NOTES ON SPECTRAL SEQUENCES

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1. LOCAL COEFFICIENT SYSTEMS

1.1. The fundamental groupoid. Recall that a groupoid is a category where all morphisms are isomorphisms.

**Lemma 1.1.** Let $F : C \to D$ be a functor between groupoids. Then $F$ is an equivalence of categories if and only if

(i) the induced map $\text{Ob}(C)/\!\!/\text{iso} \to \text{Ob}(D)/\!\!/\text{iso}$ of sets of isomorphism classes is a bijection,

(ii) the induced homomorphisms $\text{Aut}_C(c) \to \text{Aut}_D(F(c))$ are isomorphisms for all $c \in \text{Ob}(C)$. □

**Definition 1.2.** Let $X$ be a topological space. The fundamental groupoid $\pi_1(X)$ has object set $\text{Ob}(\pi_1(X)) = X$, morphisms are given by

$$\text{Hom}_{\pi_1(X)}(x_0, x_1) = \{ \lambda : I \to X \mid \lambda(0) = x_0, \lambda(1) = x_1 \}/(\sim \text{ rel } \partial I).$$

and composition is given by concatenation $[\lambda_1] \circ [\lambda_0] = [\lambda_0 \ast \lambda_1]$ if $\lambda_0(1) = \lambda_1(0)$.

If $f : X \to Y$ is continuous, let $f_* = \pi_1(f) : \pi_1(X) \to \pi_1(Y)$ be the functor given on objects by $f_*(x) = f(x)$ and on morphisms by $f_*([\lambda]) = [f \circ \lambda]$.

Note that $\text{Ob}(\pi_1(X))/\!\!\text{isom} = \pi_0(X)$ and $\text{End}_{\pi_1(X)}(x) = \pi_1(X, x)$. In particular, a continuous map $f : X \to Y$ induces an equivalence of categories $\pi_1(X) \to \pi_1(Y)$ if and only if it induces a weak equivalence $\tau_{\leq 1}X \to \tau_{\leq 1}Y$.

**Definition 1.3.** The category of simplices $\text{Simp}(X)$ has objects pairs $(p, \sigma)$, where $p \geq 0$ and $\sigma : \Delta^p \to X$. Morphisms $(p, \sigma) \to (q, \tau)$ are maps $\theta : \Delta^p \to \Delta^q$ such that
Example 1.9. \(x\) which sends a point \(\pi\) may be regarded as a diagram \((x)\).

Definition 1.4. Let \(F: \text{Simp}(X) \to C\) be the functor, which on objects sends \(\sigma: \Delta^p \to X\) to the value \(u(\sigma) = \sigma(c_p)\) at the barycenter \(c_p = (1/p+1, \ldots, 1/p+1) \in \Delta^p\), and sends a morphisms \(\theta: (\sigma: \Delta^p \to X) \to (\tau: \Delta^q \to X)\) to \(\tau \circ \lambda\), where \(\lambda: I \to \Delta^q\) is any path from \(c_q\) to \(\theta(c_p)\).

Lemma 1.5. The functor \(u: \text{Simp}(X)^{op} \to \pi_1(X)\) is the universal functor into a groupoid: if \(F: \text{Simp}(X)^{op} \to C\) is any functor into a groupoid, there exists a functor \(G: \pi_1(X) \to C\) and a natural isomorphism \(\phi: G \circ u \cong F\). The pair \((G, \phi)\) is unique in the sense that if \((G', \phi')\) is another such pair, then there exists a unique isomorphism \(\psi: G \to G'\) with \(\phi' \circ (\psi \circ u) = \phi\).

Proof sketch. Let us explain how to define \(G\) on objects and morphisms. Each object \(x \in \pi_1(X)\) gives an object \((\Delta^0 \to X) \in \text{Simp}(X)\) which we also denote \(x\), and on objects we set \(G(x) = F(x)\).

A morphisms in \(\pi_1(X)\) from \(x_0\) to \(x_1\), represented by \(\lambda: I \to X\) gives an object \(\Delta^1 \approx I \to X\) using any choice of homeomorphism \(\Delta^1 \approx I\), which we denote by \(\lambda\).

The diagram

\[
\begin{array}{ccc}
\Delta^0 & \xrightarrow{[e_0]} & \Delta^1 & \xrightarrow{[e_1]} & \Delta^0 \\
\downarrow x_0 & & \downarrow \lambda & & \downarrow x_1 \\
X & \xrightarrow{\lambda} & X & \xrightarrow{\lambda} & X
\end{array}
\]

may be regarded as a diagram \((x_0 \to \lambda \leftarrow x_1)\) in \(\text{Simp}(X)\) so by applying \(F\) we get a diagram \(G(x_0) = F(x_0) \leftarrow F(\lambda) \to F(x_1) = G(x_1)\) in \(C\), and since all morphisms in \(X\) are isomorphisms we may let \(G([\lambda]): G(x_1) \to G(x_0)\) be the composition. \(\square\)

1.2. Coefficient systems.

Definition 1.6. A coefficient system on \(X\) is a functor \(A: \pi_1(X) \to \text{Ab}\), where \(\text{Ab}\) is the category of abelian groups. A morphism of coefficient systems is a natural transformation of such functors.

If \(f: X \to Y\) is continuous and \(A: \pi_1(Y) \to \text{Ab}\) is a coefficient system on \(Y\), define \(f^*A: \pi_1(X) \to \text{Ab}\) by precomposing with \(\pi_1(f): \pi_1(X) \to \pi_1(Y)\).

Remark 1.7. A coefficient system determines a functor \(A \circ u: \text{Simp}(X)^{op} \to \text{Ab}\). By the previous subsection any functor \(V: \text{Simp}(X)^{op} \to \text{Ab}\) which sends all morphisms to isomorphisms conversely determines a coefficient system. Hence the category of coefficient systems is equivalent to the category of functors \(V: \text{Simp}(X)^{op} \to \text{Ab}\) with this property.

Let us give a few examples of naturally occurring coefficient systems.

Example 1.8. If \(A\) is an abelian group, then we have the constant functor \(\mathbb{A}: \pi_1(X) \to \text{Ab}\) which takes all objects to \(A\) and all morphisms to the identity map of \(A\).

Example 1.9. Let \(x_0 \in X\) be a point, and consider the functor \(\pi_1(X) \to \text{Ab}\) which sends a point \(x \in X\) to \(\mathbb{Z}\text{Hom}_{\pi_1(X)}(x_0, x)\), the free abelian group on the set
For a path $\gamma : I \to B$, we shall associate a coefficient system $H_q = H_q(\pi; \mathbb{Z}) : \pi_1(B) \to \text{Ab}$ to a Serre fibration $\pi : E \to B$. On objects it is defined as $H_q(x) = H_q(\pi^{-1}(x))$. For a morphism $\theta : \Delta^p \to \Delta^q$ from $\sigma \circ \theta$ to $\sigma$, the pullback diagram of Serre fibrations

$$
\begin{array}{ccc}
\sigma \circ \theta^* E & \to & \sigma^* E \\
\downarrow & & \downarrow \\
\Delta^q & \to & \Delta^p
\end{array}
$$

and the five lemma implies that the top horizontal map is a weak equivalence and hence induces an isomorphism $H_p((\sigma \circ \theta)^* E) \to H_p(\sigma^* E)$ in integral homology. We define $H_p(\theta) : H_p(\sigma^* E) \to H_p((\sigma \circ \theta)^* E)$ as the inverse of that isomorphism.

The same recipe applies for homology with other coefficients, or even with coefficients in another coefficient system $A : \pi_1(E) \to \text{Ab}$, giving rise to coefficient systems $H_q(\pi; A) : \pi_1(B) \to \text{Ab}$.

1.3. Path connected based spaces. Let $x_0 \in X$. Then for any coefficient system $V : \pi_1(X) \to \text{Ab}$ the value $V(x_0)$ is naturally a (left) module over the group ring $\mathbb{Z}[\pi_1(X, x_0)]$. Conversely, if $M$ be a right module over the group ring $\mathbb{Z}[\pi_1(X, x_0)]$, we may use the representable coefficient system from Example 1.9 to define a new coefficient system

$$x \mapsto M \otimes_{\mathbb{Z}[\pi_1(X, x_0)]} \text{ZHom}_{\pi_1(X)}(x_0, x).$$

Lemma 1.11. If $X$ is path connected and $x_0 \in X$, these processes give an equivalence of categories between coefficient systems on $X$ and right modules over the group ring $\mathbb{Z}[\pi_1(X, x_0)]$. \hfill $\square$

In older literature, a coefficient systems is sometimes defined to be a $\mathbb{Z}[\pi_1(X, x)]$-module. The definition we have given avoids an unnatural choice of base points and hence has nicer functoriality properties. It also works better for non-connected spaces.

1.4. Singular chains with local coefficients. Let $A : \pi_1(X) \to \text{Ab}$ be a coefficient system, and let us denote the corresponding functor $\text{Simp}(X) \to \text{Ab}$ by the same letter $A$. We define chain groups

$$C_p(X; A) = \bigoplus_{\sigma : \Delta^p \to X} A(\sigma).$$

For any object $\sigma : \Delta^p \to X$ of $\text{Simp}(X)$ we have new objects $\sigma|_{[e_0, \ldots, \hat{e}_i, \ldots, e_p]} : \Delta^{p-1} \to X$, and the $i$th face inclusion $\delta^i = [e_0, \ldots, \hat{e}_i, \ldots, e_p] : \Delta^{p-1} \to \Delta^p$ defines a
We then define a morphism in \( \text{Simp}(X) \) from \( \sigma|_{[e_0,\ldots,e_i,\ldots,e_p]} \) to \( \sigma \). Applying \( \mathcal{A} \) we get homomorphisms of abelian groups

\[
d_i : \mathcal{A}(\sigma) \xrightarrow{\partial(i)} \mathcal{A}(\sigma|_{[e_0,\ldots,e_i,\ldots,e_p]}) \subseteq \bigoplus_{\tau: \Delta^{p-1} \to X} \mathcal{A}(\tau) = C_{p-1}(X; \mathcal{A}).
\]

We then define

\[
\partial = \sum_{i=0}^{p} d_i : C_p(X; \mathcal{A}) \to C_{p-1}(X; \mathcal{A}).
\]

1.5. **Homology with local coefficients.** The usual proof shows that \( \partial \partial = 0 \) and \( H_1(X; \mathcal{A}) \) is then defined as the homology of this chain complex. If \( \mathcal{A} = \mathcal{A} : \pi_1(X) \to \text{Ab} \) is constant, this agrees with the usual definition of the chain complex \( (C_*(X; \mathcal{A}), \partial) \) and singular homology \( H_*(X; \mathcal{A}) \).

Many of the usual constructions and theorems about singular homology have analogues for local coefficients. Here are some (which can be proved by the obvious modifications of the usual proofs).

- A morphism of coefficient systems \( \mathcal{A} \to \mathcal{B} \) induces a map of singular chains and in turn homology \( H_*(X; \mathcal{A}) \to H_*(X; \mathcal{B}) \). If \( \mathcal{A} \to \mathcal{B} \to \mathcal{C} \) is a short exact sequence of coefficient systems (i.e. short exact when evaluated at any \( x \in X \)), then there is an induced short exact sequence of chains and hence long exact sequence in homology with twisted coefficients.

- Given a continuous map \( f : X \to Y \) and a coefficient system \( \mathcal{A} : \pi_1(Y) \to \text{Ab} \), there is an induced homomorphism \( H_*(X; f^* \mathcal{A}) \to H_*(Y; \mathcal{A}) \).

- If \( i : Y \to X \) is the inclusion of a subspace and \( \mathcal{A} : \pi_1(X) \to \text{Ab} \) is a coefficient system, we define \( C_*(X,Y; \mathcal{A}) \) as the cokernel of the induced injection of chain complexes \( i_* : C_*(Y; i^* \mathcal{A}) \to C_*(X; \mathcal{A}) \). Its homology is denoted \( H_*(X,Y; \mathcal{A}) \) and sits in a long exact sequence with \( H_*(X; \mathcal{A}) \) and \( H_*(Y,i^* \mathcal{A}) \).

- If \( Z \subset Y \) is a subspace whose closure is contained in the interior of \( X \), then the appropriate generalization of excision holds: if \( i : X \setminus Z \to X \) denotes the inclusion, then the map \( H_n(X \setminus Z,Y \setminus Z; i^* \mathcal{A}) \to H_n(X,Y; \mathcal{A}) \) induced by the inclusions is an isomorphism. (Note however that excision does not allow us to express \( H_*(X,A; \mathcal{A}) \) in terms of homology of \( X/A \) with local coefficients.)

Homotopy invariance is a bit more subtle. If \( f, g : X \to Y \) are homotopic maps and \( \mathcal{A} : \pi_1(Y) \to \text{Ab} \) is a coefficient system, then we have induced maps

\[
f_* : H_*(X; f^* \mathcal{A}) \to H_*(Y)
g_* : H_*(X; g^* \mathcal{A}) \to H_*(Y),
\]

but it does not make sense to claim that these are “equal”, since they don’t even have the same domain. Let \( h : I \times X \to Y \) be a homotopy from \( f = h(0,-) \) to \( g = h(1,-) \). We have induced functors of fundamental groupoids

\[
\pi_1(X) \xrightarrow{f_*} \pi_1(Y) \xrightarrow{g_*} \pi_1(Y)
\]
and for each \( x \in X \) we have a path \( h(-, x) : I \to Y \) from \( f(x) \) to \( g(x) \). The homotopy class of the path \( h(-, x) \) relative to \( \partial I \) may be regarded as a morphism

\[
[h(-, x)] \in \text{Hom}_{\pi_1(Y)}(f(x), g(x)),
\]

and if \( A : \pi_1(Y) \to \text{Ab} \) is a functor, we obtain an morphism of abelian groups

\[
A([h(-, x)]) : (f^*A)(x) \to (g^*A)(x).
\]

We leave it as an exercise to verify that these homomorphisms are actually isomorphisms, and that the collection of all of them, as \( x \in X \), form a natural isomorphism between the functors \( f^*A, g^*A : \pi_1(X) \to \text{Ab} \). Notice that we have produced a specific natural isomorphism, not just proved that the functors are isomorphic. This natural isomorphism depends on \( h \), and we shall denote it \( T_h : f^*A \Rightarrow g^*A \).

