NOTES ON SPECTRAL SEQUENCES

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1. Exact couples and spectral sequences

We define what spectral sequences are, and introduce the most important tool for constructing them, and give an example.

1.1. Spectral sequences.

Definition 1.1. A spectral sequence is a sequence \((E^r, d^r)_{r \geq r_0}\), where each \(E^r\) is an abelian group, \(d^r : E^r \to E^r\) is a homomorphism satisfying \(d^rd^r = 0\), and \(E^{r+1} \cong \text{Ker}(d^r)/\text{Im}(d^r)\) is a specified isomorphism. (The specified isomorphism is usually suppressed from the notation, and/or pretended to be an equality.)

Spectral sequences are the objects of a category: A morphism from a spectral sequence \((E^r, d^r)_{r \geq r_0}\) to a spectral sequence \((E'^r, d'^r)_{r \geq r_0}\) is a sequence of homomorphisms \(E^r \to E'^r\) commuting with the \(d^r\) and the specified isomorphisms \(E^{r+1} \cong \text{Ker}(d^r)/\text{Im}(d^r)\).

All interesting spectral sequences can be constructed by the method of exact couples, introduced by Massey ([1]).

1.2. Exact couples. An exact couple is a triple \((A, E, i, j, k)\), consisting of two abelian groups \(A\) and \(E\), and homomorphisms

\[
\begin{array}{ccc}
A & \xrightarrow{i} & A \\
\downarrow{k} & & \downarrow{j} \\
E & & 
\end{array}
\]

making the triangle exact at each vertex. Given this, the homomorphism \(d = jk : E \to E\) satisfies \(dd = jkjk = 0\) since \(kj = 0\).

The derived couple of \((A, E, i, j, k)\) is \((A', E', i', j', k')\), where \(E' = \text{Ker}(d)/\text{Im}(d)\), \(A' = iA\), \(i'(a) = ia\), \(j'(ia) = [ja]\), \(k'[e] = ke\). To see that \(j'\) is well defined, suppose \(ia = ib\): then exactness implies that \(b = a + ke\) for some \(e \in E\), and then
[jb] = [ja] + [jke] = [ja] + [de] = [ja]. To see that \( k' \) is well defined we first note that for \([e] \in E', de = jke = 0\), so \( ke \in \text{Ker}(j) = \text{Im}(i) = A'\). If \([e] = [e + df]\), then \( k(e + df) = ke + kjk(f) = ke \) since \( kj = 0\). We have defined the groups and maps in the triangle

\[
\begin{array}{ccc}
A' & \xrightarrow{i'} & A' \\
\downarrow{k'} & & \downarrow{j'} \\
E' & & \\
\end{array}
\]

**Theorem 1.2.** The derived couple of an exact couple is an exact couple.

**Proof.** We’ll prove the inclusions “\( \text{Im} \subset \text{Ker} \)” and “\( \text{Im} \supset \text{Ker} \)” thrice.

**Upper left corner.** \( \subset \): \( i'k'[e] = ike = 0 \). If \( a \in A' \) has \( i'a = ia0 \), then \( a = ke \) for some \( e \). Writing \( a = ib, de = jke = ja = jib = 0 \), so \([e] \in E'\) has \( k'[e] = ke = a \).

**Upper right corner.** \( \subset \): if \( a = ib \in A' \), \( j'i'(a) = j'(ia) = ja = jib = 0 \). If \( a = ib \in A' \) has \( j'(a) = [jb] = 0 \), then \( jb = de = jke \) for some \( e \in E \). Then \( j(b - ke) = 0 \), so \( b - ke \in \text{Ker}(j) = \text{Im}(i) \), so \( b = ke + ie \) for some \( e \in A \), and hence \( a = ib = i(ke + ie) = i^2(e) = \text{Im}(i') \).

**Lower corner.** \( \subset \): if \( a = ib \in A' \), \( k'j'(a) = k'[jb] = kjb = 0 \). If \( k'[e] = ke = 0 \) for \([e] \in E'\), then \( e \in \text{Ker}(k) = \text{Im}(j) \). Hence \( e = ja \) for some \( a \) and hence \([e] = [ja] = j'(ia) \). \( \Box \)

Iterating this construction, we get a sequence of exact couples

\[
\begin{array}{ccc}
A^r & \xrightarrow{i_r} & A^r \\
\downarrow{k_r} & & \downarrow{j_r} \\
E^r & & \\
\end{array}
\]

which for \( r = r_0 \) is the one we started with, and where each exact couple is the derived couple of the previous one. The choice of \( r_0 \) is just a convention, but typically \( r_0 \) is either 2, 1 or 0.

1.3. **The spectral sequence associated to an exact couple.** For an exact couple \((E, A, i, j, k)\) we may form all iterated derived couples. The resulting \((E^r, d^r)\) is a spectral sequence with \( E^{r_0} = E, d^{r_0} = jk \).

In applications, the groups \( A \) and \( E \) are usually (bi)graded, and the maps \( i, j, k \) have some fixed degree. (More generally, these could be objects and morphisms in any abelian category.) By the formulas for \( i', j' \) and \( k' \), it is clear that \( \deg(i') = \deg(i), \deg(k') = \deg(k) \), and that \( \deg(j') = \deg(j) - \deg(i) \). We get that

\[
\begin{align*}
\deg(d'^o) &= \deg(j) + \deg(k) \\
\deg(d^r) &= \deg(d'^o) - (r - r_0) \deg(i) \\
&= \deg(j) + \deg(k) - (r - r_0) \deg(i).
\end{align*}
\]

1.4. **Example: the spectral sequence of a filtered space.** A *filtered space* is a space \( X \) with specified subspaces \( X_p \subset X, p \in \mathbb{Z} \) such that \( X_p \subset X_{p+1} \) is a closed subspace and \( \lim X_p \to X \) is a homeomorphism.

For such \( X \), we may set

\[
\begin{align*}
A_{p,q} &= H_{p+q}(X_p) \\
E_{p,q} &= H_{p+q}(X_p, X_{p-1}).
\end{align*}
\]
let \( i : A \to A \) be induced by the inclusions \( X_{p-1} \to X_p \), let \( j : A \to E \) be induced by the maps of pairs \((X_p, \emptyset) \to (X_p, X_{p-1})\), and let \( k : E \to A \) be the connecting homomorphisms \( H_{p+q}(X_p, X_{p-1}) \to H_{p+q-1}(X_{p-1}) \). We see that in this case

\[
\begin{align*}
\deg(i) &= (1, -1) \\
\deg(j) &= (0, 0) \\
\deg(k) &= (-1, 0).
\end{align*}
\]

We get a bigraded spectral sequence with

\[
\deg(d') = (-r, r - 1).
\]

The \( E^1 \) term is given by \( E^1_{p,q} = H_{p+q}(X_p, X_{p-1}) \), and the differential \( d^1 \) is the composition

\[
H_{p+q}(X_p, X_{p-1}) \to H_{p+q-1}(X_{p-1}) \to H_{p+q-1}(X_{p-1}, X_{p-2}).
\]

Remark 1.3. If \( X \) is a CW complex and \( X^p \) is the \( p \)-skeleton, we see that \( E^3_{p,q} = 0 \) for \( q \neq 0 \) and that \( (E^1_{p,0}, d^1) \) is the cellular chain complex. It follows that \( E^2_{p,q} = 0 \) for \( q \neq 0 \) and that \( E^2_{p,0} = H_p(X) \). The bidegree of \( d^r \) is \((-r, r - 1)\), so for \( r \geq 2 \) we must by induction have \( E^r_{p,q} = 0 \) for \( q \neq 0 \) and \( d^r = 0 \).

