, 1)^i)
\]

And here, \( Z_{tr}((G_m, 1)^n) \) is the cokernel of the map

\[
\sum Z_{tr}(G_m \times G_m \times \cdots \times pt \times G_m \times \cdots \times G_m) \to Z_{tr}(G_m \times \cdots \times G_m)
\]

This is reminiscent of the behavior of the smash product of pointed topological spaces: if we replace \( G_m \) by the topological space \( S^1 \) and the pair \( (G_m, 1) \) of schemes with the pointed topological space \( (S^1, 1) \), then the \( n \)-fold “smash product” is the \( n \)-sphere \((S^1, 1)^\wedge n \simeq S^n\). On the level of singular chains, we have a splitting \( C_\ast(S^1) \cong \mathbb{Z} \oplus C_\ast(S^1, 1) \) and a similar chain level “binomial formula” (up to quasi-isomorphism at least) for \( C_\ast((S^1)^\wedge n) \simeq (C_\ast(S^1))^\wedge n \cong (\mathbb{Z} \oplus C_\ast(S^1, 1))^\wedge n \).

We can’t form anything like an actual smash product in the category of schemes, but in PST we may nevertheless form something which behaves like the chains on the smash product.

**Definition 6.2.** We say that \( F \in \text{PST} \) is a homotopy invariant object if \( \pi^* : F(X) \to F(X \times A^1) \) is an isomorphism for all \( X \).

We want \( X \times A^1 \to X \) to be a “homotopy equivalence”. How do we make a functor \( F : \text{Sm}_F^{\text{op}} \to \text{Ab} \) homotopy invariant?

We define:
Definition 6.3. The standard $n$-simplex is $\Delta^n = \text{Spec} \mathbb{Z}[t_0, \ldots, t_n]/(\sum_i t_i = 1) \simeq \mathbb{A}^n_\mathbb{Z}$. We also define $\Delta^n_X = X \times \mathbb{Z} \Delta^n$.

We have face maps $\delta^0, \ldots, \delta^n : \Delta^{n-1} \to \Delta^n$ and degeneracy maps in the other direction, as usual for simplicial sets. The $\Delta^i$ fit together into a co-simplicial scheme, i.e. a functor from the simplex category $\Delta$ into the category of schemes.

Given $F : \text{Sm}^{\text{op}}_F \to \text{Ab}$, we define:

Definition 6.4. $(C^*_n F)(X) = F(X \times \mathbb{Z} \Delta^n)$. This is a simplicial abelian group with maps $d_i = \delta^*_i : F(\Delta^n_X) \to F(\Delta^{n-1}_X)$. We define a differential $\delta = \sum_i (-1)^i d_i : C^*_n F \to C^*_{n-1} F$.

Remark 6.5. Sometimes, we may consider instead the normalized chain complex given by $N^*_n F = \bigcap_{i=1}^n \ker(d_i)$, with differential induced by $d_0$. For general reasons of simplicial abelian groups, this is quasi-isomorphic to the one above.

Now, this gives us a functor $C_* F : \text{Sm}^{\text{op}}_F \to \text{Ch}$, where $\text{Ch}$ is the category of $\mathbb{N}$-graded chain complexes.

We have:

Lemma 6.6. If $i_0, i_1 : X \to X \times \mathbb{A}^1$ are the maps induced by the inclusion of the points $0, 1 \in \mathbb{A}^1(F)$, the corresponding maps $(i_0)^*, (i_1)^* : (C_* F)(X \times \mathbb{A}^1) \to C_* F(X)$ are chain homotopic. In particular, they induce the same maps on homology.

These give 1-sided inverses to the map $\pi^* : (C_* F)(X) \to (C_* F)(X \times \mathbb{A}^1)$, and a two-sided inverse up to chain homotopy.

We have:

Corollary 6.7. $\pi^*$ is a quasi-isomorphism, i.e. the functor $H_n((C_* F)(-))$ is a homotopy-invariant object.

Note that we always have a surjective map from $F(X) = F(X \times \mathbb{A}^0) = C_0 F$ (where $\mathbb{A}^0 = \text{Spec} F$ is just a point) to $H_0(C_* F)$. Similarly, given $\mathcal{F}_* : \text{Sm}^{\text{op}}_F \to \text{Ch}$, we obtain a map from this to the complex $\text{Tot}(C_* \mathcal{F}_*)$.