**Theorem 1.12** (Homotopy invariance of singular homology with local coefficients). Let \( h : I \times X \to Y \) be a homotopy from \( f \) to \( g \) and let \( A : \pi_1(Y) \to \text{Ab} \) be a coefficient system. Then the diagram of homology groups

\[
\begin{array}{ccc}
H_*(X; f^*A) & \xrightarrow{f_*} & H_*(Y; A) \\
(T_h)_* & & \downarrow \\
H_*(X; g^*A) & \xrightarrow{g_*} & H_*(Y; A),
\end{array}
\]

where the horizontal maps are induced by the maps \( f \) and \( g \) of spaces, and the vertical map is induced by the map \( T_h \) of coefficient systems, is commutative. If \( A \subset X \) and \( B \subset Y \) and \( h \) restricts to a homotopy of maps \( A \to B \), then a similar statement holds in relative homology.

**Proof sketch.** There is a similar diagram on the chain level, and it suffices to prove that it commutes up to a chain homotopy. This is proved in the same way as for constant coefficients (using the “prism operators”, cf. e.g. Hatcher’s textbook).

**Corollary 1.13.** If \( f, g : X \to Y \) are homotopic and \( A : \pi_1(Y) \to \text{Ab} \), then \( f_* : H_*(X; f^*A) \to H_*(Y; A) \) is an isomorphism/injective/surjective/zero if and only if \( g_* : H_*(X; g^*A) \to H_*(Y; A) \) has that property.

If \( f : X \to Y \) is a homotopy equivalence, then \( f_* : H_*(X; f^*A) \to H_*(Y; A) \) is an isomorphism.

**Proof.** The first part is because the map \((T_h)_* : H_*(X; f^*A) \to H_*(X; g^*A)\) is an isomorphism. In particular we see that any map which is homotopic to the identity induces an isomorphism of homology with local coefficients.

For the second part, if \( g : Y \to X \) is a homotopy inverse to \( f \), we have compositions

\[
H_*(X; f^*g^*f^*A) \xrightarrow{f_*} H_*(Y; g^*f^*A) \xrightarrow{g_*} H_*(X; f^*A) \xrightarrow{f_*} H_*(Y; A).
\]

The composition of the first two maps is induced by a self-map of \( X \) homotopic to the identity and is hence an isomorphism, and similarly for the composition of the last two maps. This implies that all three maps are isomorphisms.

With this slightly more complicated homotopy invariance in place, we may continue the list of properties of singular homology which generalize to local coefficients.
• If $X$ is a $\Delta$-complex, then the singular chains with local coefficients have a subcomplex of twisted simplicial chains (where the map $\sigma$ is required to be the characteristic map of a simplex in $X$) whose homology is the simplicial homology $H^*_{\Delta}(X;\mathcal{A})$. The inclusion of this subcomplex into all singular chains induces an isomorphism $H^*_\Delta(X;\mathcal{A}) \to H_*(X;\mathcal{A})$.

• If $\mathcal{A} \subset X$ and the inclusion $\mathcal{A} \to X$ is a weak equivalence, then $H_*(X,\mathcal{A};\mathcal{A}) = 0$ for all $\mathcal{A} : \pi_1(X) \to \text{Ab}$. This is because (as for constant coefficients) any cycle in $C_*(X,\mathcal{A};\mathcal{A})$ is in the image of a cycle in $C_*(K,L;f^*\mathcal{A})$ for a map $f : (K,L) \to (X,\mathcal{A})$ where $K$ is a $\Delta$-complex and $L$ a subcomplex. Then $f$ is homotopic to a map which factors through $(\mathcal{A},\mathcal{A})$ and hence $f$ induces the zero map in relative homology.

• If $f : X \to Y$ is a weak equivalence, then $f_\ast : H_*(X,f^*\mathcal{A}) \to H_*(Y;\mathcal{A})$ is an isomorphism for all $\mathcal{A} : \pi_1(Y) \to \text{Ab}$. (Proof sketch: use homotopy invariance to replace $Y$ by $Mf$ and use the long exact sequence of the pair $(Mf,X)$. Be careful with what happens to the coefficient system when pulled back to $Mf$.)

1.6. Universal covers. Let us briefly discuss the meaning of homology with coefficients in the representable coefficient systems $x \mapsto \mathbb{Z}\text{Hom}_{\pi_1(X)}(x_0,x)$. In general, the group $C_p(X;\mathbb{Z}\text{Hom}_{\pi_1(X)}(x_0,-))$ is free abelian on the set of pairs $(\sigma,[-])$, where $\sigma : \Delta^p \to X$ is a map, $\lambda : I \to X$ is a path from $\sigma(c_p)$ to $x_0$, and $[-]$ is its homotopy class relative to $\partial I$. Now let $f : (Y,y_0) \to (X,x_0)$ be a map of pointed spaces and assume $Y$ is simply connected. Then for each map $\sigma : \Delta^p \to Y$ there exists a path $\lambda : I \to Y$ from $\sigma(c_p)$ to $y_0$ and $\lambda$ whose homotopy class $[\lambda]$ relative to $\partial I$ is unique. Hence there is a natural map

$$C_p(Y;\mathbb{Z}) \to C_p(X;\mathbb{Z}\text{Hom}_{\pi_1(X)}(x_0,-))$$

$$\sigma \mapsto (f \circ \sigma, [f \circ \lambda])$$

and we leave it to the reader to verify that this defines a chain map.

**Lemma 1.14.** If $Y \to X$ is a universal cover of the path component containing $x_0$, then the map above is an isomorphism of chain complexes $C_*(Y;\mathbb{Z}) \to C_*(X;\mathbb{Z}\text{Hom}_{\pi_1(X)}(x_0,-))$.

**Proof.** Both are free abelian and the map sends a basis to a basis. $\square$

In particular, the induced map in homology is also an isomorphism. Hence $H_*(X;\mathbb{Z}\text{Hom}_{\pi_1(X)}(x_0,-))$ calculates the homology of a universal cover of the path component containing $x_0$, in case one exists. The local homology is also defined for pathological spaces, and the following holds in complete generality.

**Corollary 1.15.** Let $\{x_0\} \to \tau\geq 2(X,x_0) \to X$ be the Moore–Postnikov factorization, i.e. $\tau\geq k(X,x_0)$ is a simply connected CW complex and the map to $X$ induces an isomorphism in $\pi_k$ for $k \geq 2$. Then the induced map

$$H_*(\tau\geq 2(X,x_0);\mathbb{Z}) \to H_*(X;\mathbb{Z}\text{Hom}_{\pi_1(X)}(x_0,-))$$

is an isomorphism.

**Proof.** If the path component of $X$ containing $x_0$ admits a universal cover $\tilde{X} \to X$, then the map from $\tau\geq 2(X,x_0)$ factors through a weak equivalence $\tau\geq 2(X,x_0) \to \tilde{X}$, in which case the result follows from the previous lemma. In particular, the
corollary is proved whenever \( X \) is a CW complex and the general case follows by CW approximation.

**Theorem 1.16.** If \( f : X \to Y \) is any map of spaces, then \( f \) is a weak equivalence if and only if \( \pi_1(X) \to \pi_1(Y) \) is an equivalence of categories and \( H_*(X; f^*A) \to H_*(X; A) \) is an isomorphism for all coefficient systems \( A : \pi_1(Y) \to \text{Ab} \).

**Proof sketch.** The assumption implies that \( \pi_0(X) \to \pi_0(Y) \) is a bijection and \( \pi_1(X, x) \to \pi_1(Y, f(x)) \) is an isomorphism for all \( x \in X \). If we let \( A \) be the representable system \( A = \mathbb{Z}\text{Hom}_{\pi_1(Y)}(\cdot, f(x)) \), the induced map of homology can be identified with

\[
H_*(\tau_{\geq 2}(X, x); \mathbb{Z}) \to H_*(\tau_{\geq 2}(Y, f(x)); \mathbb{Z}),
\]

so by the usual Whitehead theorem we deduce that \( \tau_{\geq 2}(X, x) \to \tau_{\geq 2}(Y, f(x)) \) is a weak equivalence for all \( x \in X \). Together with the fact that \( f \) induce bijections in \( \pi_0 \) and \( \pi_1 \), this implies that \( f \) is a weak equivalence. \( \square \)
2. Exact couples and spectral sequences

We define what spectral sequences are, and introduce the most important tool for constructing them, and give an example.

2.1. Spectral sequences.

Definition 2.1. A spectral sequence is a sequence \((E^r, d^r)_{r \geq 0}\), where each \(E^r\) is an abelian group, \(d^r : E^r \to E^r\) is a homomorphism satisfying \(d^r d^r = 0\), and \(E^{r+1} \cong \ker(d^r)/\im(d^r)\) is a specified isomorphism. (The specified isomorphism is usually suppressed from the notation, and/or pretended to be an equality.)

Spectral sequences are the objects of a category: A morphism from a spectral sequence \((E^r, d^r)_{r \geq 0}\) to a spectral sequence \((E'^r, d'^r)_{r \geq 0}\) is a sequence of homomorphisms \(E^r \to E'^r\) commuting with the \(d^r\) and the specified isomorphisms \(E^{r+1} \cong \ker(d^r)/\im(d^r)\).

All interesting spectral sequences can be constructed by the method of exact couples, introduced by Massey ([11]).

2.2. Exact couples. An exact couple is a triple \((A, E, i, j, k)\), consisting of two abelian groups \(A\) and \(E\), and homomorphisms

\[
\begin{array}{ccc}
A & \xrightarrow{i} & A \\
\downarrow{k} & & \downarrow{j} \\
E & & 
\end{array}
\]

making the triangle exact at each vertex. Given this, the homomorphism \(d = jk : E \to E\) satisfies \(dd = jkjk = 0\) since \(kj = 0\).

The derived couple of \((A, E, i, j, k)\) is \((A', E', i', j', k')\), where \(E' = \ker(d)/\im(d)\), \(A' = iA\), \(i'(a) = ia\), \(j'(ia) = [ja]\), \(k'[e] = ke\). To see that \(j'\) is well defined, suppose \(ia = ib\): then exactness implies that \(b = a + ke\) for some \(e \in E\), and then \([jb] = [ja] + [jke] = [ja] + [de] = [ja]\). To see that \(k'\) is well defined we first note that for \([e] \in E'\), \(de = jke = 0\), so \(ke \in \ker(j) = \im(i) = A'\). If \([e] = [e + df]\), then \(k(e + df) = ke + kjk(f) = ke\) since \(kj = 0\). We have defined the groups and maps in the triangle

\[
\begin{array}{ccc}
A' & \xrightarrow{i'} & A' \\
\downarrow{k'} & & \downarrow{j'} \\
E' & & 
\end{array}
\]

Theorem 2.2. The derived couple of an exact couple is an exact couple.

Proof. We'll prove the inclusions \(\im(i) \subseteq \ker(k)\) and \(\im(k) \supseteq \ker(i)\) thrice.

Upper left corner. \(\subset\): \(i'k'[e] = ike = 0\). \(\supset\): if \(a \in A'\) has \(i'a = ia0\), then \(a = ke\) for some \(c\). Writing \(a = ib\), \(de = jke = ja = jib = 0\), so \([e] \in E'\) has \(k'[e] = ke = a\).

Upper right corner. \(\subset\): if \(a = ib \in A'\), \(j'i'(a) = j'(ia) = ja = jib = 0\). \(\supset\): if \(a = ib \in A'\) has \(j'(a) = [jb] = 0\), then \(jb = de = jke\) for some \(e \in E\). Then \(j(b - ke) = 0\), so \(b - ke \in \ker(j) = \im(i)\), so \(b = ke + ic\) for some \(c \in A\), and hence \(a = ib = i(ke + ic) = i^2 \in \im(i')\).

Lower corner. \(\subset\): if \(a = ib \in A'\), \(k'j'(a) = k'[jb] = kjb = 0\). \(\supset\): if \(k'[e] = ke = 0\) for \([e] \in E'\), then \(e \in \ker(k) = \im(j)\). Hence \(e = ja\) for some \(a\) and hence \([e] = [ja] = j'(ia)\). \(\square\)
Iterating this construction, we get a sequence of exact couples

\[
\begin{array}{c}
A^r \\
\downarrow i_r \\
A^r \\
\downarrow j_r
\end{array}
\]

which for \( r = r_0 \) is the one we started with, and where each exact couple is the derived couple of the previous one. The choice of \( r_0 \) is just a convention, but typically \( r_0 \) is either 2, 1 or 0.

2.3. **The spectral sequence associated to an exact couple.** For an exact couple \((E, A, i, j, k)\) we may form all iterated derived couples. The resulting \((E^r, d^r)\) is a spectral sequence with \( E^{r_0} = E, d^{r_0} = jk \).

In applications, the groups \( A \) and \( E \) are usually (bi)graded, and the maps \( i, j, k \) have some fixed degree. (More generally, these could be objects and morphisms in any abelian category.) By the formulas for \( i', j', k' \), it is clear that \( \deg(i') = \deg(i), \deg(k') = \deg(k) \), and that \( \deg(j') = \deg(j) - \deg(i) \). We get that

\[
\begin{align*}
\deg(d^{r_0}) &= \deg(j) + \deg(k) \\
\deg(d^r) &= \deg(d^{r_0}) - (r - r_0) \deg(i) \\
&= \deg(j) + \deg(k) - (r - r_0) \deg(i).
\end{align*}
\]

2.4. **Example: the spectral sequence of a filtered space.** A filtered space is a space \( X \) with specified subspaces \( X_p \subset X, p \in \mathbb{Z} \) such that \( X_p \subset X_{p+1} \) is a closed subspace and \( \varprojlim X_p \rightarrow X \) is a homeomorphism.