The spectral sequence for a general filtered space should be thought of as attempting to “calculate” \( H_*(X) \) starting from \( H_*(X^p, X^{p-1}) \). We shall return to that that means.

Remark 1.4. This need not be a “first quadrant” spectral sequence. In general, all we can say is that \( E^1_{p,q} = 0 \) when \( p + q < 0 \).

References


2. Pages of the spectral sequence

The groups \( E^r \) are called the pages (or terms) of the spectral sequence. We give an explicit formula for the pages of the spectral sequence associated to an exact couple \((A, E, i, j, k)\) (as opposed to the inductive formula used to define it).

2.1. Iterated subquotients. Recall that a subquotient of an abelian group \( G \) is a quotient of a subgroup.

“A subquotient of a subquotient is a subquotient”: If \( A \subset B \subset G \), and \( A' \subset B' \subset B/A \), then \( B'/A' \) is canonically isomorphic to \( B''/A'' \), where \( A'' \subset G \) and \( B'' \subset G \) are the unique subgroups containing \( A \) such that \( (A''/A) = A' \) and \( B''/A = B' \).

(The isomorphism sends \([b] \in B''/A''\) to \([bA] \in B'/A'\).)

In the derived couple of an exact couple, \( E' = \ker(d)/\text{im}(d) \) is a subquotient of \( E \), so iteratively \( E^r \) is a subquotient of \( E \). Thus, there are canonical subgroups and isomorphisms

\[
B^r \subset Z^r \subset E \quad E^r \cong Z^r/B^r
\]

The meaning of \( Z^r \) can best be explained if we introduce some (slight) abuse of notation: An element \( e \in E \) with \( de = 0 \) represents an element \([e] \in E^r \). If also \( d^r[e] = 0 \in E^r \), then \([e] \) represents an element which should properly be written
[(e)] ∈ E'. In this case we will write just [e] instead of [(e)]. Confusion between the two meanings of [e] can be avoided by emphasizing the group: [e] ∈ E' or [e] ∈ E''. Similarly, starting with an element e ∈ E, we could potentially get an element of E' which should be written with (r − 1) pairs of brackets. We will again write just [e].

Then, start with e ∈ E^1, we are allowed to write [e] ∈ E^2 if d^2[e] = 0. If also d^3[e] = 0 we are allowed to write [e] ∈ E^3. Etc. Then Z^r is the subset consisting of e ∈ E for which we are allowed to write [e] ∈ E^r. In that case e ∈ B^r precisely if [e] = 0 ∈ E^r. Equivalently, [e] = d^{r−1}[x] for some x ∈ Z^{r−1}.

2.2. Formula for the rth page. The following lemma gives explicit formulas for B^r ⊂ Z^r ⊂ E, and also for the maps j_r : A^r → E^r and k_r : E^r → A^r in terms of the identification E^r = Z^r/B^r. Let us assume that we start with the zeroth, i.e. r_0 = 0. (Otherwise, replace r by r − r_0 in suitable places).

**Lemma 2.1.** In the rth derived couple of an exact couple (A, E, i, j, k), we have

\[ A^r = \text{Im}(i^r) \]
\[ Z^r = k^{-1}(\text{Im}(i^r)) \]
\[ B^r = j(\text{Ker}(i^r)). \]

With respect to the canonical isomorphism E^r = Z^r/B^r, we have

\[ i_r(a) = ia \]
\[ j_r(i^r a) = [ja] \]
\[ k_r[e] = ke. \]

**Proof.** The statement is tautological for r = 0, and we proceed by induction. Assume the lemma is proved for r and let d^r = j_r k_r. Given an element e ∈ E with ke = i^r a (i.e. e ∈ Z^r), we have

\[ d^r[e] = j_r k_r[e] = j_r i^r a = [ja] \in Z^r/B^r \]

If ke = i^{r+1} b, then d^{r−1}[e] = [jb] = 0. Conversely, assume ke = i^r a and that d^r[e] = [ja] = 0. Using the induction hypothesis we get ja = jb for some b ∈ Ker(i^r). Then a − b ∈ Ker(j) = \text{Im}(i), so a = b + ix, and we have

\[ ke = i^r a = i^r (b + ix) = i^{r+1} x. \]

and we have proved the formula for Z^{r+1}.

B^{r+1} is the set of b ∈ E such that [b] = d^r[e] for some [e] ∈ E^r = Z^r/B^r. Writing again ke = i^r a we have d^r[e] = [ja] and hence

\[ b − ja ∈ B^r = j(\text{Ker}(i^r)), \]

so b = j(a + x) with i^r x = 0. Then

\[ i^{r+1}(a + x) = i^{r+1} a = ike = 0, \]

so b ∈ j(\text{Ker}(i^{r+1})). Conversely, if b = ja with i^{r+1} a = 0, then i^r a ∈ Ker(i) = \text{Im}(k), so we can write i^r a = ke and then d^r[e] = [ja] = [b]. This proves the induction formula for B^r.

The formula for i_{r+1} is obvious, and j_{r+1} is easy: j_{r+1}(i(i^r a)) = [j_r i^r a] = [ja], where the first equality is the definition of derived couple and the second is the induction hypothesis. The formula for k_r is also easy: k_{r+1}[e] = k_r[e] = ke where
the first equality is the definition of derived couple and the second is the induction hypothesis.

2.3. Filtered spaces. In the spectral sequence of a filtered space (for which we will set \( r_0 = 1 \)), the element

\[ e \in E^1_{p,q} = H_{p+q}(X_p, X_{p-1}) \]

is in \( Z^r \) if its image in \( H_{p+q-1}(X_{p-1}) \) comes from \( H_{p+q-1}(X_p) \). The resulting class \([e] \in E^r_{p,q}\) is zero if \( e \) comes from an element in \( H_{p+q}(X_p) \) which vanishes in \( H_{p+r}(X_{p+r-1}) \).

We shall study in what sense the spectral sequence “calculates” \( H_*(X) \). This phenomenon is called convergence, and under suitable assumptions the spectral sequence will converge to \( H_*(X) \). We discuss convergence in more generality.

3. Convergence

We define the \( E^\infty \) page of a spectral sequence, and explain how the \( E^\infty \) page of the spectral sequence of an exact couple is related to the map \( i : A \to A \).

3.1. Simplest case: first quadrant spectral sequences. A bigraded spectral sequence \( (E^r_{p,q}, d^r) \) where \( d^r \) has bidegree \((-r, r-1)\) is said to be first quadrant if \( E^r_{p,q} = 0 \) for unless \( p \geq 0 \) and \( q \geq 0 \). For such spectral sequences, if we fix any \((p, q)\) both the differential \( E^r_{p,q} \to E^r_{p-r,q+r-1} \) and \( E^r_{p+r,q-r+1} \to E^r_{p,q} \) must be zero, as long as \( r \geq \max(p+1, q+2) \), and hence \( E^r_{p,q} = E^{r+1}_{p,q} = \ldots \). This common value is denoted \( E^\infty_{p,q} \).

As we’ll see, it makes sense to talk about \( E^\infty \) even without this stabilization phenomenon (and even without gradings), by taking an appropriate “limit” of the groups \( E^r \) as \( r \to \infty \).

3.2. The \( E^\infty \) page of a spectral sequence. The abelian group \( E \) has subgroups \( Z^r, B^r \) arranged as follows

\[
0 = B^1 \subset B^2 \subset \cdots \subset B^r \subset \cdots \subset Z^1 \subset Z^2 \subset Z^3 = E,
\]

and \( E^r = Z^r/B^r \) is the quotient of a group that becomes smaller and smaller, by a group that becomes bigger and bigger. Thus we may define

\[
Z^\infty = \bigcap_r Z^r, \quad B^\infty = \bigcup_r B^r, \quad E^\infty = Z^\infty/B^\infty.
\]

This agrees with the definition in the bigraded first-quadrant case, where the sequences \( Z^r_{p,q} \) and \( B^r_{p,q} \) stabilize for fixed \( p,q \).