Also, if $F : \text{Corr}^{\text{op}} \to \text{Ab}$ is defined on the correspondence category, i.e. $F \in \text{PST}$, then we may similarly consider $C_* F : \text{Corr}^{\text{op}} \to \text{Ch}$.

We define:

Definition 6.8. $Z(n) = C_* Z_{\text{tr}}((G_m, 1)^n)[-n] \in \text{Ch}(\text{PST})$.

This is a cochain complex of PST’s concentrated in cohomological degrees $(-\infty, n]$, i.e. it looks like:

$$0 \longleftarrow Z(n)^n \longleftarrow Z(n)^{n-1} \longleftarrow \cdots \longleftarrow Z(n)^0 \longleftarrow Z(n)^{-1} \longleftarrow \cdots$$

Now, we may define the motivic cohomology groups:
**Definition 6.9.** For $X \in \text{Sm}_F$, the *motivic cohomology groups* $H^{p,q}(X) = H^{p,q}(X; \mathbb{Z})$ are defined to be the Zariski sheaf (hyper-) cohomology $H^p(X; \mathbb{Z}(q))$. More generally, for an abelian group $A$ we define $A(q) \in \text{Ch}(\text{PST})$ as $A \otimes \mathbb{Z}(q)$. Then we have $H^{p,q}(X; A) := H^p(X; A(q))$. (We shall see later that $\mathbb{Z}(q)$ and more generally $A(q)$ are in fact cochain complexes of sheaves, not just presheaves. This boils down to $Z_{tr}(X)$ being a Zariski sheaf for any $X$.)

The easiest case to understand is $q = 0$. We have $Z_{tr}((\mathbb{G}_m, 1)^0) = Z_{tr}(\text{pt})$. As a functor, this sends $X$ to the constant sheaf $\mathbb{Z}$ on $X$. Since this already respects the “homotopy equivalence” $X \times \mathbb{A}^1 \to X$, taking chains does nothing (because $\mathbb{A}^1$ and more generally $\Delta^0_\mathbb{A}$ is connected, so locally constant maps out of $X \times \Delta^0_\mathbb{A}$ are the same as locally constant maps out of $X$), i.e. the homology of $C_\ast Z_{tr}(\text{pt})$ is concentrated in degree 0. (The normalized chain complex is concentrated in degree 0 even on the chain level.) Then essentially the same computation as that of the singular cohomology of a point shows us that $H^{p,0}(X; A) = H^p_{\text{Zar}}(X, A)$. Up next, we will see that there is a quasi-isomorphism $Z_{tr}(\mathbb{G}_m) \xrightarrow{\sim} \mathbb{Z} \oplus \mathcal{O}^\times$, and this shows that $H^{p,1}(X) \simeq H^{p-1}_{\text{Zar}}(X, \mathcal{O}^\times_\mathcal{X})$, as predicted. Also, we will see that for a field extension $F \hookrightarrow k$, we have an isomorphism $H^{p,0}(\text{Spec } k) \simeq K^M_p(k)$.

### 7 1/24/18

Last time, we defined the objects $Z(n) = C_\ast Z_{tr}(\mathbb{G}_m, 1)^n[-n]$. This is a cochain complex in the category PST of presheaves with transfer on the category of smooth $F$-schemes. Actually, the $Z(n)$ are really complexes of sheaves for the Zariski topology. To see this, first note that a representable functor in this category, i.e. a functor $\text{Corr}_F \to \text{Ab}$ of the form $U \mapsto \text{Corr}_F(U, Y)$ for a smooth $F$-scheme $Y$, is a Zariski sheaf. This is because an elementary correspondence $X \sim Y$ is uniquely determined by the restriction to any non-empty dense open subset $U \subseteq X$:

**Lemma 7.1.** The restriction map $\text{Corr}_F(X, Y) \to \text{Corr}_F(U, Y)$ is injective with free cokernel. Moreover, if $X = U \cup V$ for $U, V$ dense open subsets, then the following sequence is exact and remains exact after applying $A \otimes \mathbb{Z}(-)$ for any abelian group $A$:

$$
0 \longrightarrow Z_{tr}(Y)(X) \longrightarrow Z_{tr}(Y)(U) \oplus Z_{tr}(Y)(V) \longrightarrow Z_{tr}(Y)(U \cap V)
$$

Since $Z_{tr}(Y)$ is a sheaf for any $Y \in \text{Sm}_F$, so are direct summands of such presheaves such as $Z_{tr}(\mathbb{G}_m, 1)$. Therefore, for any abelian group $A$, $A(n)$ is a cochain complex of sheaves.