For such \( X \), we may set

\[
\begin{align*}
A_{p,q} &= H_{p+q}(X_p) \\
E_{p,q} &= H_{p+q}(X_p, X_{p-1})
\end{align*}
\]

let \( i : A \rightarrow A \) be induced by the inclusions \( X_{p-1} \rightarrow X_p \), let \( j : A \rightarrow E \) be induced by the maps of pairs \((X_p, \emptyset) \rightarrow (X_p, X_{p-1})\), and let \( k : E \rightarrow A \) be the connecting homomorphisms \( H_{p+q}(X_p, X_{p-1}) \rightarrow H_{p+q-1}(X_{p-1}) \). We see that in this case

\[
\begin{align*}
\deg(i) &= (1, -1) \\
\deg(j) &= (0, 0) \\
\deg(k) &= (-1, 0)
\end{align*}
\]

We get a bigraded spectral sequence with

\[
\deg(d^r) = (-r, r - 1).
\]

The \( E^1 \) term is given by \( E^1_{p,q} = H_{p+q}(X_p, X_{p-1}) \), and the differential \( d^1 \) is the composition

\[
H_{p+q}(X_p, X_{p-1}) \rightarrow H_{p+q-1}(X_{p-1}) \rightarrow H_{p+q-1}(X_{p-1}, X_{p-2}).
\]

**Remark 2.3.** If \( X \) is a CW complex and \( X^p \) is the \( p \)-skeleton, we see that \( E^1_{p,q} = 0 \) for \( q \neq 0 \) and that \((E^1_{p,0}, d^1)\) is the cellular chain complex. It follows that \( E^2_{p,q} = 0 \) for \( q \neq 0 \) and that \( E^2_{p,0} = H_p(X) \). The bidegree of \( d^r \) is \((-r, r - 1)\), so for \( r \geq 2 \) we must by induction have \( E^r_{p,q} = 0 \) for \( q \neq 0 \) and \( d^r = 0 \).
The spectral sequence for a general filtered space should be thought of as attempting to “calculate” $H_*(X)$ starting from $H_*(X^p, X^{p-1})$. We shall return to what that means.

Remark 2.4. This need not be a “first quadrant” spectral sequence. In general, all we can say is that $E^1_{p,q} = 0$ when $p + q < 0$.

References


3. Pages of the spectral sequence

The groups $E^r$ are called the pages (or terms) of the spectral sequence. We give an explicit formula for the pages of the spectral sequence associated to an exact couple $(A, E, i, j, k)$ (as opposed to the inductive formula used to define it).

3.1. Iterated subquotients. Recall that a subquotient of an abelian group $G$ is a quotient of a subgroup.

“A subquotient of a subquotient is a subquotient”: If $A \subset B \subset G$, and $A' \subset B' \subset B/A$, then $B'/A'$ is canonically isomorphic to $B''/A''$, where $A'' \subset G$ and $B'' \subset G$ are the unique subgroups containing $A$ such that $(A''/A) = A'$ and $B''/A = B'$.

(The isomorphism sends $[b] \in B''/A''$ to $[[b]] \in B'/A'$.)

In the derived couple of an exact couple, $E' = \ker(d)/\text{im}(d)$ is a subquotient of $E$, so iteratively $E''$ is a subquotient of $E$. Thus, there are canonical subgroups and isomorphisms

$$B^r \subset Z^r \subset E$$

$$E^r \cong Z^r/B^r$$

The meaning of $Z^r$ can best be explained if we introduce some (slight) abuse of notation: An element $e \in E$ with $de = 0$ represents an element $[e] \in E^r$. If also $d^r[e] = 0 \in E'$, then $[e]$ represents an element which should properly be written $[[e]] \in E''$. In this case we will write just $[e]$ instead of $[[e]]$. Confusion between the two meanings of $[e]$ can be avoided by emphasizing the group: $[e] \in E'$ or $[e] \in E''$.

Similarly, starting with an element $e \in E$, we could potentially get an element of $E^r$ which should be written with $(r - 1)$ pairs of brackets. We will again write just $[e]$.

Then, start with $e \in E^1$, we are allowed to write $[e] \in E^2$ if $d^1[e] = 0$. If also $d^2[e] = 0$ we are allowed to write $[e] \in E^3$. Etc. Then $Z^r$ is the subset consisting of $e \in E$ for which we are allowed to write $[e] \in E^r$. In that case $e \in B^r$ precisely if $[e] = 0 \in E^r$. Equivalently, $[e] = d^{r-1}[x]$ for some $x \in Z^{r-1}$.

3.2. Formula for the $r$th page. The following lemma gives explicit formulas for $B^r \subset Z^r \subset E$, and also for the maps $j_r : A^r \to E^r$ and $k_r : E^r \to A^r$ in terms of the identification $E^r \cong Z^r/B^r$. Let us assume that we start with the zeroth, i.e. $r_0 = 0$. (Otherwise, replace $r$ by $r - r_0$ in suitable places).
Lemma 3.1. In the $r$th derived couple of an exact couple $(A, E, i, j, k)$, we have

$$
A^r = \text{Im}(i^r)
$$

$$
Z^r = k^{-1}(\text{Im}(i^r))
$$

$$
B^r = j(\text{Ker}(i^r)).
$$

With respect to the canonical isomorphism $E^r = Z^r/B^r$, we have

$$
i_r(a) = ia
$$

$$
j_r(\iota^r a) = [ja]
$$

$$
k_r[e] = ke.
$$

Proof. The statement is tautological for $r = 0$, and we proceed by induction. Assume the lemma is proved for $r$ and let $d^r = j_r k_r$. Given an element $e \in E$ with $ke = \iota^r a$ (i.e. $e \in Z^r$), we have

$$
d^r[e] = j_r k_r[e] = j_r \iota^r a = [ja] \in Z^r/B^r
$$

If $ke = \iota^r b$, then $d^r[e] = jib = 0$. Conversely, assume $ke = \iota^r a$ and that $d^r[e] = [ja] = 0$. Using the induction hypothesis we get $ja = jb$ for some $b \in \text{Ker}(i^r)$. Then $a - b \in \text{Ker}(j) = \text{Im}(i)$, so $a = b + ix$, and we have

$$
ke = \iota^r a = \iota^r (b + ix) = \iota^{r+1} x.
$$

and we have proved the formula for $Z^{r+1}$.

$B^{r+1}$ is the set of $b \in E$ such that $[b] = d^r[e]$ for some $e \in E^r = Z^r/B^r$. Writing again $ke = \iota^r a$ we have $d^r[e] = [ja]$ and hence

$$
b - ja \in B^r = j(\text{Ker}(\iota^r))
$$

so $b = j(a + x)$ with $\iota^r x = 0$. Then

$$
\iota^{r+1}(a + x) = \iota^{r+1} a = ike = 0,
$$

so $b \in j(\text{Ker}(\iota^{r+1}))$. Conversely, if $b = ja$ with $\iota^{r+1} a = 0$, then $\iota^r a \in \text{Ker}(i) = \text{Im}(k)$, so we can write $\iota^r a = ke$ and then $d^r[e] = [ja] = [b]$. This proves the induction formula for $B^r$.

The formula for $i_{r+1}$ is obvious, and $j_{r+1}$ is easy: $j_{r+1}(i(\iota^r a)) = [ja]$, where the first equality is the definition of derived couple and the second is the induction hypothesis. The formula for $k_r$ is also easy: $k_{r+1}[e] = k_r[e] = ke$ where the first equality is the definition of derived couple and the second is the induction hypothesis.

\[\square\]

3.3. Filtered spaces. In the spectral sequence of a filtered space (for which we will set $r_0 = 1$), the element

$$
e \in E^1_{p,q} = H_{p+q}(X_p, X_{p-1})
$$

is in $Z^r$ if its image in $H_{p+q-1}(X_{p-1})$ comes from $H_{p+q-1}(X_{p-r})$. The resulting class $[e] \in E^r_{p,q}$ is zero if $e$ comes from an element in $E^r_{p,q}(X_p)$ which vanishes in $H_{p+q}(X_{p+r-1})$.

We shall study in what sense the spectral sequence “calculates” $H_*(X)$. This phenomenon is called convergence, and under suitable assumptions the spectral sequence will converge to $H_*(X)$. We discuss convergence in more generality.
4. Convergence

We define the $E^\infty$ page of a spectral sequence, and explain how the $E^\infty$ page of the spectral sequence of an exact couple is related to the map $i : A \to A$.

4.1. Simplest case: first quadrant spectral sequences. A bigraded spectral sequence $(E^r_{p,q}, d^r)$ where $d^r$ has bidegree $(-r, r-1)$ is said to be first quadrant if $E^r_{p,q} = 0$ for unless $p \geq 0$ and $q \geq 0$. For such spectral sequences, if we fix any $(p, q)$ both the differential $E^r_{p,q} \to E^{r-r}_{p-r,q+r-1}$ and $E^{r+r}_{p,q-r+1} \to E^r_{p,q}$ must be zero, as long as $r \geq \max(p+1, q+2)$, and hence $E^r_{p,q} = E^{r+1}_{p,q} = \ldots$. This common value is denoted $E^\infty_{p,q}$.

As we'll see, it makes sense to talk about $E^\infty$ even without this stabilization phenomenon (and even without gradings), by taking an appropriate “limit” of the groups $E^r$ as $r \to \infty$.

4.2. The $E^\infty$ page of a spectral sequence. The abelian group $E$ has subgroups $Z^r, B^r$ arranged as follows

$$0 = B^1 \subset B^2 \subset \ldots \subset B^r \subset \ldots \subset Z^r \subset \ldots \subset Z^2 \subset Z^1 = E,$$

and $E^r = Z^r / B^r$ is the quotient of a group that becomes smaller and smaller, by a group that becomes bigger and bigger. Thus we may define

$$Z^\infty = \bigcap_r Z^r, \quad B^\infty = \bigcup_r B^r, \quad E^\infty = Z^\infty / B^\infty.$$

This agrees with the definition in the bigraded first-quadrant case, where the sequences $Z^r_{p,q}$ and $B^r_{p,q}$ stabilize for fixed $p, q$.

Lemma 3.1 immediately gives a completely general formula for $E^\infty$, namely

$$E^\infty = \frac{k^{-1}(\cap_r \text{Im}(i^r))}{j(\cup_r \text{Ker}(i^r))}.$$

In this generality the formula is of little use, but under additional assumptions we can rewrite it in a more useful way.

4.3. $E^\infty$ for exact couples. Let us rewrite the formula (4.1) under the additional assumption on the map $i : A \to A$ that

$$\text{Ker}(i) \cap \bigcap_r i^r A = 0$$

i.e. no non-zero element $a \in A$ has $ia = 0$ and can be written as $a = i^r b_r$ for arbitrarily large $r$. In most examples it will even be true that $\cap_r i^r A = 0$, for example in the exact couple associated to a filtered spaces this holds if $X_{-1} = \emptyset$.

The direct limit $A^\infty = A[i^{-1}] = A \otimes_{\mathbb{Z}[i]} \mathbb{Z}[i, i^{-1}] = \text{lim}_r (A \xrightarrow{i} A \xrightarrow{i} \ldots)$ is filtered in the following way. Let us write $F \subset A^\infty$ for the image of the natural map $A \to A^\infty$. Then we have subgroups $i^r F \subset A^\infty$ for every $r \in \mathbb{Z}$, giving a filtration

$$\ldots \subset i^{r+1} F \subset i^r F \subset \ldots$$

which is exhaustive in the sense that $A^\infty = \cup_r i^r F$. The filtration quotients $i^r F / i^{r+1} F$, $r \in \mathbb{Z}$, are all isomorphic to $F / iF$, since $i : A^\infty \to A^\infty$ is an isomorphism.
Proposition 4.1. The spectral sequence associated to an exact couple \((A, E, i, j, k)\) satisfying condition (4.2) has a natural isomorphism 
\[ E^\infty \cong F/iF, \]
Proof. Assuming (4.2), we have
\[ E^\infty = \frac{\text{Ker}(k)}{j(\bigcup_r \text{Ker}(i^r))} = \frac{j(A)}{j(\bigcup_r \text{Ker}(i^r))}. \]
The map \(j : A \to E\) induces an isomorphism \(A/iA \cong jA \subset E\) which descends to an isomorphims
\[ \frac{A}{iA + \bigcup_r \text{Ker}(i^r)} \cong E^\infty. \]

On the other hand, \(F\) is the image of the map \(A \to A^\infty\) whose kernel is precisely \(\bigcup_r \text{Ker}(i^r)\), so we get an isomorphism \(A/(\bigcup_r \text{Ker}(i^r)) \cong F\), compatible with multiplication with \(i\). We can therefore mod out by \(iA/(\bigcup_r \text{Ker}(i^r)) \cong iF\) on both sides to get the desired isomorphism \(E^\infty \cong F/iF\). \(\square\)

The isomorphism of the proposition is very explicit: It is the composition
\[ E \xrightarrow{\phi} A \to A^\infty, \]
of \(j^{-1}\) (restricted to \(Z^\infty\)) with the canonical map \(A \to \lim_{\to} A = A^\infty\).