Lemma 2.1 immediately gives a completely general formula for \( E^\infty \), namely

\[
E^\infty = \frac{k^{-1}(\cap_r \text{Im}(i^r))}{j(\cup_r \text{Ker}(i^r))}
\]

In this generality the formula is of little use, but under additional assumptions we can rewrite it in a more useful way.
3.3. $E^\infty$ for exact couples. Let us rewrite the formula (3.1) under the additional assumption on the map $i : A \to A$ that

$$(3.2) \quad \ker(i) \cap \bigcap_r i^r A = 0$$

i.e. no non-zero element $a \in A$ has $ia = 0$ and can be written as $a = i^r b_r$ for arbitrarily large $r$. In most examples it will even be true that $\cap_r i^r A = 0$, for example in the exact couple associated to a filtered spaces this holds if $X_{-1} = 0$.

The direct limit $A^\infty = A[i^{-1}] = A \oplus \bigoplus Z[i, i^{-1}] = \lim(A \xrightarrow{j} A \xrightarrow{j} \ldots)$ is filtered in the following way. Let us write $F \subset A^\infty$ for the image of the natural map $A \to A^\infty$. Then we have subgroups $i^r F \subset A^\infty$ for every $r \in \mathbb{Z}$, giving a filtration

$$\ldots \subset i^{r+1} F \subset i^r F \subset \ldots$$

which is exhaustive in the sense that $A^\infty = \cup_r i^r F$. The filtration quotients $i^r F/i^{r+1} F, \ r \in \mathbb{Z}$, are all isomorphic to $F/iF$, since $i : A^\infty \to A^\infty$ is an isomorphism.

**Proposition 3.1.** The spectral sequence associated to an exact couple $(A, E, i, j, k)$ satisfying condition (3.2) has a natural isomorphism

$$E^\infty \cong F/iF.$$

**Proof.** Assuming (3.2), we have

$$E^\infty = \frac{\ker(k)}{j(\cup_r \ker(i^r))} = \frac{j(A)}{j(\cup_r \ker(i^r))}.$$  

The map $j : A \to E$ induces an isomorphism $A/iA \cong jA \subset E$ which descends to an isomorphisms

$$A/iA + \cup_r \ker(i^r) \cong E^\infty.$$  

On the other hand, $F$ is the image of the map $A \to A^\infty$ whose kernel is precisely $\cup_r \ker(i^r)$, so we get an isomorphism $A/(\cup_r \ker(i^r)) \cong F$, compatible with multiplication with $i$. We can therefore mod out by $iA/(\cup_r \ker(i^r)) \cong iF$ on both sides to get the desired isomorphism $E^\infty \cong F/iF$. \qed

The isomorphism of the proposition is very explicit: It is the composition

$$E \xrightarrow{j} A \to A^\infty.$$

of $j^{-1}$ (restricted to $Z^\infty$) with the canonical map $A \to \lim A = A^\infty$.

3.4. The spectral sequence of a filtered space, $E^\infty$ page. Let us calculate the $E^\infty$ page of the spectral sequence of a filtered space using Proposition 3.1. The assumptions imply that any compact subset of $X$ is contained in some $X_p \subset X$ which implies that $\lim C_*(X_p) \cong C_*(X)$ and in turn that the natural map

$$\lim H_n(X_p) \to H_n(X)$$

is an isomorphism. Let us further assume that $X_{-1} = 0$, or more generally that (3.2) holds. Recall that we have $A_{p,q} = H_{p+q}(X_p)$ and that $i : A_{p,q} \to A_{p+1,q-1}$ is the map induced from the inclusion $X_p \to X_{p+1}$. Then the group $A^\infty_{p,q}$ is the direct limit of the system

$$H_{p+q}(X_p) \to H_{p+q}(X_{p+1}) \to \ldots,$$
A spectral sequence converges to $H$ spectral sequence with maps $\partial$ with maps $F\subset C$.

If we further assume that $X$ be interpreted in the diagram

$\partial^p \colon F^p H_{p+q}(X) \to F^{p+1} H_{p+q}(X)$.

In symbols we often write

$E^1_{p,q} = H_{p+q}(X_p, X_{p-1}) \Rightarrow H_{p+q}(X)$,

meaning that we are considering a spectral sequence with the specified $E^1$ page, and where the $E^\infty$ page is the filtration quotients in a filtration of $H_\ast(X)$. In words, we say that the spectral sequence “converges” to $H_\ast(X)$. Note that this spectral sequence need not be first quadrant. If $X_1 = \emptyset$, we see that $E^1_{p,q}$ vanishes unless $p \geq 0$ and $q \geq -p$, so the $E^1$ page can occupy 37.5% of the plane in this case.

Let us briefly discuss more explicitly what the spectral sequence does between $E^r$ and $E^\infty$. That an element $x \in E^r_{p,q}$ is in $Z^r$ means that $d^r x = 0$, $d^r [x] = 0$, ..., $d^{r-1} [x] = 0$, so $x$ defines an element in $E^r$. The formula $Z^r = k^{-1}(\text{Im}(d^{r-1}))$ can be interpreted in the diagram

$$
\begin{array}{cccc}
H_{p+q}(X_p, X_{p-r}) & \longrightarrow & H_{p+q}(X_p, X_{p-1}) & \longrightarrow & H_{p+q-1}(X_{p-1}, X_{p-r}) \\
\downarrow & & \downarrow k & & \downarrow \\
H_{p+q-1}(X_{p-r}) & \longrightarrow & H_{p+q-1}(X_{p-1}) & \longrightarrow & H_{p+q-1}(X_{p-1}, X_{p-r})
\end{array}
$$

with exact rows. It follows from the diagram that $Z^r_{p,q} \subset E^1_{p,q}$ consists of the elements $x \in H_{p+q}(X_p, X_{p-1})$ that come from $H_{p+q}(X_p, X_{p-r})$. By a similar argument, $B^r_{p,q}$ is the group of elements $x \in H_{p+q}(X_p, X_{p-1})$ that vanish when mapped to $H_{p+q}(X_{p+r-1}, X_{p-1})$.

$Z^\infty$ is the elements that come from $H_{p+q}(X_p)$, and the isomorphism $E^\infty_{p,q} \to F^p H_{p+q}(X) / F^{p-1} H_{p+q}(X)$ is given by lifting to $H_{p+q}(X)$ and mapping to $H_{p+q}(X)$.

3.5. Spectral sequences of filtered chain complexes and double complexes.

**Definition 3.2.** A filtered chain complex $C_\ast = (C_\ast, \partial)$ equipped with subcomplexes $C^n_\ast \subset C^{n+1}_\ast \subset \cdots \subset C_\ast$, such that $C_\ast = \lim_{\to n} C^n_\ast$.

In the same way as for filtered spaces, a filtered chain complex gives rise to a spectral sequence with

$E^2_{p,q} = H_{p+q}(C^p_\ast, C^{p-1}_\ast)$.

If we further assume that $C^{p-1}_\ast = 0$ (or more generally that (3.2) holds) then the spectral sequence converges to $H_{p+q}(C_\ast)$.

**Definition 3.3.** A double complex is a bigraded abelian group $C = \bigoplus C_{p,q}$ together with maps $\partial' : C_{p,q} \to C_{p-1,q}$ and $\partial'' : C_{p,q} \to C_{p,q-1}$ satisfying $(\partial')^2 = (\partial'')^2 = \partial' \partial'' + \partial'' \partial' = 0$.