Now, if $\mathcal{F}$ is a sheaf of abelian groups, the complex $C_\ast \mathcal{F}$ is a cochain complex of sheaves with the property that $C_\ast \mathcal{F}(X \times \mathbb{A}^1) \to C_\ast \mathcal{F}(X)$ is a quasi-isomorphism. Thus, the presheaf $X \mapsto H_n(C_\ast \mathcal{F}(X))$ is a homotopy-invariant presheaf. However, it may not be a sheaf in general, and this does not in any way formally imply that the hypercohomology $H^p(X; C_\ast \mathcal{F})$ is also homotopy-invariant. Voevodsky proved the latter statement when $\mathcal{F}$ has transfers, but this requires a long and careful argument and is special to the Zariski (and Nisnevich) topologies.\(^{10}\)

We collect some “elementary” (i.e. not Fields Medal-winning) properties of the motivic cohomology:

**Proposition 7.2.** The following properties hold for the motivic cohomology groups $H^{p,q}(X) = H^p(X; \mathbb{Z}(q))$:

\(^{10}\)see the last 4 chapters of [6]
(i) There are Mayer-Vietoris sequences.

(ii) There is a Thom isomorphism.

(iii) \( H^{p,q}(X) = H^p(X; \mathbb{Z}(q)) = 0 \) for \( p > q + \dim(X) \). (This follows easily from Grothendieck’s theorem that \( H^k(F) = 0 \) for \( k > \dim(X) \) for any abelian sheaf \( F \) and topological space \( X \).)

(iv) Motivic cohomology \( H^{p,q}(X) \) is functorial on \( X \in \text{Corr}_F \), i.e. there are pullbacks \( f^* \) for any \( f : X \to Y \) and transfers \( f_* \) for any finite \( f : X \to Y \).

(v) Motivic cohomology is “independent of \( F \)”; this makes sense since \( G_m \) is defined over \( \mathbb{Z} \) and \( X \times_F (G_m)_F = X \times \mathbb{Z} G_m \), so \( \text{Corr}_F(X, G_m) \) does not depend on \( F \).

We also have products. There is a map

\[
\text{Corr}_F(X, Y) \otimes \text{Corr}_F(X', Y') \to \text{Corr}_F(X \times X', Y \times Y')
\]

defined by \( Z \otimes Z' \to Z \times Z' \).

Taking \( X = X' \), we get a map of presheaves:

\[
Z_{tr}(Y) \otimes Z_{tr}(Y') \to Z_{tr}(Y \times Y')
\]

where the tensor product is taken (objectwise) in the category \( \text{PST} \). This sends \( Z \otimes Z' \) to \( Z \times Z'|\Delta \).

This gives a pairing:

\[
Z_{tr}\left( (G_m, 1)^n \right) \otimes Z_{tr}\left( (G_m, 1)^{n'} \right) \to Z_{tr}\left( (G_m, 1)^{n+n'} \right)
\]

Evaluating this on \( X \times \Delta^p \) gives a level-wise product. We can use the Eilenberg-Zilber formula to get a map \( Z(n) \otimes Z(n') \to Z(n+n') \): let \( A_*, B_* \) be simplicial abelian groups (i.e. simplicial objects in the category of abelian group). Then the Eilenberg-Zilber formula gives a quasi-isomorphism between \( C(A_*) \otimes C(B_*) \) and \( C(A_* \otimes B_*) \). The tensor product on the left is on the category of chain complexes, and the one on the right is on the category of simplicial abelian groups. All of this put together means that the group \( \bigoplus_{p,q} H^p(\mathbb{Z}; \mathbb{Z}(q)) \) admits the structure of a bi-graded ring which is graded-commutative with respect to \( p \).