4.4. The spectral sequence of a filtered space, \(E^\infty\) page. Let us calculate the \(E^\infty\) page of the spectral sequence of a filtered space using Proposition 4.1. The assumptions imply that any compact subset of \(X\) is contained in some \(X_p \subset X\) which implies that \(\lim_{\to} C_*(X_p) \cong C_*(X)\) and in turn that the natural map
\[ \lim_{\to} H_n(X_p) \to H_n(X) \]
is an isomorphism. Let us further assume that \(X_{-1} = \emptyset\), or more generally that (4.2) holds. Recall that we have \(A_{p,q} = H_{p+q}(X_p)\) and that \(i : A_{p,q} \to A_{p+1,q-1}\) is the map induced from the inclusion \(X_p \to X_{p+1}\). Then the group \(A^\infty_{p,q}\) is the direct limit of the system
\[ H_{p+q}(X_p) \to H_{p+q}(X_{p+1}) \to \ldots, \]
i.e. \(A^\infty_{p,q} = H_{p+q}(X)\) by our assumptions. The subspace \(F_{p,q}\) is the image of \(H_{p+q}(X_p)\), and \(iF_{p-1,q+1}\) is the image of \(H_{p+q}(X_{p-1})\) in \(H_{p+q}(X)\). Changing notation to \(F^p H_{p+q}(X) = F_{p,q}\), we see that
\[ E^\infty_{p,q} = F^p H_{p+q}(X)/F^{p-1} H_{p+q}(X). \]
In symbols we often write
\[ E^1_{p,q} = H_{p+q}(X_p, X_{p-1}) \Rightarrow H_{p+q}(X), \]
meaning that we are considering a spectral sequence with the specified \(E^1\) page, and where the \(E^\infty\) page is the filtration quotients in a filtration of \(H_*(X)\). In words, we say that the spectral sequence “converges” to \(H_*(X)\). Note that this spectral sequence need not be first quadrant. If \(X_{-1} = \emptyset\), we see that \(E^1_{p,q}\) vanishes unless \(p \geq 0\) and \(q \geq -p\), so the \(E^1\) page can occupy 37.5% of the plane in this case.
Let us briefly discuss more explicitly what the spectral sequence does between \(E^1\) and \(E^\infty\). That an element \(x \in E^1_{p,q}\) is in \(Z^1\) means that \(d^1 x = 0, d^2 [x] = 0, \ldots, \)
$d^{r-1}[x] = 0$, so $x$ defines an element in $E^r$. The formula $Z^r = k^{-1}(\text{Im}(i^{r-1}))$ can be interpreted in the diagram

$$
\begin{array}{c}
H_{p+q}(X_p, X_{p-r}) \xrightarrow{i_r} H_{p+q}(X_p, X_{p-1}) \xrightarrow{k} H_{p+q-1}(X_{p-1}, X_{p-r}) \\
\downarrow \downarrow \\
H_{p+q-1}(X_{p-r}) \xrightarrow{i_{r-1}} H_{p+q-1}(X_{p-1}) \xrightarrow{k} H_{p+q-1}(X_{p-1}, X_{p-r})
\end{array}
$$

with exact rows. It follows from the diagram that $Z_{p,q}^r \subseteq E_{p,q}^1$ consists of the elements $x \in H_{p+q}(X_p, X_{p-1})$ that come from $H_{p+q}(X_p, X_{p-r})$. By a similar argument, $B_{p,q}^r$ is the group of elements $x \in H_{p+q}(X_p, X_{p-1})$ that vanish when mapped to $H_{p+q}(X_{p-r-1}, X_{p-1})$.

$Z^\infty$ is the elements that come from $H_{p+q}(X_p)$, and the isomorphism $E_{p,q}^\infty \rightarrow F^p H_{p+q}(X)/F^{p-1} H_{p+q}(X)$ is given by lifting to $H_{p+q}(X_p)$ and mapping to $H_{p+q}(X)$.

4.5. Spectral sequences of filtered chain complexes and double complexes.

**Definition 4.2.** A filtered chain complex is a chain complex $C_\ast = (C_\ast, \partial)$ equipped with subcomplexes $\subseteq C^n_\ast \subseteq C^{n+1}_\ast \subseteq \cdots \subseteq C_\ast$, such that $C_\ast = \varprojlim C^n_\ast$.

In the same way as for filtered spaces, a filtered chain complex gives rise to a spectral sequence with

$$E_{p,q}^2 = H_{p+q}(C^p_\ast, C_\ast^{p-1}).$$

If we further assume that $C^{-1}_\ast = 0$ (or more generally that (4.2) holds) then the spectral sequence converges to $H_{p+q}(C_\ast)$.

**Definition 4.3.** A double complex is a bigraded abelian group $C = \bigoplus C_{p,q}$ together with maps $\partial^r : C_{p,q} \rightarrow C_{p-1,q}$ and $\partial^s : C_{p,q} \rightarrow C_{p,q-1}$ satisfying $(\partial^r)^2 = (\partial^s)^2 = \partial^r \partial^s + \partial^s \partial^r = 0$.

The associated total complex has $T_n \oplus_{p+q=n} C_{p,q}$ with boundary map $\partial = \partial^r + \partial^s$.

The total complex is canonically filtered by subcomplexes defined by

$$T^m_n = \bigoplus_{n-q = p \leq m} C_{p,q} \subset T_n.$$

This filtered chain complex gives rise to a spectral sequence with

$$E_{p,q}^2 = H_p(H_q(C_\ast, \partial^s), \partial^r),$$

i.e. first compute “homology in the $q$-direction”, then compute homology with respect to the map induced by $\partial^r$. The exact couple automatically satisfies (4.2), so the spectral sequence converges to homology of the total complex.

Reversing the meaning of $p$ and $q$ gives another spectral sequence

$$E_{p,q}^2 = H_q(H_p(C_\ast, \partial^r), \partial^s) \Rightarrow H_{p+q}(T_\ast, \partial).$$

The two spectral sequences converge to the same groups, but of course the two filtrations on $H_{p+q}(T_\ast, \partial)$ will likely be quite different.
4.6. Appendix: Detecting isomorphisms.

**Definition 4.4.** Let $G$ be an abelian group equipped with a filtration $G \supset \cdots \supset F^n \supset F^{n-1} \supset \ldots$. The filtration is **exhaustive** if $\cup_n F^n = G$, **Hausdorff** if $\cap_n F^n = 0$, and **complete** if $G \to \varprojlim G/F^n$ is surjective.

The reason for the latter two words comes from the fact that the filtration induces a topology on $G$ with basis $g + F^n$, and the topology is Hausdorff if and only if the filtration is. If the filtration is also exhaustive, the topology comes from a metric $d(g, g') = (\max\{n|g - g' \in F^n\})^{-1}$, and the inverse limit $G^\wedge = \varprojlim G/F^n$ is precisely the completion of $G$ as a metric space.

Let us briefly discuss the question of reconstructing an abelian group $G$ with a filtration $G \supset \cdots \supset F^n \supset F^{n-1} \supset \ldots$, from its **associated graded**

$$Gr^F(G) = \bigoplus_{n \in \mathbb{Z}} F^n/F^{n-1}.$$ 

In general it is of course not possible to reconstruct $G$ from the $F^n/F^{n-1}$, even up to isomorphism. Let us ask a weaker and more reasonable question: given a homomorphism $\phi : G \to \overline{G}$ of filtered groups (i.e. $\phi(F^n) \subset \overline{F^n}$), for which the induced maps of associated graded is an isomorphism (i.e. $F^n/F^{n-1} \to \overline{F^n}/\overline{F^{n-1}}$ is an isomorphism for all $n$), can we deduce that $\phi$ is an isomorphism?

A (counter-)example to keep in mind: filter the $p$-adic integers $\mathbb{Z}_p$ by the ideals $p^n\mathbb{Z}_p$ and filter $\mathbb{Z}$ by the ideals $p^n\mathbb{Z}$. The inclusion $\mathbb{Z} \to \mathbb{Z}_p$ induces an isomorphism on filtration quotients and both filtrations are exhaustive and Hausdorff. The filtration on $\mathbb{Z}_p$ is complete, but the one on $\mathbb{Z}$ is not (the completion is exactly $\mathbb{Q}_p$).

For a homomorphism $G \to \overline{G}$ as above, we may use the 5-lemma to deduce that $\phi$ induces isomorphisms $F^n/F^{n-2} \to \overline{F^n}/\overline{F^{n-2}}$ for all $n$, and inductively that $F^n/F^m \to \overline{F^n}/\overline{F^m}$ is an isomorphism for all $m \leq n$. If the filtrations are exhaustive, we may take the direct limit $n \to \infty$ to see that $\phi$ induces an isomorphism $G/F^n \to \overline{G}/\overline{F^n}$ for all $m$. If the filtrations are complete and Hausdorff, we may then take inverse limits and deduce that the original $\phi$ is indeed an isomorphism.

It is natural to assume a condition slightly stronger on $i : A \to A$ than (4.2). To the filtration $A \supset iA \supset i^2A \supset \ldots$ we may associate the completion

$$(4.3) \quad A \to A^\wedge = \varprojlim A/i^rA,$$

and it is convenient to replace condition (4.2) by the stronger condition that (4.3) be an isomorphism. The kernel of (4.3) is $\cap i^rA$, so injectivity already implies (4.2), and therefore an isomorphism between the $E^{\infty}$ page of the spectral sequence and the filtration quotients of $A^{\infty}$.

The stronger assumption that (4.3) be an isomorphism can now be phrased as the filtration of $A$ by the $i^rA$, $r \geq 0$ being complete and Hausdorff (it is automatically exhaustive). This implies that the filtration of $A^{\infty}$ by the $i^rF$, $r \in \mathbb{Z}$ is complete and Hausdorff. The spectral sequence may then be used to detect isomorphisms: if a map of exact couples induces an isomorphism of $E^{\infty}$ pages then it also induces an isomorphism between the filtered objects $A^{\infty}$.

If for example $X = \cup X_p$ and $Y = \cup Y_p$ are filtered spaces and $f : X \to Y$ is a filtered map (i.e. $f(X_p) \subset Y_p$), such that $f_* : H_\ast(X_p, X_{p-1}) \to H_\ast(Y_p, Y_{p-1})$ is an isomorphism for all $p$, then the induced map of spectral sequences is an isomorphism
5. The Serre spectral sequence

Recall that in Example 1.10 we associated to a Serre fibration $p : E \to B$ a coefficient system $\mathcal{H}_q$ given on objects by $b \mapsto H_q(p^{-1}(b))$. We shall use the same notation for the corresponding functor on simplices of $B$, given on objects as $(\sigma : \Delta^p \to B) \mapsto H_q(\sigma^*E)$. We will construct the homology Serre spectral sequence in the following form.

**Theorem 5.1.** For any Serre fibration $p : E \to B$ there is naturally a spectral sequence with

$$E^2_{p,q} = H_p(B; \mathcal{H}_q)$$

where $\mathcal{H}_q : \pi_1(B) \to \text{Ab}$ is the coefficient system $b \mapsto H_q(p^{-1}(b))$. The spectral sequence has

$$E^\infty_{p,q} = \frac{F^pH_{p+q}(E)}{F^{p-1}H_{p+q}(E)},$$

where $F^pH_n(E)$ is a filtration $0 = F^{-1}H_n(E) \subset F^0H_n(E) \subset F^1H_n(E) \subset \cdots \subset F^nH_n(E) = H_n(E)$.

This is the main result about the (homological) Serre spectral sequence, and is often summarized as “$E^2_{p,q} = H_p(B; \mathcal{H}_q) \Rightarrow H_{p+q}(E)$”. In the special case where $B$ is path connected, and for some $b \in B$ we write $F = p^{-1}(b)$ and assume that the action of $\pi_1(B, b)$ on $H_*(F)$ is trivial, the coefficient system is equivalent to the constant system $H_*(F)$. In this case we get the following corollary.

**Corollary 5.2.** Let $p : E \to B$ be a Serre fibration with $B$ path connected and let $F = p^{-1}(b)$ for some $b \in B$. Assume that the action of $\pi_1(B, b)$ on $H_*(F)$ is trivial (for example, if $B$ is simply connected). Then there is a spectral sequence with

$$E^2_{p,q} = H_p(B; H_q(F)) \Rightarrow H_{p+q}(E).$$

We now embark on the construction of the spectral sequence.

### 5.1. The coefficient system associated to a Serre fibration

For a continuous map $\sigma : \Delta^p \to B$ we shall write $E_\sigma = \sigma^*E \to \Delta^p$ for the pullback of $p$ along $\sigma$. The definitions imply that

$$C_p(B; \mathcal{H}_q) = \bigoplus_{\sigma : \Delta^p \to B} H_q(E_\sigma),$$

where the sum is over all continuous maps, and $\partial = \sum (-1)^i d_i : C_p(B; \mathcal{H}_q) \to C_{p-1}(B; \mathcal{H}_q)$, where $d_i : H_q(E_\sigma) \to H_q(E_{\sigma |_{[e_{i-1}, \ldots, e_i, e_{i+1}, \ldots, e_p]}})$ is the inverse to the isomorphism induced by the weak equivalence $E_{\sigma |_{[e_0, \ldots, e_i, \ldots, e_p]}} \leftrightarrow E_\sigma$. 

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**REFERENCES**

5.2. The Serre spectral sequence: $E^1$ page and convergence. Let $p : E \to B$ be a Serre fibration and assume for the moment that $B$ is a $\Delta$-complex. The Serre spectral sequence is a special case of the spectral sequence of a filtered space, namely we filter $E$ by $E_n = p^{-1}(B^{(n)})$, the inverse images of the skeleta of $B$. This gives a spectral sequence with

$$E^1_{p,q} = H_{p+q}(E_p, E_{p-1}).$$

For each simplex $\sigma : \Delta^p \to B$, we write $E_\sigma$ for the total space of the pullback fibration $\sigma^*(E) \to \Delta^p$ and $E_{\partial \sigma}$ for the restriction of $E_\sigma$ to $\partial \Delta^p$. For each $p$-simplex $\sigma$ there is a natural map of pairs $(E_\sigma, E_{\partial \sigma}) \to (E_p, E_{p-1})$, and it is not hard to see (using excision and the LES a couple of times) that these maps induce isomorphisms

$$\bigoplus_{\sigma : \Delta^p \to B} H_{p+q}(E_\sigma, E_{\partial \sigma}) \xrightarrow{\cong} H_{p+q}(\bigsqcup_{\sigma} E_\sigma, \bigsqcup_{\sigma} E_{\partial \sigma}) = E^1_{p,q},$$

where the sum is over all $\sigma : \Delta^p \to B$ in the $\Delta$-complex structure on $B$. We have the diagram

$$
\begin{array}{ccc}
E_\sigma & \xrightarrow{(1, p)} & E_\sigma \times \Delta^p \\
p \downarrow & & \downarrow \text{proj} \\
\Delta^p & \xrightarrow{\text{proj}} & \Delta^p
\end{array}
$$

in which the top horizontal map is obviously a homotopy equivalence, and the vertical maps are both Serre fibrations. The LES in homotopy groups implies that the spectral sequence for filtered spaces implies that the spectral sequence converges in $E^\infty$. The first differential and the $E^2$ page. To the Serre fibration $p : E \to B$, $B$ a $\Delta$-complex, we associated a spectral sequence and a natural isomorphism $E^1_{p,q} \cong \oplus_\tau H_q(E_\sigma)$. The differential $d^1 : E^1_{p,q} \to E^1_{p-1,q}$ therefore corresponds to a map

$$d^1 : \bigoplus_{\sigma} H_q(E_\sigma) \to \bigoplus_{\tau} H_q(E_\tau).$$
We want to describe this more explicitly. To state the result, we first define the candidate answer. Let \( d : \bigoplus \sigma H_q(E_\sigma) \to \bigoplus H_q(E_\sigma) \) be the homomorphism given by the matrix whose \((\sigma, \tau)\)th entry is 0 unless \( \tau \) is a face of \( \sigma \); if \( \tau \) is the ith face of \( \sigma \), then the \((\sigma, \tau)\)th entry of \( d \) is \((-1)^i\) times the isomorphism \( H_q(E_\sigma) \cong H_q(E_\tau) \) induced by the inclusion \( E_\tau \to E_\sigma \). It is clear that both \( d \) and the actual differential are natural transformations.