The associated total complex has $T_\ast = \bigoplus_{p+q=n} C_{p,q}$ with boundary map $\partial = \partial' + \partial''$.

The total complex is canonically filtered by subcomplexes defined by

$$
T^m_n = \bigoplus_{n-q=p \leq m} C_{p,q} \subset T_n.
$$
This filtered chain complex gives rise to a spectral sequence with
\[
E^2_{p,q} = H_p(H_q(C_{*,*}, \partial'), \partial'),
\]
i.e. first compute “homology in the q-direction”, then compute homology with respect to the map induced by \( \partial' \). The exact couple automatically satisfies (3.2), so the spectral sequence converges to homology of the total complex.

Reversing the meaning of \( p \) and \( q \) gives another spectral sequence
\[
E^2_{p,q} = H_q(H_p(C_{*,*}, \partial'), \partial') \Rightarrow H_{p+q}(T_*, \partial).
\]
The two spectral sequences converge to the same groups, but of course the two filtrations on \( H_{p+q}(T_*, \partial) \) will likely be quite different.

3.6. Appendix: Detecting isomorphisms.

**Definition 3.4.** Let \( G \) be an abelian group equipped with a filtration \( G \supset \cdots \supset G^n \supset G^{n-1} \supset \cdots \). The filtration is exhaustive if \( \cup_n G^n = G \), Hausdorff if \( \cap_n G^n = 0 \), and complete if \( G \to \lim_{\leftarrow} G/F^n \) is surjective.

The reason for the latter two words comes from the fact that the filtration induces a topology on \( G \) with basis \( g + F^n \), and the topology is Hausdorff if and only if the filtration is. If the filtration is also exhaustive, the topology comes from a metric \( d(g, g') = (\max\{n | g - g' \in F^n\})^{-1} \), and the inverse limit \( G^n = \lim_{\to} G/F^n \) is precisely the completion of \( G \) as a metric space.

Let us briefly discuss the question of reconstructing an abelian group \( G \) with a filtration \( G \supset \cdots \supset G^n \supset G^{n-1} \supset \cdots \), from its associated graded
\[
\text{Gr}^F(G) = \bigoplus_{n \in \mathbb{Z}} F^n/F^{n-1}.
\]
In general it is of course not possible to reconstruct \( G \) from the \( F^n/F^{n-1} \), even up to isomorphism. Let us ask a weaker and more reasonable question: given a homomorphism \( \phi : G \to \overline{G} \) of filtered groups (i.e. \( \phi(F^n) \subset F^n \)), for which the induced maps of associated graded is an isomorphism (i.e. \( F^n/F^{n-1} \to F^n/F^{n-1} \) is an isomorphism for all \( n \)), can we deduce that \( \phi \) is an isomorphism?

A (counter-) example to keep in mind: filter the \( p \)-adic integers \( \mathbb{Z}_p \) by the ideals \( p^n \mathbb{Z}_p \) and filter \( \mathbb{Z} \subset \mathbb{Z}_p \) by the ideals \( p^n \mathbb{Z} \). The inclusion \( \mathbb{Z} \to \mathbb{Z}_p \) induces an isomorphism on filtration quotients and both filtrations are exhaustive and Hausdorff. The filtration on \( \mathbb{Z}_p \) is complete, but the one on \( \mathbb{Z} \) is not (the completion is exactly \( \mathbb{Z}_p \)).

For a homomorphism \( G \to \overline{G} \) as above, we may use the 5-lemma to deduce that \( \phi \) induces isomorphisms \( F^n/F^{n-2} \to F^n/F^{n-2} \) for all \( n \), and inductively that \( F^n/F^m \to F^n/F^m \) is an isomorphism for all \( m \leq n \). If the filtrations are exhaustive, we may take the direct limit \( n \to \infty \) to see that \( \phi \) induces an isomorphism \( G/F^m \to \overline{G}/F^m \) for all \( m \). If the filtrations are complete and Hausdorff, we may then take inverse limits and deduce that the original \( \phi \) is indeed an isomorphism.

It is natural to assume a condition slightly stronger on \( i : A \to A \) than (3.2). To the filtration \( A \supset iA \supset i^2A \supset \cdots \) we may associate the completion
\[
A \to A^\wedge = \lim A/i^n A,
\]
and it is convenient to replace condition (3.2) by the stronger condition that (3.3) be an isomorphism. The kernel of (3.3) is \( \cap i^n A \), so injectivity already implies (3.2),
and therefore an isomorphism between the $E^\infty$ page of the spectral sequence and the filtration quotients of $A^\infty$.

The stronger assumption that (3.3) be an isomorphism can now be phrased as the filtration of $A$ by the $i^rA$, $r \geq 0$ being complete and Hausdorff (it is automatically exhaustive). This implies that the filtration of $A^\infty$ by the $i^rF$, $r \in \mathbb{Z}$ is complete and Hausdorff. The spectral sequence may then be used to detect isomorphisms: if a map of exact couples induces an isomorphism of $E^\infty$ pages then it also induces an isomorphism between the filtered objects $A^\infty$.

If for example $X = \cup X_p$ and $Y = \cup Y_p$ are filtered spaces and $f : X \to Y$ is a filtered map (i.e. $f(X_p) \subset Y_p$), such that $f_* : H_*(X_p, X_{p-1}) \to H_*(Y_p, Y_{p-1})$ is an isomorphism for all $p$, then the induced map of spectral sequences is an isomorphism on the $E^1$ page, hence inductively on all $E^r$ pages, and hence on the $E^\infty$ page. If the filtrations of $H_*(X)$ and $H_*(Y)$ are complete and Hausdorff, we may deduce that $f_* : H_*(X) \to H_*(Y)$ is an isomorphism. (If $X_{-1} = \emptyset$ then the filtration of $H_*(X)$ is complete and Hausdorff for trivial reasons.)

### References


### 4. Homology with local coefficients

In order to construct the Serre spectral sequence, it is helpful to first review a slight generalization of singular homology. The singular chains $C_n(X; A)$ of a space $X$ with coefficients in a group $A$ consists of formal linear combinations of maps $\sigma : \Delta^n \to X$ with coefficients in the group $A$. It is useful to consider more general linear combinations, where the coefficients depend on where in $X$ you are.

#### 4.1. Local coefficients.

Recall that the *fundamental groupoid* of a space $X$ has the points of $X$ as objects, and the morphisms $x \to y$ is the set of paths $\gamma : I \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$, up to homotopy rel $\partial I$. Composition is concatenation of paths. Let us denote this category as $\pi(X)$. A continuous map $f : X \to Y$ induces a functor $\pi(f) : \pi(X) \to \pi(Y)$.

**Definition 4.1.** A *coefficient system* on $X$ is a functor $A : \pi(X) \to \text{Ab}$, where $\text{Ab}$ is the category of abelian groups.

Coefficient systems are the objects of a category, whose morphisms are the natural transformations.

If $f : X \to Y$ is continuous and $A : \pi(Y) \to \text{Ab}$ is a coefficient system on $Y$, then we may pull it back to $f^*A : \pi(X) \to \text{Ab}$, defined by precomposing with $\pi(f)$. Let us give a few examples of naturally occurring coefficient systems.

**Example 4.2.** If $A$ is an abelian group, then we have the constant functor $A : \pi(X) \to \text{Ab}$ which takes all objects to $A$ and all morphisms to the identity map of $A$.

**Example 4.3.** A Serre fibration $p : E \to B$ induces for each integer $q \geq 0$ a coefficient system $\mathcal{H}_q : \pi(B) \to \text{Ab}$ given on objects as $\mathcal{H}_q(b) = H_q(p^{-1}(b))$. For a path $\gamma : I \to B$, the inclusions $p^{-1}(\gamma(0)) \to \gamma^* E \to p^{-1}(\gamma(1))$
are both weak equivalences, and hence induce isomorphisms \((i_0)_*\) and \((i_1)_*\) in \(\mathcal{H}_p\).