Now, we will discuss some special cases. We saw that \( H^p(X; \mathbb{Z}(0)) = H^p(X; \mathbb{Z}) \) last time. We also have an isomorphism \( H^1(X; \mathbb{Z}(1)) \cong \mathcal{O}_X^\times \) is done in e.g. [6]. In addition, when \( X = \text{Spec} F \), we’ll see that \( H^n(X; \mathbb{Z}(n)) \cong K_n^M(F) \).

First, for \( X = \text{Spec} F \), \( H^n(\mathbb{Z}(n))(X) = H^*(X; \mathbb{Z}(n)) := \text{Cok}(d : \mathbb{Z}(n)^{n-1}(X) \to \mathbb{Z}(n)^n(X)) \).

This uses the fact that \( X = \text{Spec} F \).

We have \( d : \mathbb{Z}(n)^{n-1}(X) \to \mathbb{Z}(n)^n(X) \) is the same thing as

\[
C_1 Z_{tr}\left( (G_m, 1)^n \right)(X) \to C_0 Z_{tr}\left( (G_m, 1)^n \right)(X)
\]

An elementary correspondence \( Z \subseteq \text{Spec} F \times G_m^n \) is the same thing as a finite field extension \( E/F \) and a map \( \text{Spec} E \to G_m^n \), i.e. an element \( x' \in E^\times \times \cdots \times E^\times \). In particular, for \( a = (a_1, \ldots, a_n) \in (F^\times)^n \), we have a class \([a_1 : \cdots : a_n] \in H^n(\text{Spec} F; \mathbb{Z}(n))\). This equals \([a_1] \cdot [a_2] \cdots [a_n] \) where the product is as defined above.

Now, this gives us a map \( K_1^M(F) = F^\times \to H^1(F; \mathbb{Z}(1)) \) sending \( a \) to \([a]\). We have:
Lemma 7.3. \[ [ab] = [a] + [b] \]

Proof. Note that \([1] = 0\) because this maps to the kernel of \(\mathbb{Z}_{tr}(\mathbb{G}_m) \to \mathbb{Z}_{tr}(\mathbb{G}_m; 1)\). Now, we claim that \([ab] + [1] = [a] + [b]\).

Note that \(\Delta^1 = \text{Spec } \mathbb{F}[[t_0, t_1]]/(t_0 + t_1 = 1) \simeq \text{Spec } \mathbb{F}[t] = \mathbb{A}_k^1\), under which the two face maps \(\Delta^0 \to \Delta^1\) correspond to the inclusions of the two points \(\{0\}\) and \(\{1\}\) into \(\mathbb{A}_k^1\). We define an interpolation from \([ab] + [1]\) to \([a] + [b]\) parametrized by the \(s\)-line \(\text{Spec } \mathbb{F}[s]\) via the formula:

\[
s(t^2 - (a + b)t + ab) = (1 - s)(t^2 - (ab + 1)t + ab) = t^2 - (\cdots)t + ab
\]

This defines a curve in the \((s, t)\)-plane \(\text{Spec } \mathbb{F}[s, t]\), and this maps to \(\text{Spec } \mathbb{F}[s] = \mathbb{A}_k^1\) and to \(\text{Spec } \mathbb{F}[t, t^{-1}] = \mathbb{G}_m\).

This shows that we obtain a well-defined homomorphism of rings

\[
T_{\mathbb{F}} F^\times = \bigoplus_n (F^\times)^{\otimes n} \to \bigoplus_n H^n(F; \mathbb{Z}(n))
\]

which sends \(a_1 \otimes \cdots \otimes a_n \to [a_1] \cdot [a_2] \cdots [a_n] = [a_1 : \cdots : a_n]\).

Next time, we will see that \([a] \cdot [1 - a] = 0\) if \(a \in F^\times \setminus \{1\}\). This will show that the above map descends to a homomorphism \(K_2^M(F) \to \bigoplus_n H^n(F; \mathbb{Z}(n))\) which sends \(\ell(a)\) to \([a]\). Furthermore, we will see that this is in fact an isomorphism.

References