**Lemma 5.3.** With respect to the isomorphism \( E_{p,q}^1 = \bigoplus \sigma H_q(E_\sigma) \) described above, the maps \( d^1 \) and \( d \) agree.

**Proof.** By naturality, it suffices to check the case where \( B = \Delta^p \) and \( p : E \to \Delta^p \) is some Serre fibration\(^1\). We can further reduce to the case where \( E = F \times \Delta^p \) and \( p \) is the projection by applying naturality to the map \((1, p) : E \to E \times \Delta^p \): it is a fiberwise map of Serre fibrations and induces an isomorphism on \( E^1 \) pages of the spectral sequence.

In this case we have the commutative diagram

\[
\begin{array}{cccc}
H_q(F) \otimes H_p(\Delta^p, \partial \Delta^p) & \xrightarrow{} & H_{p+q}(E_{p+1}, E_{p-1}) & \\
\downarrow & & \downarrow k & \\
H_q(F) \otimes H_{p-1}(\partial \Delta^p) & \xrightarrow{} & H_{p+q-1}(E_{p-1}) & \\
\downarrow & & \downarrow j & \\
H_q(F) \otimes H_{p-1}(\partial \Delta^p, (\Delta^p)^{(p-2)}) & \xrightarrow{} & H_{(p-1)+q}(E_{p-1}, E_{p-2}) & \\
\cong & & \cong & \\
H_q(F) \otimes \bigoplus_j H_{p-1}(\Delta^{p-1}, \partial \Delta^{p-1}) & \xrightarrow{} & \bigoplus_j H_{(p-1)+q}(E_\tau, E_{\partial \tau}), &
\end{array}
\]

where all horizontal maps are induced by the cross product. The composition of the vertical maps on the right is \( d^1 \) (by definition), and on the left the composition is the identity on \( H_q(F) \) tensored with the sequence

\[
H_p(\Delta^p, \partial \Delta^p) \to H_{p-1}(\partial \Delta^p) \to H_{p-1}(\partial \Delta^p, (\Delta^p)^{(p-2)}) \cong \bigoplus \tau H_{p-1}(\Delta^{p-1}, \partial \Delta^{p-1}),
\]

which looks like \( \mathbb{Z} = \mathbb{Z} \to \bigoplus \tau \mathbb{Z} = \bigoplus \tau \mathbb{Z} \). Since the face maps \( F_1 : \Delta^p \to \partial \Delta^p \) are orientation preserving if and only if \( i \) is even, we see that the resulting map \( \mathbb{Z} \to \bigoplus \tau \mathbb{Z} \) is \( 1 \mapsto (1, -1, 1, -1, \ldots) \), and hence the left hand side of the diagram above is precisely \( d \). \( \square \)

**5.4. A canonical CW approximation.** We now construct the Serre spectral sequence for an arbitrary Serre fibration \( p : E \to B \), i.e. with no assumption on \( B \). The main trick is to find a \( \Delta \)-complex \( B' \) and a weak equivalence \( f : B' \to B \). Then the pull back \( f^* E \to B' \) is again a Serre fibration, with the same fibers as \( p \), and the long exact sequence in homotopy groups implies that \( f^* E \to E \) is a weak equivalence. There are several ways to construct such a \( B' \), but here’s a canonical one (which often appears in the literature, under the name “the realization of the total singular complex”, denoted \([\text{Sin}_\bullet(B)]\)). The construction is very simple, in a certain sense the \( B' \) that we construct is the universal \( \Delta \)-complex mapping to \( B \).

\(^1\)I learned this trick from notes by Dan Dugger
Construction 5.4. For an arbitrary topological space $B$, let $\Gamma B$ be the $\Delta$-complex with one $n$-cell for each continuous map $\sigma : \Delta^n \to B$. The $i$th face of an $n$-cell $\sigma$ is glued to the $(n - 1)$-cell labeled by the continuous map $\sigma \circ F_i : \Delta^{n-1} \to \Delta^n \to B$. In other words, $\Gamma B$ is the space

$$\Gamma B = \left( \bigcoprod_{n \geq 0} \text{Map}(\Delta^n, B) \times \Delta^n \right) / \sim,$$

where $\text{Map}(\Delta^n, B)$ is the set of continuous maps (regarded as a set), and $\sim$ is the equivalence relation generated by $(\sigma \circ \delta^i, t) \sim (\sigma, \delta^i(t))$, where again $\delta^i = [\epsilon_0, \ldots, \epsilon_{i-1}, \epsilon_i, \ldots, \epsilon_p] : \Delta^{p-1} \to \Delta^p$ denotes the $i$th face map for $i = 0, \ldots, p$.

**Theorem 5.5** (Milnor). The natural map $c : \Gamma B \to B$ is a weak equivalence.

*Proof.* It is easy to see that $c$ induces an isomorphism in singular homology with coefficients in any $A : \pi_1(B) \to \text{Ab}$. Indeed, the composition

$$C_\ast^\Delta(\Gamma B; c^\ast A) \subset C_\ast(\Gamma B; c^\ast A) \xrightarrow{\sim} C_\ast(B; A)$$

is an isomorphism of chain complexes, and the first inclusion induces an isomorphism after passing to homology.

It remains to see that the induced map on fundamental groupoids is an equivalence, or equivalently that $c_* : \pi_0(\Gamma B) \to \pi_0(B)$ is a bijection and that $\pi_1(\Gamma B, b) \to \pi_1(B, c(b))$ is an isomorphism for one $b \in \Gamma B$ in each path component. This is easy for $\pi_0$ and left as an exercise for $\pi_1$. □

*Proof of Theorem 5.1.* Any Serre fibration $p : E \to B$ may be pulled back to a Serre fibration $c^\ast E \to \Gamma B$, and the LES in homotopy groups imply that $c^\ast E \to E$ is a weak equivalence. Since $\Gamma B$ is a $\Delta$-complex, we have constructed a spectral sequence converging to $H_\ast(c^\ast E) \cong H_\ast(E)$, with $E^1$ page given by

$$E^1_{p,q} = \bigoplus_{\sigma : \Delta^p \to B} H_q(E_\sigma),$$

where the direct sum is now over all continuous maps $\sigma : \Delta^p \to B$, the first differential is again given by Lemma 5.3, and the $E^\infty$ page is given by precisely the subquotients of $H_\ast(c^\ast E) \cong H_\ast(E)$ appearing in Theorem 5.1. Comparing with (5.1), we see that the chain complex $(E^1_{x,q}, d^1)$ is isomorphic to $C_\ast(B; \mathcal{H}_q)$, inducing an isomorphism $E^2_{p,q} = H_p(B; \mathcal{H}_q)$ as claimed. □

**Remark 5.6.** With the same proof, we could start with a coefficient system $A : \pi(E) \to \text{Ab}$. Restricting this to each fiber gives a local system $\mathcal{H}_q(A)$ on $B$ whose value at $b$ is $H_q(p^{-1}(b); A)$, and we get a spectral sequence with

$$E^2_{p,q} = H_p(B; \mathcal{H}_q(A)) \Rightarrow H_{p+q}(E; A)$$

In particular, $A$ could be the constant system $A$ for some abelian group $A$, and we get a spectral sequence converging to $H_\ast(E; A)$.

In the special case where $A = k$ is a field, $B$ is path connected and $\pi_1(B, b)$ acts trivially on $H_\ast(F; k)$, where $b \in B$ is a point and $F = p^{-1}(b)$, then the coefficient system is isomorphic to the constant coefficient system $H_\ast(F; k)$. Since $k$ is a field, we may use the universal coefficient system to rewrite the $E^2$ page as

$$E^2_{p,q} = H_p(B; k) \otimes_k H_q(F; k).$$
Over a field, the filtration $0 \subset F^0_n \subset \cdots \subset F^n_n = H_n(E;k)$ is automatically split. Thus we may pick isomorphisms $F^p \cong F^p/F^{p-1} \oplus F^{p-1}$ and inductively get a non-canonical isomorphism $H_n(E) \cong \oplus_{p+q=n} E_{p,q}^\infty$ (Over $\mathbb{Z}$ we may make a similar conclusion provided the abelian groups $E_{p,q}^\infty$ are free.)

6. Examples

6.1. The loop space of a sphere. Let’s calculate the homology of $\Omega S^n$ using the Serre spectral sequence of the fibration $\Omega S^n \to PS^n \to S^n$. We have

$$E^2_{p,q} = H_p(S^n; H_q(\Omega S^n)) \Rightarrow H_{p+q}(PS^n)$$

We can calculate the $E^2$ term using the universal coefficient theorem. Since $H_p(S^n)$ has no torsion, the Tor term vanishes, and we have an isomorphism

$$E^2_{p,q} = H_p(S^n) \otimes H_q(\Omega S^n) = \begin{cases} H_q(\Omega S^n) & p = 0, n \\ 0 & \text{otherwise} \end{cases}$$

We can immediately see that $E^2_{0,0} = \mathbb{Z}$, generated by the fundamental class $\sigma = [S^n] \in H_n(S^n)$. Since $d^r$ has bidegree $(-p, p-1)$, the only possible non-zero differential is

$$d^n : E^n_{0,q} \to E^n_{0,q+n-1}.$$ 

This map has to be an isomorphism (except when $q = -(n-1)$), since otherwise we would have non-zero classes in $E^n_{n+1} = E^\infty$, in contradiction with $H_n(PX) = 0$ for $n \neq 0$. Combining this with the formula for $E^2 = E^n$, we can deduce that

$$H_k(\Omega S^n) \cong \begin{cases} \mathbb{Z} & (n-1)|k \\ 0 & \text{otherwise}. \end{cases}$$

Indeed, this clear holds for $k \leq 0$. Assume for contradiction that it fails for some $k$ and pick the smallest such: then $H_k(\Omega S^n)$ and $H_{k-1}(\Omega S^n)$ would be non-isomorphic groups, and the differential $d^n : E^n_{n,k-1} \to E^n_{0,k}$ could not be an isomorphism, contradicting what we argued above.

If the generator in $E^n_{0,k(n-1)} = H_k(n-1)(\Omega S^n)$ is denoted $x_k$, then the generator of $E^n_{n,k(n-1)}$ is $\sigma \otimes x_k \in H_n(S^n) \otimes H_k(n-1)(\Omega S^n)$, and the differential is given by $d(\sigma \otimes x_k) = x_{k+1}$.

6.2. Homology of $SU(3)$. The quotient map $SU(3) \to SU(3)/SU(2)$ is a fiber bundle and hence a Serre fibration. The fibers are diffeomorphic to $SU(2) \cong S^3$ and the base can be identified with $S^3$ via the action of $SU(3)$ on $S^3 \subset \mathbb{C}^3$. The Serre spectral sequence now has

$$E^2_{p,q} = H_p(S^3) \otimes H_q(S^3) \Rightarrow H_{p+q}(SU(3)).$$

For degree reasons, all differentials vanish, and we see that

$$H_k(SU(3)) \cong \begin{cases} \mathbb{Z} & k = 0, 3, 5, 8 \\ 0 & \text{otherwise}. \end{cases}$$
6.3. **Trivial fibrations.** The projection $p : B \times F \to B$ is a Serre fibration. The spectral sequence has $E^2_{p,q} = H_p(B; H_q(F))$ and no differentials. In this case the spectral sequence contains essentially the same information as the Künneth theorem.

For an arbitrary Serre fibration $p : E \to B$ with fibers weakly equivalent to $F$ and $\pi_q(B, b)$ acting trivially on $H_*(F)$ for all $b \in B$, we see that the $E^2$ page is the same as for the trivial fibration. From this point of view, the differentials in the spectral sequence measure the failure of $p : E \to B$ being weakly equivalent to a trivial fibration.

6.4. **The Hopf fibration.** For $p : S^3 \to S^2$ with fibers $S^1$, we already know the homology of all three spaces. The spectral sequence has $E^2_{p,q} = H_p(S^2) \otimes H_q(S^1)$ and converges to $H_{p+q}(S^3)$. The only possible differential is $d^2_0 : E^2_{0,0} \to E^2_{1,1}$, which must be an isomorphism, since $E^3_{0,1} = E^\infty_{0,1}$ vanishes.

6.5. **Euler characteristic.** Let $p : E \to B$ be a Serre fibration with $B$ path connected, and write $F = p^{-1}(b)$ for some $b \in B$. Let $k$ a field and assume that $\pi_1(B)$ acts trivially on $H_*(F; k)$. If $H_*(B; k)$ and $H_*(F; k)$ are both finite dimensional vector spaces (where we write $H_* = \oplus_n H_n$), then $H_*(E; k)$ is also finite dimensional and

$$\chi_k(E) = \chi_k(B)\chi_k(F)$$

where $\chi_k(\cdot)$ is the usual alternating sum of the dimensions of $H_p(\cdot; k)$.