We then send the morphisms \([\gamma]\) in \(\pi(B)\) to the isomorphism \(H_q(i_1) \circ (H_q(i_0))^{-1} : \mathcal{H}_q(\gamma_0) \to \mathcal{H}_q(\gamma(1))\).

### 4.2. Path connected based spaces.

If \(X\) is path connected and \(x \in X\), then the category of coefficient systems on \(X\) is equivalent to the category of modules over the group ring \(\mathbb{Z}[\pi_1(X, x)]\). Indeed, the group \(\pi_1(X, x)\), regarded as a groupoid with one object, is a full subcategory of \(\pi(X)\), and modules over \(\mathbb{Z}[\pi_1(X, x)]\) are the same thing as functors \(\pi_1(X, x) \to \text{Ab}\). The inclusion functor \(\pi_1(X, x) \to \pi(X)\) induces a functor from the category of functors \(\pi(X) \to \text{Ab}\) to the category of functors \(\pi_1(X, x) \to \text{Ab}\). If \(X\) is path connected then the inclusion functor \(\pi_1(X, x) \to \pi(X)\) is an equivalence of categories, and hence induces an equivalence between the functor categories.

For a functor \(\mathcal{A} : \pi(X) \to \text{Ab}\), the associated \(\mathbb{Z}[\pi_1(X, x)]\)-module is obtained by restriction. In the other direction, a right \(\mathbb{Z}[\pi_1(X, x)]\)-module \(M\) gives rise to a functor \(\pi(X) \to \text{Ab}\) which on objects is given by

\[
y \mapsto M \otimes_{\mathbb{Z}[\pi_1(X, x)]} \mathbb{Z}[\pi(X)(x, y)],
\]

where \(\mathbb{Z}[\pi(X)(x, y)]\) is the free abelian group generated by the set \(\pi(X)(x, y)\).

In older literature, a coefficient systems is sometimes defined to be a \(\mathbb{Z}[\pi_1(X, x)]\)-module. The definition we have given avoids an unnatural choice of base points and hence has nicer functoriality properties. It also works better for non-connected spaces.

### 4.3. Singular chains with local coefficients.

To a coefficient system \(\mathcal{A} : \pi(X) \to \text{Ab}\) and a continuous map \(\sigma : \Delta^p \to X\), we shall associate an abelian group \(\mathcal{A}_\sigma\) as the limit (in the sense of category theory) of the functor \(\sigma^*\mathcal{A} : \pi(\Delta^p) \to \text{Ab}\). More explicitly, the group \(\mathcal{A}_\sigma\) is a subgroup

\[
\mathcal{A}_\sigma \subset \prod_{t \in \Delta^p} \mathcal{A}(\sigma(t)),
\]

namely the subgroup consisting those \(x_t\) such that for any \(\gamma : I \to \Delta^p\), the isomorphism \(\mathcal{A}([\sigma \circ \gamma]) : \mathcal{A}(\sigma \circ \gamma(0)) \to \mathcal{A}(\sigma \circ \gamma(1))\) sends \(x_{\sigma \circ \gamma(0)}\) to \(x_{\sigma \circ \gamma(1)}\).

The projection \(\mathcal{A}_\sigma \to \mathcal{A}(\sigma(t))\) is an isomorphism for all \(t \in \Delta^p\), and it is convenient to think of \(\mathcal{A}_\sigma\) as the “common value” of the isomorphic groups \(\mathcal{A}(\sigma(t))\), \(t \in \Delta^p\).

Any continuous map \(F : \Delta^m \to \Delta^n\) induces an isomorphism \(F^* : \mathcal{A}_\sigma \to \mathcal{A}_{\sigma \circ F}\). For the \(i\)th face map \(F_i : \Delta^{n-1} \to \Delta^n\), this isomorphism shall be written \(d_i : \mathcal{A}_\sigma \to \mathcal{A}_{\sigma \circ F_i}\). We now define the twisted singular chains of a space \(X\) with coefficients in \(\mathcal{A}\) as the direct sum

\[
C_p(X; \mathcal{A}) = \bigoplus_{\sigma : \Delta^p \to X} \mathcal{A}_\sigma,
\]

define \(d_i : C_p(X; \mathcal{A}) \to C_{p-1}(X; \mathcal{A})\) for \(i = 0, \ldots, p\) as the sum of the maps \(d_i : \mathcal{A}_{\sigma} \to \mathcal{A}_{\sigma \circ F_i}\), and define the boundary operator as

\[
\partial = \sum_{i=0}^{p} (-1)^i d_i : C_p(X; \mathcal{A}) \to C_{p-1}(X; \mathcal{A}).
\]

### 4.4. Homology with local coefficients.

The usual proof shows that \(\partial \partial = 0\) and \(H_*(X; \mathcal{A})\) is then defined as the homology of this chain complex. If \(\mathcal{A} = \mathcal{A} : \pi(X) \to \text{Ab}\) is constant, this agrees with the usual definition of the chain complex \((C_*(X; \mathcal{A}), \partial)\) and singular homology \(H_*(X; \mathcal{A})\).
Many of the usual constructions and theorems about singular homology have analogues for local coefficients. Here are some (which can be proved by modifications of the usual proofs).

- A morphism of coefficient systems $\mathcal{A} \to \mathcal{B}$ induces a map of singular chains and in turn homology $H_*(X; \mathcal{A}) \to H_*(X; \mathcal{B})$. If $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ is a short exact sequence of coefficient systems (i.e. short exact when evaluated at any $x \in X$), then there is an induced short exact sequence of chains and hence long exact sequence in homology with twisted coefficients.

- Given a continuous map $f : X \to Y$ and a coefficient system $\mathcal{A} : \pi(Y) \to \text{Ab}$, there is an induced coefficient system $f^* \mathcal{A} : \pi(X) \to \text{Ab}$ and an induced homomorphism $H_*(X; f^* \mathcal{A}) \to H_*(Y; \mathcal{A})$.

- If $i : Y \to X$ is the inclusion of a subspace and $\mathcal{A} : \pi(X) \to \text{Ab}$ is a coefficient system, we define $C_*(X, Y; \mathcal{A})$ as the cokernel of the induced injection of chain complexes $i_* : C_*(Y; i^* \mathcal{A}) \to C_*(X; \mathcal{A})$. Its homotopy is denoted $H_*(X, Y; \mathcal{A})$ and sits in a long exact sequence with $H_*(X; \mathcal{A})$ and $H_*(Y, i^* \mathcal{A})$.

- If $Z \subset Y$ is a subspace whose closure is contained in the interior of $X$, then the appropriate generalization of excision holds: if $i : X \setminus Z \to X$ denotes the inclusion, then $H_n(X \setminus Z, Y \setminus Z; i^* \mathcal{A}) \to H_n(X, Y; \mathcal{A})$ is an isomorphism.

- If $X$ is a $\Delta$-complex, then the twisted singular chains have a subcomplex of twisted simplicial chains (where the map $\sigma$ is required to be the characteristic map of a simplex in $X$), and the inclusion of this into all singular chains $C_*(X; \mathcal{A})$ induces an isomorphism on homology.

- If $f : X \to Y$ is a weak equivalence and $\mathcal{A} : \pi(Y) \to \text{Ab}$ is a coefficient system, then $f_* : H_*(X; f^* \mathcal{A}) \to H_*(Y; \mathcal{A})$ is an isomorphism.