*Proof.* Under the assumption, the universal coefficient theorem applied to the $E^2$ page of the spectral sequence gives $E^2_{p,q} = H_p(B; k) \otimes H_q(E; k)$, and hence

$$\chi_k(B)\chi_k(F) = \left(\sum_p (-1)^p \dim H_p(B; k)\right)\left(\sum_q (-1)^q \dim H_q(F; k)\right)$$

$$= \sum_{p,q} (-1)^{p+q} \dim E^2_{p,q}$$

Since $d^r$ changes $p + q$ by $-1$, passing from $E^r$ to $E^{r+1}$ does not change the number $\sum_{p,q} (-1)^{p+q} \dim E^r_{p,q}$ and after finitely many steps all differentials must vanish, so $\chi_k(B)\chi_k(F) = \sum_{p,q} (-1)^{p+q} \dim E^\infty_{p,q}$.

On the other hand, there is a (non-canonical) isomorphism $H_*(E; k) \cong \bigoplus_{p+q = n} E^\infty_{p,q}$, proving that $H_*(E; k)$ is finite dimensional and the Euler characteristic is as claimed.

\[\square\]

7. ** Cohomology spectral sequences **

WARNING(1/24/16): Notes not updated from this point on. Notation likely not consistent with previous parts of the notes.

7.1. **Cohomology spectral sequence of a filtered space.** In the same sense as we can use a spectral sequence to "calculate" the homology of a filtered space $X$ from the groups $H_*(X_p, X_{p-1})$, there is a spectral sequence starting with $H^*(X_p, X_{p-1})$ and calculating $H^*(X)$, at least in good cases. Namely, we may form an exact couple with $A = \bigoplus_p H^{p,q}(X, X_p)$ and $E = \bigoplus_p H^{p+q}(X_p, X_{p-1})$, by summing the the long exact sequences in cohomology from the triples $(X, X_p, X_{p-1})$, over all $p$. The
degrees in the resulting spectral sequence work out a bit differently, which we will indicate by using upper indices, such as $E_1^{p,q}$, etc. The spectral sequence has

$$E_1^{p,q} = H^{p+q}(X_p, X_{p-1}),$$

the differential $d^1 : E_1^{p,q} \to E_1^{p+1,q}$ is the composition $H^{p+1}(X_p, X_{p-1}) \to H^{p+q-1}(X, X_p) \to H^{p+q+1}(X_{p+1}, X_p)$, and the map $i : H^{p+q}(X, X_p) \to H^{p+q}(X, X_{p-1})$ is induced by the inclusion $X_{p-1} \to X_p$. In general the differential $d' : E_r^{p,q} \to E_r^{p+r,q-r-1}$.

For this spectral sequence to converge to $H^*(X)$, we must impose slightly stronger conditions on the filtration than in the homological case (arising from the fact that inverse limits are not as well behaved as direct limits), viz. we shall make the following two assumptions

(i) The map

$$\lim_{p \to -\infty} H^*(X, X_p) \to H^*(X)$$

is an isomorphism. This is automatic if $X_{-1} = \emptyset$.

(ii) The maps $i^r : H^n(X, X_{p+r}) \to H^n(X, X_p)$ have $\cap \text{Im}(i^r) = 0$. This holds if $X = \bigcup X_p$ with the direct limit topology, and $H^n(X, X_{p+r})$ is independent of $r$ for large $r$.

In that case the exact couple satisfies (4.2), and the spectral sequence converges to $\lim H^*(X, X_p) = H^*(X)$. This means that each group $H^n(X)$ comes with a filtration $F^p = \text{Im}(H^n(X, X_p) \to H^n(X))$ which is now increasing: $F^p \supset F^{p+1} \supset \ldots$, and there is an isomorphism $E_\infty^{p,q} = F^p/F^{p+1}$.

7.2. Cohomology with local coefficients. We will use the cohomology spectral sequence of a filtered space to deduce a Serre spectral sequence in cohomology. Let us briefly discuss cochains of a space $X$ with coefficients in a coefficient system $A : \pi(X) \to \text{Ab}$. Cochains are defined as

$$C^n(X; A) = \prod_{\sigma : \Delta^n \to X} A_\sigma,$$

where the product is over all continuous maps and $A_\sigma$ is the group from § 1.5. (It would perhaps be more natural to define $A^\sigma$ as the colimit of the functor $\sigma^*: \pi(\Delta^n) \to \text{Ab}$ and work with that instead. The natural map from the colimit to the limit is an isomorphism in this case, so we shall not emphasize this distinction.)

To each face map $F_1 : \Delta^{n-1} \to \Delta^n$ we associated an isomorphism $d_1 : A_\sigma \cong A_{\sigma \circ F_1}$, and we shall denote its inverse by $d^1 : A_{\sigma \circ F_1} \to A_\sigma$. These assemble to homomorphisms $d^i : C^{p-1}(X; A) \to C^p(X; A)$ for $i = 0, \ldots, p$, and we shall define a coboundary as

$$\delta = \sum_{i=0}^p (-1)^i d^i : C^{p-1}(X; A) \to C^p(X; A).$$

7.3. The cohomology Serre spectral sequence. The cohomological Serre spectral sequence associated to a Serre fibration $p : E \to B$ calculates the cohomology of $E$ starting from the cohomology of $B$ with coefficients in the fibers. It is constructed similarly to the homology spectral sequence: When $B$ is a $\Delta$-complex we consider the spectral sequence associated to the filtration of $E$ by inverse images of skeleta of $B$, and for general $B$ we first pull back along $c : \Gamma B \to B$. 
Theorem 7.1. For any Serre fibration \( p : E \to B \) there is a spectral sequence with
\[
E_2^{p,q} = H^p(B; H^q),
\]
where \( H^q : \pi(B) \to \text{Ab} \) is the coefficient system \( b \mapsto H^q(p^{-1}(b)) \). The \( d_r \) differential is a homomorphism \( d_r : E_r^{p,q} \to E_r^{p+r,q-r-1} \). The spectral sequence has
\[
E_\infty^{p,q} = F^p H^{p+q}(E)/F^{p+1} H^{p+q}(E),
\]
where \( H^n(E) = F^0 H^n(E) \supset F^1 H^n(E) \supset \cdots \supset F^0 H^n(E) \supset F^{n+1} H^n(E) = 0 \) is a filtration of \( H^n(E) \).

The cohomology version of the Serre spectral sequence has several advantages over the homology version. As we shall see, the whole spectral sequence can be endowed with a product structure, which relates the cup products on \( H^*(E) \), \( H^*(B) \) and each \( H^*(p^{-1}(b)) \).

The fact that cohomology is contravariant and that the differentials go in the opposite direction is useful in itself, as we shall see.

8. Examples

8.1. The Euler class of a spherical fibration. A spherical fibration is a Serre fibration \( p : E \to B \) such that all fibres \( p^{-1}(b) \) are weakly equivalent to \( S^n \), and thus each \( H^n(p^{-1}(b)) \approx \mathbb{Z} \). An orientation of \( p \) is a “continuous” choice of isomorphism \( H^n(p^{-1}(b)) \approx \mathbb{Z} \), i.e. an isomorphism from \( H^n \) to the constant coefficient system \( \mathbb{Z} \).

These may or may not exist. An orientation gives an isomorphism \( E_2^{0,n} = H^0(B) \) and in particular \( E_2^{0,n} = H^0(B) \).

The Serre spectral sequence is then concentrated on the horizontal lines \( E_2^{0,0} \) and \( E_2^{2,n} \) and hence the only possible non-zero differentials are \( d_{n+1} : E_{n+1}^{0,n} \to E_{n+1}^{2,n+1} \).

We then let \( e(p) \in H^{n+1}(B) \) be the image of \( 1 \in E_2^{0,n} = H^0(B) \).

8.2. Pushforward in cohomology. Let \( p : E \to B \) be a fibration and assume that each \( p^{-1}(b) \) is weakly equivalent to an oriented closed \( n \)-manifold for each \( b \). Then \( H^n(p^{-1}(b)) \approx \mathbb{Z} \) for all \( b \), and we shall in addition assume given a “continuous choice” of such isomorphisms: that is, an isomorphism of coefficient systems \( H^n \approx \mathbb{Z} \).

We can then use the Serre spectral sequence to define a map
\[
p_* : H^{n+k}(E) \to H^k(B)
\]
in the following way. The fact that the groups \( E_2^{p,q} \) vanish for \( q > n \) imply firstly that the filtration on the group \( H^{k+n}(E) \) has \( F^0 H^{n+k}(E) = \cdots = F^k H^{n+k}(E) = H^{n+k}(E) \) and secondly that no differentials go into \( E_r^{k,n} \) and hence \( E_\infty^{k,n} \subset E_2^{k,n} \).

We may then define \( p_* \) as the composition
\[
H^{n+k}(E) \to H^{n+k}(E)/F^{k+1} H^{n+k}(E) = E_\infty^{k,n} \to E_2^{k,n} = H_k(B; \mathbb{Z}).
\]

8.3. Chern classes. We shall use the cohomological Serre spectral sequence to define the Chern classes of a complex vector bundle (at least up to a sign).

The homeomorphism \( U(k)/U(k-1) \approx S^{2k-1} \) proves that the inclusion \( U(k-1) \to U(k) \) is \( (2k-2) \)-connected. For \( n > k \), the inclusion \( S^{2k-1} = U(k)/U(k-1) \to U(n)/U(k-1) \) is then \( 2k \)-connected. It follows that \( U(n)/U(k-1) \) is \( (2k-2) \)-connected and \( H_{2k-1}(U(n)/U(k-1)) = \pi_{2k-1}(U(n)/U(k-1)) = \mathbb{Z} \) and hence \( H^{2k-1}(U(n)/U(k-1)) = \mathbb{Z} \).

Let \( V \to X \) be a complex \( n \)-dimensional vector bundle, and let \( P \to X \) be the corresponding principal \( U(n) \)-bundle. (I.e. \( P \) is the space of orthonormal frames in
V with respect to some chosen Hermitian metric.) This is a fiber bundle with fiber $U(n)$, and $p_k : P/U(k - 1) \to X$ is a fiber bundle with fiber $U(n)/U(k - 1)$. The spectral sequence has $E^{p,q}_2 = 0$ for $0 < q < 2k - 1$ and

$$E^{0,2k-1}_2 = H^0(X; H^{2k-1}(U(n)/U(k - 1))) = H^0(B) \otimes H^{2k-1}(U(n)/U(k - 1)).$$

If we let $\sigma \in H^{2k-1}(U(n)/U(k - 1)) \approx \mathbb{Z}$ denote a generator, we have the canonical element $1 \otimes \sigma \in E^{0,2k-1}_2$. There are no possible differentials coming into this group, and the first possible differential going out of it is $d_{2k} : E^{0,2k-1}_2 \to E^{2k,0}_2 = H^{2k}(X)$. We then set $c_k(V) = \pm d_{2k}(1 \otimes \sigma)$. (The sign depends on the choice of generator $\sigma$.)

8.4. Stiefel-Whitney classes. For a real vector bundle $V \to X$, the Stiefel-Whitney classes $w_k(V) \in H^k(X; \mathbb{F}_2)$ can be defined similarly to the Chern classes, using $O(n)$ in place of $U(n)$.

9. Products in spectral sequences

9.1. Definitions.

**Definition 9.1.** Let $R$ be a ring. A filtration $R \supset \cdots \supset F^n R \supset F^{n+1} R \supset \cdots$ is multiplicative if $(F^n R)(F^m R) \subset F^{n+m} R$. The associated graded

$$\bigoplus_{n \geq 0} F^n R/F^{n+1} R$$

then inherits the structure of a graded ring, where $(F^n R/F^{n+1} R) \otimes (F^m R/F^{m+1} R) \to F^{n+m} R/F^{n+m+1} R$ is defined as $[a][b] = [ab]$.

**Definition 9.2.** Let $R = \bigoplus R^{p,q}$ be a bigraded ring. A differential $d : R \to R$ is a derivation if $d(xy) = (dx)y + (-1)^{|x|}x(dy)$, where $|x| = p + q$ if $x \in R^{p,q}$. The homology groups $H^{p,q}(R,d) = \text{Ker}(d : R^{p,q} \to R^{p,q+1})/\text{Im}(d : R^{p,q} \to R^{p,q+1})$ then inherit a product, given by $[a][b] = [ab]$.

9.2. Cup products in the Serre spectral sequence. We have explained all ingredients in the following theorem, except that in (iv) we must explain how the cup products in the cohomology of the fibers and the base combine to a product on $H^p(B; H_q)$. We do this in a special case in Remark 9.4 below, and in full generality in Section 9.4.

**Theorem 9.3.** The cohomology Serre spectral sequence of a Serre fibration $p : E \to B$ admits a natural “cup product” map

$$E^{p,q}_r \otimes E^{p',q'}_r \to E^{p+p',q+q'}_r$$

with the following properties

(i) $E^{*,*}_r$ is bigraded ring for each $r \geq 2$.
(ii) $d_r : E_r \to E_r$ is a derivation
(iii) The isomorphism $E^{p,q+1}_r \cong H^{p,q}(E_r, d_r)$ is an isomorphism of bigraded rings.
(iv) The isomorphism $E^{2,q}_r \cong H^p(B; H^q)$ is an isomorphism of bigraded rings.
(v) The filtration of $H^*(E)$ is multiplicative.
(vi) The isomorphism $E^{0,q}_\infty \cong F^p H^{p+q}(E)/F^{p+1} H^{p+q}$ defines an isomorphism of bigraded rings.
Remark 9.4. In the special case where the coefficient systems \( b \mapsto H^q(p^{-1}(b)) \) are constant and the Ext term in the universal coefficient theorem vanishes, we have an additive isomorphism

\[
H^p(B) \otimes H^q(F) \to H^p(B; \mathcal{H}^q)
\]

(9.1)

where \( F \) is the fiber over some chosen basepoint. When added over all \( p, q \), the left hand side of this isomorphism is naturally a ring, whose product is given by \((x \otimes y)(x' \otimes y') = (-1)^{|x||y|} (xx') \otimes (yy')\). The isomorphism (9.1) thus induces a product on \( H^p(B; \mathcal{H}^q) \), and the meaning of statement (iv) of the theorem is now that the isomorphism to \( E_2^{p,q} \) is multiplicative.

9.3. Products in homology spectral sequences.

Theorem 9.5. Let \( X, Y \) and \( Z \) be filtered spaces, and let \( \mu : X \times Y \to Z \) be a filtered map (i.e. \( \mu(X_p \times Y_{p'}) \subset Z_{p+p'} \)). In the resulting spectral sequences there are products

\[
E^r_{p,q} \otimes E^r_{p',q'} \to E^r_{p+p',q+q'}
\]

such that \( d^r \) satisfies the Leibniz rule and the isomorphism \( E_\infty^{p,q} \cong F_p^p/F_{p-1} \) preserves products.

Corollary 9.6. Let \( \nu F \to \nu E \xrightarrow{\nu \mu} \nu B \) be Serre fibrations, \( \nu = 1, 2, 3 \) and let \( \nu\mathcal{H}_q : \pi(\nu B) \to \text{Ab} \) be the corresponding coefficient systems. Then a commutative diagram

\[
\begin{array}{ccc}
1E & \times & 2E \\
1p \times 2p & \mu & 3E \\
1B \times 2B & \mu & 3B
\end{array}
\]

induces a product on spectral sequences

\[
E^r_{p,q} \otimes E^r_{p',q'} \to E^r_{p+p',q+q'}
\]

satisfying the Leibniz rule, and such that the isomorphism \( E_\infty^{p,q} \cong F_p^p/F_{p-1} \) preserves products. On the \( E^2 \) terms, the product

\[
H_p(1B; \mathcal{H}_q) \otimes H_p(2B; \mathcal{H}_q) \to H_{p+p'}(3B; \mathcal{H}_{q+q'})
\]

agrees with the one induced by cross products in fibers and base.

A typical application would be to the case when \( p : E \to B \) is obtained by looping another Serre fibration. Then we can apply the corollary with all three fibrations being \( p : E \to B \), and the maps \( \mu \) being concatenation of loops.

9.4. Appendix: cup products with local coefficients in graded rings. The coefficient system \( \mathcal{H}^q : b \mapsto H^q(p^{-1}(b)) \) attached to the Serre fibration \( p : E \to B \) has a product induced from cup product, i.e. the direct sum \( \oplus_q \mathcal{H}^q \) lifts to a functor

\[
\mathcal{H}^* : \pi(B) \to \text{GRing},
\]

where \( \text{GRing} \) is the category of graded rings. In this section we explain why \( \mathcal{H}^*(X; \mathcal{R}^*) \) inherits a cup product for any functor \( \mathcal{R}^* : \pi(X) \to \text{GRing} \).
Definition 9.7. Let $X$ be a space, and $A: \pi(X) \to \text{Ab}$ and $B: \pi(X) \to \text{Ab}$ be coefficient systems. Let $\mathcal{A} \otimes \mathcal{B}: \pi(X) \to \text{Ab}$ be the coefficient system given on objects by $(\mathcal{A} \otimes \mathcal{B})(x) = (\mathcal{A}(x)) \otimes (\mathcal{B}(x))$. Let

$$C^p(X; A) \otimes C^{p'}(X; B) \xrightarrow{\cup} C^{p+p'}(X; A \otimes B) = \prod_{\sigma: \Delta^{p+p'} \to X} (A \otimes B)_\sigma$$

be the map whose $\sigma$th coordinate is the natural isomorphism

$$A_{\sigma[e_0, \ldots, e_p]} \otimes B_{\sigma[e_p, \ldots, e_{p+p'}]} \to (A \otimes B)_\sigma,$$

where $(\sigma[e_0, \ldots, e_p]) : \Delta^p \to X$ and $(\sigma[e_p, \ldots, e_{p+p'}]) : \Delta^{p'} \to X$ denote the restriction to the “front” and “back” faces of $\Delta^{p+p'}$. (Both sides of the isomorphism are canonically isomorphic to $\mathcal{A}(\sigma(e_p)) \otimes \mathcal{B}(\sigma(e_p))$.)

This construction is obviously natural in both $X$, $A$ and $B$. If $R: \pi(B) \to \text{Rings}$ is a coefficient system of rings, we therefore obtain a homomorphism

$$C^p(X; R) \otimes C^{p'}(X; R) \xrightarrow{\cup} C^{p+p'}(X; R \otimes R) \to C^{p+p'}(X; R),$$

where the last homomorphism is induced by the multiplication in $R$.

If $\text{GRing}$ denotes the category of graded rings, i.e. rings $R$ with a decomposition $R = \bigoplus_{p,q} R^{pq}$ such that $(R^p)^q \subseteq R^{pq}$, then for a coefficient system $R: \pi(X) \to \text{GRing}$ we shall define a product on $C^*(X; R) = \bigoplus_{p,q} C^p(X; R^{pq})$ as

$$C^p(X; R^q) \otimes C^{p'}(X; R^{q'}) \xrightarrow{\cup} C^{p+p'}(X; R^q \otimes R^{q'}) \to C^{p+p'}(X; R^{p+q'}),$$

where the last homomorphism is induced by the multiplication $R^q \otimes R^{q'} \to R^{p+q'}$ times the sign $(-1)^{p'q}$.

Remark 9.8. When $R = \bigoplus R^q$ is a constant coefficient system, the sign $(-1)^{p'q}$ makes natural maps $H^p(X) \otimes R^q \to H^{p+q}(X; R^q)$ given by $[\phi] \otimes r \mapsto [r\phi]$ assemble into a homomorphism of bigraded rings.

References


10. Examples

10.1. Rational cohomology of $K(\mathbb{Z}; n)$. We know that $K(\mathbb{Z}, 1) \simeq S^1$ and $K(\mathbb{Z}, 2) \simeq \mathbb{C}P^\infty$ and hence $H^*(K(\mathbb{Z}, 1); \mathbb{Q}) \cong \Lambda_2[x_1]$ with $|x_1| = 1$ and $H^*(K(\mathbb{Z}, 2); \mathbb{Q}) \cong \mathbb{Q}[x_2]$ with $|x_2| = 2$. Here $\Lambda_2[x]$ denotes the exterior algebra on a generator $x$ of odd degree.

Let us use the path-loop fibration to calculate $H^*(K(\mathbb{Z}, 3); \mathbb{Q})$. By the Hurewicz theorem and the universal coefficient theorem, we see that

$$H^k(K(\mathbb{Z}, 3); \mathbb{Q}) = \begin{cases} \mathbb{Q} & k = 0, 3 \\ 0 & k = 1, 2 \end{cases}$$

and we claim that $H^k(K(\mathbb{Z}, 3); \mathbb{Q}) = 0$ for $k > 3$. If this were not the case, there would exist some $z \neq 0 \in H^{k+1}(K(\mathbb{Z}, 3); \mathbb{Q})$ for $|z| > 3$. Let us pick such a $z$ with $|z|$ minimal.
The Serre spectral sequence in rational cohomology for the path-loop fibration has

\[ E_2^{p,q} = H^p(K(\mathbb{Z}, 3); H^q(\mathbb{C}P^\infty; \mathbb{Q})) \cong H^p(K(\mathbb{Z}, 3); \mathbb{Q}) \otimes H^q(K(\mathbb{Z}, 2); \mathbb{Q}) \]

For degree reasons, we must have \( d_2(1 \otimes \iota_2) = 0 \) and \( E_2^{3,0} = E_3^{3,0} = \mathbb{Q}, \iota_3 \otimes 1 \) and \( E_2^{3,2} = E_3^{3,2} = \mathbb{Q} \cdot 1 \otimes t_2 \). The only option consistent with the vanishing of \( E_\infty \) is that \( d_3(\iota_2 \otimes 1) = \lambda 1 \otimes \iota_3 \) for a non-zero \( \lambda \in \mathbb{Q} \). Writing \( x = \iota_3 \otimes 1 \) and \( y = 1 \otimes \iota_2 \), the fact that \( d_r \) is a derivation then implies that \( d_2(1 \otimes \iota_2^k) = d_2(y^k) = 2ky^{k-1}d_2(y) = 0 \), \( d_2(\iota_3 \otimes \iota_2^k) = d_2(xy^k) = 0 \) and

\[ d_3(1 \otimes \iota_2^k) = d_3(y^k) = ky^{k-1}d_3(y) = k\lambda y^{k-1}, \]

and hence \( d_3 : E_3^{3,2k+2} \rightarrow E_3^{3,2k} \) is an isomorphism for all \( k \geq 0 \). We have now completely understood the structure of \( (E_3^{p,q}, d_r) \) for all \( r \geq 2 \), all \( q \) and all \( p < |z| \).

The element \( z \otimes 1 \neq 0 \in E_2^{p,0} \) would have to be hit by a non-zero differential \( d_r \) for some \( r \), but that would violate the known structure of \( E_2^{p,q} \).

By a similar argument, we can show that \( H^*(K(\mathbb{Z}, n); \mathbb{Q}) \cong \Lambda q[t_n] \) for all odd \( n \) and \( H^*(K(\mathbb{Z}, n); \mathbb{Q}) \cong \mathbb{Q}[t_n] \) for all even \( n > 0 \).

10.2. Pushforward in cohomology. Elaborating on the example in Section 8.2, the multiplicativity of the spectral sequence implies that if the fibers of \( p : E \rightarrow B \) are closed oriented manifolds (with orientations varying “continuously” as in Section 8.2), then

\[ p_n(x \cup p^*(y)) = (p_n(x)) \cup y. \]

10.3. The Gysin sequence. Elaborating on the example in Section 8.1, we can see that under the assumptions in that section, the entire structure of the spectral sequence follows from the value of \( e(p) \in H^{n+1}(B) \). If we let \( \sigma \in H^n(p^{-1}(b)) \cong \mathbb{Z} \) denote the generator corresponding to \( 1 \in \mathbb{Z} \) under the chosen isomorphism, we have defined \( e(p) \) by \( d_{n+1}(1 \otimes \sigma) = (e(p)) \otimes 1 \). It then follows that \( d_{n+1}(x \otimes \sigma) = ((e(p)) \otimes 1)(x \otimes 1) = xe(p) \otimes 1 \). The filtration on \( H^*(E) \) has precisely two non-zero filtration quotients, and the convergence of the spectral sequence translates to the long exact sequence

\[ \ldots \rightarrow H^p(B) \xrightarrow{p^*} H^p(E) \xrightarrow{-\cdot e(p)} H^{p+n+1}(E) \xrightarrow{p_n} H^{p+1}(B) \rightarrow \ldots \]

10.4. Homology of the loop space of \( S^n \). This example uses products in the homology spectral sequence.

The map \( \Omega S^n \times \Omega S^n \rightarrow \Omega S^n \) given by concatenation of loops induces a product on \( H_*(\Omega S^n) \). Let’s use the Serre spectral sequence to calculate the ring structure of \( H_*(\Omega S^n) \). The map \( \Omega S^n \rightarrow \ast \) can be considered a fibration with fiber \( \Omega S^n \). Concatenation of paths gives a map \( \Omega S^n \times PS^n \) that fits in a commutative diagram

\[ \begin{array}{ccc} \Omega S^n \times PS^n & \longrightarrow & PS^n \\ \downarrow & & \downarrow \\ \ast \times S^n & \longrightarrow & S^n. \end{array} \]

The induced map on fibers is concatenation of loops. By the general theory, this diagram gives a product on spectral sequences \( ^1E^r \otimes ^2E^r \rightarrow ^3E^r \), where \( ^1E^r \) is the spectral sequence of the fibration \( \Omega S^n \rightarrow \Omega S^n \rightarrow \ast \), and both \( ^2E^r \rightarrow ^3E^r \) is the
It is not hard to see that the product \( x \) has \( E \). Thus, the induced product of spectral sequences in this case is theorem). Thus we have

\[
\pi_n \otimes \pi_n = \pi_{n-1}
\]

and we had names for the generators: If \( \sigma \in E^2_{n,0} = H_n(S^n) \) is the fundamental class, then we have generators

\[
x_1 = d\sigma \in E^2_{0,n-1} \quad \text{and} \quad x_1 = \sigma \in E^2_{n,0-1}
\]

\[
x_2 = d(\sigma \otimes x_1) \in E^2_{2n-2} \quad \text{and} \quad x_2 = \sigma \otimes x_2 \in E^2_{n,2}
\]

\[
x_3 = d(\sigma \otimes x_2) \in E^2_{3n-3} \quad \text{and} \quad x_3 = \sigma \otimes x_3 \in E^2_{n,3n-3}
\]

\[
\ldots
\]

Now, the Serre spectral sequence of the fibration \( \Omega S^n \to \Omega S^n \to * \) is very boring: It has \( E^0_{n,q} = H_q(\Omega S^n) \) for all \( q \) and \( E^p_{n,q} = 0 \) for \( p > 0 \), and all differentials vanish. Thus, the induced product of spectral sequences in this case is

\[ H_q(\Omega S^n) \otimes E^r_{p,q} \Rightarrow E^r_{p+q,q}. \]

It is not hard to see that the product \( x_k \times \sigma \in E^2_{n,n-1} \) is the same element that we previously denoted \( \sigma \otimes x_k \) (you have to trace through what the product on \( E^2 \) is, and check that it is the same as the one appearing in the universal coefficient theorem). Thus we have

\[
x_2 = d(x_1 \times \sigma) = (dx_1) \times \sigma \pm x_1 \times (d\sigma) = \pm x_1^2
\]

\[
x_3 = \pm d(x_1 \times \sigma) = (dx_1^2) \times \sigma \pm x_1 \times (d\sigma) = \pm x_1^3
\]

\[
\ldots
\]

\[
x_k = \pm x_1^k,
\]

where the power \( x_1^k \) is calculated with respect to the product on \( H_*(\Omega S^n) \) induced form concatenation of loops. Thus, we might as well use \( x_1 \in H_0(\Omega S^n) \) as a generator, and we have proved the following theorem.