Some warnings:

- If $f, g : X \to Y$ are homotopic and $\mathcal{A} : \pi(Y) \to \text{Ab}$ is a coefficient system, we cannot conclude that the induced maps $f_*$ and $g_*$ in homology with local coefficients are equal. The reason for this is that $f_*$ and $g_*$ don’t even have the same domain!

- Even if $Y \subset X$ is a nice subspace, excision does not imply that we can replace twisted relative homology of $(X, Y)$ with that of the quotient $X/Y$. The reason is that $\mathcal{A}$ need not induce a local system on $X/Y$.

4.5. Appendix: Detecting weak equivalences. If $X$ is a space and $x \in X$, we have a “representable” coefficient system $\mathcal{A}_x : \pi(X) \to \text{Ab}$ given on objects by

$$\mathcal{A}_x(y) = \mathbb{Z}[\pi(X)(x, y)],$$

the free abelian group generated by the set $\pi(X)(x, y)$. Let us ask for an interpretation of $H_*(X; \mathcal{A}_x)$.

If $f : E \to X$ is a covering space with $E$ simply connected and $x \in f(E)$, then for each $e \in E$ there is a preferred element in $\pi(x, f(e))$, represented by a path which lifts to $Y$. This defines a morphism of coefficient systems $\mathbb{Z} \to \mathcal{A}_x$, and in turn a chain map

$$C_*(E; \mathbb{Z}) \to C_*(X; \mathcal{A}_x).$$

It is not hard to verify that this is in fact an isomorphism of chain complexes (exercise!), and hence that $H_*(X; \mathcal{A}_x) \cong H_*(E)$. In words, $H_*(X; \mathcal{A}_x)$ is the homology of the universal cover of the path component of $X$ containing $x$. 


Theorem 4.4. If \( f : X \to Y \) is any map of spaces, then \( f \) is a weak equivalence if and only if \( \pi(X) \to \pi(Y) \) is an equivalence of categories and \( H_\ast(X; f^*A) \to H_\ast(X; A) \) is an isomorphism for all coefficient systems \( A : \pi(Y) \to \text{Ab} \).

Proof sketch. For the "if" direction, it suffices to consider \( X \) and \( Y \) path connected, (otherwise, restrict attention to one path component at a time). Without loss of generality \( X \) and \( Y \) admit simply connected covering spaces \( \tilde{X} \) and \( \tilde{Y} \) (otherwise, replace by CW approximations) and by the Hurewicz theorem \( f \) is a weak equivalence if and only if the lifted map \( \tilde{X} \to \tilde{Y} \) induces an isomorphism in integral homology. As explained above, the integral homology of a simply connected covering space is a special case of homology with local coefficients. \( \square \)

5. The Serre spectral sequence

Recall that in Example 4.3 we associated to a Serre fibration \( p : E \to B \) a coefficient system \( \mathcal{H}_q \) given on objects by \( b \mapsto H_\ast(p^{-1}(b)) \). We will construct the homology Serre spectral sequence in the following form.

Theorem 5.1. For any Serre fibration \( p : E \to B \) there is naturally a spectral sequence with
\[
E^2_{p,q} = H_p(B; \mathcal{H}_q)
\]
where \( \mathcal{H}_q : \pi(B) \to \text{Ab} \) is the coefficient system \( b \mapsto H_\ast(p^{-1}(b)) \). The spectral sequence has
\[
E^\infty_{p,q} = F^pH_{p+q}(E)/F^{p-1}H_{p+q}(E),
\]
where \( F^pH_n(E) \) is a filtration \( 0 = F^{-1}H_n(E) \subset F^0H_n(E) \subset F^1H_n(E) \subset \cdots \subset F^pH_n(E) = H_n(E) \).

This is the main result about the (homological) Serre spectral sequence, and is often summarized as "\( E^2_{p,q} = H_p(B; \mathcal{H}_q) \Rightarrow H_{p+q}(E) \)". In the special case where \( B \) is path connected, and for some \( b \in B \) we write \( F = p^{-1}(b) \) and assume that the action of \( \pi_1(B, b) \) on \( H_\ast(F) \) is trivial, the coefficient system is equivalent to the constant system \( H_\ast(F) \). In this case we get the following corollary.

Corollary 5.2. Let \( p : E \to B \) be a Serre fibration with \( B \) path connected and let \( F = p^{-1}(b) \) for some \( b \in B \). Assume that the action of \( \pi_1(B, b) \) on \( H_\ast(F) \) is trivial (for example, if \( B \) is simply connected). Then there is a spectral sequence with
\[
E^2_{p,q} = H_p(B; H_q(F)) \Rightarrow H_{p+q}(E).
\]

We now embark on the construction of the spectral sequence.

5.1. The coefficient system associated to a Serre fibration. As explained in Example 4.3, a Serre fibration \( p : E \to B \) gives rise to a coefficient system \( \mathcal{H}_q : \pi(B) \to \text{Ab} \) given on objects by \( b \mapsto H_\ast(B) \). For later use, we shall spell out the definition of the chain complex \( C_\ast(B; \mathcal{H}_q) \) in this case.

For a continuous map \( \sigma : \Delta^p \to B \) we shall write \( E_\sigma = \sigma^*E \to \Delta^p \) for the pullback of \( p \) along \( \sigma \). For any \( t \in \Delta^p \), the contractibility of \( \Delta^p \) together with the long exact sequence in homotopy groups show that the inclusion \( p^{-1}(\sigma(t)) \to E_\sigma \) is a weak equivalence, and hence an isomorphism \( H_\ast(p^{-1}(\sigma(t))) \cong H_\ast(E_\sigma) \). This isomorphism is compatible with the isomorphisms induced by paths in \( \Delta^p \), so in other words we have constructed a natural isomorphism from \( \mathcal{H}_q : \pi(\Delta^p) \to \text{Ab} \) to
the constant functor \( H_q(E_\sigma) \), and in turn an isomorphism \( (H_q)_\sigma \cong H_q(E_\sigma) \). This proves the isomorphism

\[
(5.1) \quad C_p(B; \mathcal{H}_q) = \bigoplus_{\sigma : \Delta^p \to B} H_q(E_\sigma),
\]

where the sum is over all continuous maps, and \( \partial = \sum (-1)^i d_i : C_p(B; \mathcal{H}_q) \to C_{p-1}(B; \mathcal{H}_q) \), where \( d_i : H_q(E_\sigma) \to H_q(E_{\sigma \circ F_i}) \) is the isomorphism inverse to the isomorphism induced by the weak equivalence \( E_{\sigma \circ F_i} \hookrightarrow E_\sigma \).

5.2. The Serre spectral sequence: \( E^1 \) page and convergence. Let \( p : E \to B \) be a Serre fibration and assume for the moment that \( B \) is a \( \Delta \)-complex. The Serre spectral sequence is a special case of the spectral sequence of a filtered space, namely we filter \( E \) by \( E_n = p^{-1}(B^{(n)}) \), the inverse images of the skeleta of \( B \). This gives a spectral sequence with

\[ E^1_{p,q} = H_{p+q}(E_p, E_{p-1}). \]

For each simplex \( \sigma : \Delta^p \to B \), we write \( E_\sigma \) for the total space of the pullback fibration \( \sigma^*(E) \to \Delta^p \) and \( E_{\sigma\partial} \) for the restriction of \( E_\sigma \) to \( \partial \Delta^p \). For each \( p \)-simplex \( \sigma \) there is a natural map of pairs \( (E_\sigma, E_{\sigma\partial}) \to (E_p, E_{p-1}) \), and it is not hard to see (using excision and the LES a couple of times) that these maps induce isomorphisms

\[
(5.2) \quad \bigoplus_{\sigma : \Delta^p \to B} H_{p+q}(E_\sigma, E_{\sigma\partial}) \cong H_{p+q} \left( \coprod_{\sigma} E_\sigma, \coprod_{\sigma} E_{\sigma\partial} \right) = E^1_{p,q},
\]

where the sum is over all \( \sigma : \Delta^p \to B \) in the \( \Delta \)-complex structure on \( B \). We have the diagram

\[
\begin{array}{ccc}
E_\sigma & \xrightarrow{(1,p)} & E_\sigma \times \Delta^p \\
p \downarrow & & \downarrow \text{proj} \\
\Delta^p & \xrightarrow{\cong} & \Delta^p
\end{array}
\]

in which the top horizontal map is obviously a homotopy equivalence, and the vertical maps are both Serre fibrations. The LES in homotopy groups implies that the induced maps of fibers is a weak equivalence, and then using it again implies that the top horizontal map restricts to a weak equivalence (incl, \( p \)) : \( E_{\sigma\partial} \to E_\sigma \times \partial \Delta^p \), giving an isomorphism \( H_{p+q}(E_\sigma, E_{\sigma\partial}) \cong H_{p+q}(E_\sigma \times \Delta^p, E_\sigma \times \partial \Delta^p) \). The identity map of \( \Delta^p \) represents a generator \( \iota \in H_p(\Delta^p, \partial \Delta^p) \cong \mathbb{Z} \), and the cross product with \( \iota \) gives the isomorphisms

\[
H_q(E_\sigma) \cong H_q(E_\sigma) \otimes H_p(\Delta^p, \partial \Delta^p) \cong H_{p+q}(E_\sigma \times \Delta^p, E_\sigma \times \partial \Delta^p)
\]

and we arrive at an isomorphism

\[
(5.3) \quad E^1_{p,q} \cong \bigoplus_{\sigma : \Delta^p \to B} H_q(E_\sigma),
\]

where the sum is over characteristic maps for \( p \)-simplices of the \( \Delta \)-complex \( B \).

The filtration of \( E \) by the \( E_n \) clearly satisfies that any \( \sigma : \Delta^p \to E \) has image in \( E_n \) for some finite \( n \), so \( H_*(E) \cong \lim H_*(E_n) \) and our general results about the spectral sequence for filtered spaces implies that the spectral sequence converges.
to \( H_*(E) \): Writing \( F^p_q \subset H_*(E) \) for the image of \( H_*(X_p) \to H_*(X) \), we have established the isomorphism

\[
E^\infty_{p,q} = F^p_q/F^p_{p+1}.
\]

5.3. The first differential and the \( E^2 \) page. To the Serre fibration \( p : E \to B, B \) a \( \Delta \)-complex, we associated a spectral sequence and a natural isomorphism \( E^1_{p,q} \cong \oplus \sigma H_q(E_\sigma) \). The differential \( E^1_{p,q} \to E^1_{p-1,q} \) therefore corresponds to a map

\[
d^1 : \bigoplus \sigma H_q(E_\sigma) \to \bigoplus \tau H_q(E_\tau).
\]

We want to describe this more explicitly. To state the result, we first define the candidate answer. Let \( d : \oplus \sigma H_q(E_\sigma) \to \oplus \tau H_q(E_\tau) \) be the homomorphism given by the matrix whose \((\sigma, \tau)\)th entry is 0 unless \( \tau \) is a face of \( \sigma \); if \( \tau \) is the \( i \)th face of \( \sigma \), then the \((\sigma, \tau)\)th entry of \( d \) is \((-1)^i \) times the isomorphism \( H_q(E_\sigma) \cong H_q(E_\tau) \) induced by the inclusion \( E_\tau \to E_\sigma \). It is clear that \( d^1 \) and \( d \) are both natural transformations.

**Lemma 5.3.** With respect to the isomorphism \( E^1_{p,q} = \oplus \sigma H_q(E_\sigma) \) described above, the maps \( d^1 \) and \( d \) agree.

**Proof.** By naturality, it suffices to check the case where \( B = \Delta^p \) and \( p : E \to \Delta^p \) is some Serre fibration (I learned this trick from notes by Dan Dugger). We can further reduce to the case where \( E = F \times \Delta^p \) and \( p \) is the projection by applying naturality to the map \((1,p) : E \to E \times \Delta^p \): it is a fiberwise map of Serre fibrations and induces an isomorphism on \( E^1 \) pages of the spectral sequence.

In this case we have the commutative diagram

\[
\begin{array}{ccc}
H_q(F) \otimes H_p(\Delta^p, \partial \Delta^p) & \xrightarrow{\times} & H_{p+q}(E_p, E_{p-1}) \\
\downarrow & & \downarrow k \\
H_q(F) \otimes H_{p-1}(\partial \Delta^p) & \xrightarrow{\times} & H_{p+q-1}(E_{p-1}) \\
\downarrow & & \downarrow j \\
H_q(F) \otimes H_{p-1}(\partial \Delta^p, (\Delta^p)^{p-2}) & \xrightarrow{\times} & H_{(p-1)+q}(E_{p-1}, E_{p-2}) \\
\cong & & \cong \\
H_q(F) \otimes \bigoplus \tau H_{p-1}(\Delta^{p-1}, \partial \Delta^{p-1}) & \xrightarrow{\times} & \bigoplus \tau H_{(p-1)+q}(E_{\tau}, E_{\partial \tau}),
\end{array}
\]

where all horizontal maps are induced by the cross product. The composition of the vertical maps on the right is \( d^1 \) (by definition), and on the left the composition is the identity on \( H_q(F) \) tensored with the sequence

\[
H_p(\Delta^p, \partial \Delta^p) \to H_{p-1}(\partial \Delta^p) \to H_{p-1}(\partial \Delta^p, (\Delta^p)^{p-2}) \cong \bigoplus \tau H_{p-1}(\Delta^{p-1}, \partial \Delta^{p-1}),
\]

which looks like \( Z = Z \to \oplus \tau Z \to \oplus \tau Z \). Since the face maps \( F_1 : \Delta^{p-1} \to \partial \Delta^p \) are orientation preserving if and only if \( i \) is even, we see that the resulting map \( Z \to \oplus \tau Z \) is 1 \( \leftrightarrow (1, -1, 1, -1, \ldots) \), and hence the left hand side of the diagram above is precisely \( d \). 

\( \square \)
5.4. **A canonical CW approximation.** We now construct the Serre spectral sequence for an arbitrary Serre fibration $p : E \to B$, i.e. with no assumption on $B$. The main trick is to find a $\Delta$-complex $B'$ and a weak equivalence $f : B' \to B$. Then the pull back $f^* E \to B'$ is again a Serre fibration, with the same fibers as $p$, and the long exact sequence in homotopy groups implies that $f^* E \to E$ is a weak equivalence. There are several ways to construct such a $B'$, but here’s a canonical one (which often appears in the literature, under the name “the realization of the total singular complex”, denoted $|\text{Sin}_n(B)|$). The construction is very simple, in a certain sense the $B'$ that we construct is the universal $\Delta$-complex mapping to $B$.

**Construction 5.4.** For an arbitrary topological space $B$, let $\Gamma B$ be the $\Delta$-complex with one $n$-cell for each continuous map $\sigma : \Delta^n \to B$. The $i$th face of an $n$-cell $\sigma$ is glued to the $(n-1)$-cell labeled by the continuous map $\sigma \circ F_i : \Delta^{n-1} \to \Delta^n \to B$.

In other words, $\Gamma B$ is the space
\[
\Gamma B = \left( \prod_{n \geq 0} \text{Map}(\Delta^n, B) \times \Delta^n \right) / \sim,
\]
where $\text{Map}(\Delta^n, B)$ is the set of continuous maps (regarded as a set), and $\sim$ is the equivalence relation generated by $(\sigma \circ F_i, t) \sim (\sigma, F_i(t))$.