**Theorem 10.1.** There is an isomorphism of rings

\[ Z[x] \to H_*(\Omega S^n), \]

where \( x \in H_{n-1}(\Omega S^n) \) is the image of the identity map under the Hurewicz homomorphism \( \pi_n(S^n) = \pi_{n-1}(\Omega S^n) \to H_{n-1}(\Omega S^n) \).

**10.5. Cohomology of the loop space of \( S^n \).** Let us calculate the ring \( H^*(\Omega S^n; Z) \) using the Serre spectral sequence of the path-loop fibration. This is very similar to the calculation of the homology algebra \( H_*(\Omega S^n) \).

The \( E_2 \) term is

\[
E^p_{0,q} = H^p(S^n; H^q(\Omega S^n)) = H^p(S^n) \otimes H^q(\Omega S^n) = \begin{cases} H^q(\Omega S^n) & p = 0, n \\ 0 & \text{otherwise} \end{cases}
\]

so the only possible differential is \( d_n : E^0_{n,n-1} \to E^1_{n-1,n-1} \). Since the \( E_\infty \) term vanishes except in degree \((0,0), d^n \) must give an isomorphism \( H^q(\Omega S^n) \to H^{q-n+1}(\Omega S^n) \).
By the same induction argument as for homology, we get

\[ H^q(\Omega S^n) = \begin{cases} \mathbb{Z} & (n-1)|q \\ 0 & \text{otherwise.} \end{cases} \]

Let \( y_k \in H^{k(n-1)}(\Omega S^n) \) and \( \sigma \in H^n(S^n) \) be generators. Then the product \( \sigma y_k \in E_2^{n,k(n-1)} = \mathbb{Z} \) is a generator. Since \( d_n : E_0^{n,k(n-1)} \to E_1^{n,(k-1)(n-1)} \) is an isomorphism, we can assume that \( d_n(y_k) = y_{k-1} \sigma \) (otherwise, change the signs of the \( y_k \)'s).

There are now two cases, depending on whether \( n \) is even or odd. Let us first treat the case where \( n \) is odd. In this case, all the elements \( y_k \) have even degree, so they commute strictly with all elements in the spectral sequence. In particular we get

\[ d_n(y_k^k) = ky_i^{k-1}d(y_1) = k(\sigma y_1^{k-1}). \]

Comparing this formula with \( d_n y_k = \sigma y_{k-1} \) it follows by induction that \( y_k^k = k! y_k \), and hence the ring structure is

\[ y_i y_k = \binom{k+i}{k} y_{k+i}. \]

The case where \( n \) is even is a little different. In this case \( y_1^2 = 0 \) by graded commutativity. Letting \( \sigma \) and \( y_k \) be additive generators as before, we first investigate multiplication by \( y_1 \). We can write \( y_1 y_k = \alpha(k)y_{k+1} \) for some function \( \alpha \). Differentiating gives

\[ \alpha(k) \sigma y_k = \alpha(k)d(y_{k+1}) = d(y_1 y_k) = \sigma y_k - y_1 \sigma y_{k-1} = (1 - \alpha(k-1)) \sigma y_k, \]

and hence \( \alpha(k) = 1 - \alpha(k-1) \). Therefore

\[ y_1 y_k = \begin{cases} y_{k+1} & k \text{ even} \\ 0 & k \text{ odd} \end{cases} \]

We now investigate the power \( y_2^k \). Since \( y_2 \) is even dimensional, it strictly commutes with everything.

\[ d(y_2^k) = ky_2^{k-1}d y_2 = ky_2^{k-1}y_1 \sigma. \]

If we write \( y_2^k = \alpha(k)y_{2k} \), this gives \( \alpha(k)d y_{2k} = k\alpha(k-1)y_{2(k-1)}y_1 \sigma \). On the other hand \( dy_{2k} = \sigma y_{2k-1} = \sigma y_1 y_{2(k-1)} \), so we get \( \alpha(k) = k\alpha(k-1) \) and hence \( \alpha(k) = k! \). From this it follows that the multiplication is given by

\[ y_{2k} y_{2l} = \binom{k+l}{k} y_{2k+2l} \]

\[ y_{2k+1} y_{2l} = y_{2k} y_{2l+1} = \binom{k+l}{k} y_{2k+2l+1} \]

\[ y_{2k+1} y_{2k+2} = 0 \]

11. Construction of multiplicative structures

The Serre spectral sequences in both homology and cohomology are special cases of the spectral sequence associated to any filtered chain complex. Let us explain a general setup in which such spectral sequences acquire a product.
11.1. **Multiplicatively filtered chain complexes.** Let us assume given three filtered chain complexes \( ^\nu C_n \), where \( \nu = 1, 2, 3 \), filtered by subcomplexes \( ^\nu C_n^p \), together with a homomorphism

\[
1^C_n \otimes 2^C_n \xrightarrow{\phi} 3^C_{n+n}
\]

which is a filtered chain map, i.e. it satisfies \( 1^C_n \otimes 2^C_n \subset 3^C_{n+n} \) and the Leibniz rule \( \partial(x \times y) = (\partial x) \times y + (-1)^{|x|} x \times \partial y \).

Out of this setup we can immediately get induced structures on associated objects. Firstly, the fact that \( \phi \) is filtered gives an induced map of quotient complexes

\[
(1^C_n \otimes 1^C_{n'}) \otimes (2^C_n \otimes 2^C_{n'}) \rightarrow (3^C_n \otimes 3^C_{n'})
\]

given by \([x] \times [y] = [x \times y] \). This is a chain map (i.e. satisfies a Leibniz rule), and hence induces a product between the homologies of the three filtration quotients.

Secondly, we can use first that \( \times \) is a chain map and get an induced product

\[
H_p(1^C_n) \otimes H_{p'}(2^C_n) \rightarrow H_{p+p'}(3^C_n),
\]

which is filtration preserving in the induced filtrations on \( H_*(^\nu C_n) \), and hence induces a product between the filtration quotients of the \( H_*(^\nu C_n) \).

In this setup, we have the following result.

**Proposition 11.1.**  
(i) The product \( 1^E_{p,q} \otimes 2^E_{p',q'} \rightarrow 3^E_{p+p',q+q} \) induced by (11.1) induces well defined products on subquotients

\[
1^E_{p,q} \otimes 2^E_{p',q'} \rightarrow 3^E_{p+p',q+q}
\]

for all \( r \) and \( r' \) satisfies the Leibnitz rule \( d'(e \times f) = (d' e) \times f + (-1)^r e \times (d' f) \) with respect to these. (Here \([e] = p + q \) if \( e \in E_{p,q} \)).

(ii) In the above situation, the product on \( E^1 \) induces a product

\[
1^E_{p,q} \otimes 2^E_{p',q'} \rightarrow 3^E_{p+p',q+q}.
\]

Letting \( ^\nu E_n^p \subset H_*(^\nu C_n) \) denote the \( p \)th filtration, the isomorphisms \( (\nu = 1, 2, 3) \)

\[
^\nu \varphi : ^\nu E_{p,q} \rightarrow ^\nu E_{p+q} / ^\nu E_{p+q-1}
\]

preserve products, i.e. satisfy \( \varphi(e \times f) = 1^\nu \varphi(e) \times 2^\nu \varphi(f) \), where the right hand side is given the product induced by (11.2).

**Proof of Proposition 11.1.** It suffices to prove the Leibniz rule for the product on each \( E^r \), since it will then follow inductively that the product on the next page is well defined. We will use the explicit formulas for \( d^r \) and for the \( E^r \) terms as subquotients of the \( E^1 \) terms.

Let \( e \in 1^E_{p,q} \) be in \( 1^Z \) i.e. \([e] \) represents an element in the subquotient \( 1^E^r \). In the exact couple, that means we can write \( ke = i^{r-1} a \) for some \( a \in H_{p+q-1}(C_{p+r-1}^\nu) \) and in that case \( d^r[e] = [ja] \). Let us write \( e = [x] \) for a representative

\[
x \in 1^C_{p+q}/(1^C_{p+q-1})
\]

and spell out how to find a chain level representative for \( d^r[e] \). Here is the recipe.

1. Lift \( x \) to \( \bar{x} \in 1^C_{p+q} \). Then \( \partial \bar{x} \in 1^C_{p+q-1} \), and in here, it is a cycle which represents \( k[e] \).
(2) Find elements $\overline{\alpha} \in 1^{p-r}C_{p+q-1}$ and $\alpha \in 1^{p-1}C_{p+q}$ such that
\[
\partial \overline{\alpha} = \overline{\alpha} + \partial \alpha.
\]

Then $\overline{\alpha}$ is a cycle which represents an element $a$ with $i^{-1}(a) = ke$.

(3) Let $\alpha \in 1^{p-r}C_{p+q-r}1^{p+q-1}$ be the reduction of $\overline{\alpha}$. This is a cycle which represents an element of $1^{p+1}E_{p-r,q+r-1}$ which in the quotient represents $d''[e]$.

Starting with an element $f \in 2E_{p',q'}1$ which is in $2Z'$, we use the same recipe to find $d'[f]$; write $f = [y]$ for a representative $y \in 2^{p'}C_{p'+q'}1^{p'/q'}$, pick a lift $\overline{y} \in 2^{p'+q'}$ and write
\[
\partial \overline{y} = \overline{\beta} + \partial \tau,
\]
for elements $\overline{\beta} \in 2^{p'-r}C_{p'+q-r-1}$ and $\tau \in 2^{p'-1}$.

Finally we use the recipe to calculate $d'[e \times f]$. Write $p' = p + p'$ and $q'' = q + q'$.

First, $x \times y \in 3^{p''}C_{p''+q''}1^{p''/q''}$, is a chain representing $e \times f$. Here are the steps in the recipe.

1. $x \times y$ lifts to $\overline{x} \otimes \overline{y} \in 3^{p''}C_{p''+q''}$.
2. We have
\[
\partial(\overline{x} \otimes \overline{y}) = (\partial \overline{x}) \otimes y + (-1)^{|x|} x \otimes (\partial \overline{y})
= (\overline{x} \otimes \partial \overline{y}) + (-1)^{|x|} x \otimes (\overline{\beta} + \partial \tau)
= (\overline{x} \otimes y) + (-1)^{|x|} x \otimes y + (\partial \overline{y}) \otimes (-1)^{|x|} x \otimes (\partial \tau)
= \overline{\gamma} + \partial \nu,
\]
where
\[
\nu = \sigma \otimes \overline{y} + (-1)^{|x|} x \otimes \tau \in 3^{p''}C_{p''+q''-1}
\]
and
\[
\overline{\gamma} = \overline{x} \otimes \overline{y} + (-1)^{|x|} x \otimes \overline{\beta} \in 3^{p''-r}C_{p''+q''-1-1}.
\]

3. Let $\gamma$ be the class of $\overline{\gamma}$ in the quotient modulo $3^{p''-r}C_{p''+q''-1}$. Then $\gamma = \alpha \times y + (-1)^{|x|} x \times \beta$ which in the homology groups $3E^1$ gives
\[
[\gamma] = [\alpha]f + (-1)^{|x|} e[\beta].
\]

In the subquotient $3E'$, the elements $[\alpha]$, $[\beta]$ and $[\gamma]$ become $d''[e]$, $d''[f]$ and $d''(e \times f)$ which is the desired Leibniz rule.

\[\square\]

Proof of Proposition 11.1.ii. That the product is defined on all $E''$ for $r < \infty$ means that
\[
1^{Z'} \times 2^{Z'} \subset 3^{Z'}
\]
\[
1^{B'} \times 2^{Z'} \subset 3^{B'}
\]
\[
1^{Z'} \times 2^{B'} \subset 3^{B'}.
\]
If these hold for all $r$, it is easy to see they hold for $r = \infty$, so the product on $E^1$ induces a product on $E^\infty$.

In the filtration on $H_*(C_*)$, $F^p$ is the image of $H_*(C^p_*)$. It is then clear that $(1^{F^p}) \times 2^{F^p} \subset 3^{F^{p+p'}}$. 


The definition of $\phi$: Starting with $e \in H_*(C_p^*/C_p^{p-1})$, lift to $H_*(C_p^*)$ and map it to $H_*(C_*)$. This obviously preserves products.

11.2. Application to the homology Serre spectral sequence. We can now prove Theorem 9.3. In that situation, the spectral sequence arises from the filtration of $C_*(X)$ is by the subcomplexes $C_*(X_n) \subset C_*(X)$ and similarly for $C_*(Y)$ and $C_*(Z)$. The chain level product is induced by the “cross product”

$$C_*(X) \otimes C_*(Y) \xrightarrow{\partial} C_*(X \times Y) \xrightarrow{\mu} C_*(Z),$$

and we check that it is multiplicative. Indeed, if $\sigma: \Delta^p \rightarrow X_n$ and $\tau: \Delta^{p'} \rightarrow Y_m$, then the cartesian product gives a map $\Delta^p \times \Delta^{p'} \rightarrow X_n \times Y_m \rightarrow Z_{n+m}$. The product of $\sigma$ and $\tau$ is the image of a universal cross product $\iota_p \times \iota_{p'} \in C_{p+p'}(\Delta^p \times \Delta^{p'})$, and hence lands in $C_*(Z_{p+p'})$.

11.3. Application to the cohomology Serre spectral sequence. This requires a bit more work. On the chain level, the cup product in the end comes down to the diagonal map $C_*(E) \rightarrow C_*(E \times E) \cong C_*(E) \otimes C_*(E)$ by dualizing. The problem is that it is not so easy to find a model in which this is a map of filtered chain complexes. The natural guess is that we should replace $B$ by a $\Delta$-complex and filter $C_*(E)$ by the chains on the inverse image of the skeleta of $B$, but that does not work without modifications.