**Lemma 5.5.** The natural map $c : \Gamma B \to B$ is a weak equivalence.

One proof uses homology with local coefficients: First prove that the map induces an equivalence of fundamental groupoids. Then prove that it induces an isomorphism in homology with any local coefficients, by observing that the cellular chains of $\Gamma B$ are isomorphic to the singular chains of $B$. Let us also give a direct proof, not using homology and the Hurewicz theorem.

**Proof.** Let $\Gamma^n B$ denote the $n$-skeleton of $\Gamma B$. We prove that the restriction $\Gamma^n B \to B$ is $n$-connected for all $n$. This is clear for $n = 0$, and we suppose inductively that $\Gamma^{n-1} B \to B$ is $(n-1)$-connected. Then $\Gamma^n B$ is obtained from $\Gamma^{n-1} B$ by attaching one $n$-cell for each continuous map $\Delta^n \to B$, and we need to show that $\pi_{n-1}(\Gamma^n B) \to \pi_{n-1}(B)$ is injective and that $\pi_n(\Gamma^n B) \to \pi_n(B)$ is surjective. To see injectivity, it suffices to prove that for any map $\partial \Delta^n \to \Gamma^{n-1} B$ such that the composition $\partial \Delta^n \to \Gamma^{n-1} B \to B$ is null homotopic, there exists an extension to $\Delta^n \to \Gamma^n B$. (Cellular approximation implies that any element in the kernel is represented by such a map.) This gives the diagram
\[
\begin{array}{ccc}
\partial \Delta^n & \longrightarrow & \Gamma^{n-1} B \\
\downarrow & & \downarrow \\
\Delta^n & \longrightarrow & B.
\end{array}
\]

We first consider the case where the original map $\partial \Delta^n \to \Gamma^{n-1} B$ is simplicial. Then the null homotopy $\Delta^n \to B$ corresponds to an $n$-cell $\Delta^n \to \Gamma^n B$ whose boundary is a map $\partial \Delta^n \to \Gamma^{n-1} B \subset \Gamma^n B$ homotopic to the one we started with (they’re equal if the original map sends $p$-simplices to $p$-simplices). In the general case we start with a map $\partial \Delta^n \to \Gamma^{n-1} B$ which need not be simplicial. By the simplicial approximation theorem we can assume it is simplicial after possibly subdividing $\Delta^n$ sufficiently many times. This gives a triangulation of $\Delta^n$, and by induction there is no obstruction to extending the map to a simplicial map from the $(n-1)$-skeleton
into $\Gamma^{n-1}B$. Then the previous step shows how to extend to a map from each $n$-cell.

Surjectivity is similar (but easier).

Proof of Theorem 5.1. Any Serre fibration $p : E \to B$ may be pulled back to a Serre fibration $c^*E \to \Gamma B$, and the LES in homotopy groups imply that $c^*E \to E$ is a weak equivalence. Since $\Gamma B$ is a $\Delta$-complex, we have constructed a spectral sequence converging to $H_*(c^*E) \cong H_*(E)$, with $E^1$ page given by

$$E^1_{p,q} = \bigoplus_{\sigma : \Delta^p \to B} H_q(E_\sigma),$$

where the direct sum is now over all continuous maps $\sigma : \Delta^p \to B$, the first differential is again given by Lemma 5.3, and the $E^\infty$ page is given by precisely the subquotients of $H_*(c^*E) \cong H_*(E)$ appearing in Theorem 5.1. Comparing with (5.1), we see that the chain complex $(E_*^{\infty}, d^1)$ is isomorphic to $C_*(B;\mathcal{H}_q)$, inducing an isomorphism $E^2_{p,q} = H_p(B;\mathcal{H}_q)$ as claimed.

Remark 5.6. With the same proof, we could start with a coefficient system $\mathcal{A} : \pi(E) \to \text{Ab}$. Restricting this to each fiber gives a local system $\mathcal{H}_q(\mathcal{A})$ on $B$ whose value at $b$ is $H_q(p^{-1}(b);\mathcal{A})$, and we get a spectral sequence with

$$E^2_{p,q} = H_p(B;\mathcal{H}_q(\mathcal{A})) \Rightarrow H_{p+q}(E;\mathcal{A}).$$

In particular, $\mathcal{A}$ could be the constant system $\mathcal{A}$ for some abelian group $A$, and we get a spectral sequence converging to $H_*(E;A)$.

In the special case where $A = k$ is a field, $B$ is path connected and $\pi_1(B,b)$ acts trivially on $H_*F(k)$, where $b \in B$ is a point and $F = p^{-1}(b)$, then the coefficient system is isomorphic to the constant coefficient system $H_*(B;\mathcal{H}_q)$. Since $k$ is a field, we may use the universal coefficient system to rewrite the $E^2$ page as

$$E^2_{p,q} = H_p(B;k) \otimes_k H_q(F;k).$$

Over a field, the filtration $0 \subset F^0_n \subset \cdots \subset F^n_n = H_n(E;k)$ is automatically split. Thus we may pick isomorphisms $F^p \cong F^q/F^{p-1} \oplus F^{p-1}$ and inductively get a non-canonical isomorphism $H_n(E) \cong \bigoplus_{p+q=n} E^\infty_{p,q}$. (Over $\mathbb{Z}$ we may make a similar conclusion provided the abelian groups $E^\infty_{p,q}$ are free.)

6. Examples

6.1. The loop space of a sphere. Let’s calculate the homology of $\Omega S^n$ using the Serre spectral sequence of the fibration $\Omega S^n \to PS^n \to S^n$. We have

$$E^2_{p,q} = H_p(S^n;\mathcal{H}_q(\Omega S^n)) \Rightarrow H_{p+q}(PS^n).$$

We can calculate the $E^2$ term using the universal coefficient theorem. Since $H_p(S^n)$ has no torsion, the Tor term vanishes, and we have an isomorphism

$$E^2_{p,q} = H_p(S^n) \otimes H_q(\Omega S^n) = \begin{cases} H_q(\Omega S^n) & p = 0, n \\ 0 & \text{otherwise}. \end{cases}$$

We can immediately see that $E^2_{n,0} = \mathbb{Z}$, generated by the fundamental class $\sigma = [S^n] \in H_q(S^n)$. Since $d^r$ has bidegree $(-p, p-1)$, the only possible non-zero differential is

$$d^n : E^n_{n,q} \to E^n_{0,q+n-1}.$$
This map has to be an isomorphism (except when \( q = -(n - 1) \)), since otherwise we would have non-zero classes in \( E^{n+1} = E^\infty \), in contradiction with \( H_n(PX) = 0 \) for \( n \neq 0 \). Combining this with the formula for \( E^2 = E^n \), we can deduce that

\[
H_k(\Omega S^n) \cong \begin{cases} 
\mathbb{Z} & (n - 1)|k \\
0 & \text{otherwise.}
\end{cases}
\]

Indeed, this clear holds for \( k \leq 0 \). Assume for contradiction that it fails for some \( k \) and pick the smallest such: then \( H_k(\Omega S^n) \) and \( H_{k-n+1}(\Omega S^n) \) would be non-isomorphic groups, and the differential \( d^n : E^n_{n,k-n+1} \to E^n_{0,k} \) could not be an isomorphism, contradicting what we argued above.

If the generator in \( E^2_{0,k(n-1)} = H_{k(n-1)}(\Omega S^n) \) is denoted \( x_k \), then the generator of \( E^2_{n,k(n-1)} \) is \( \sigma \otimes x_k \in H_n(S^n) \otimes H_{k(n-1)}(\Omega S^n) \), and the differential is given by \( d(\sigma \otimes x_k) = x_{k+1} \).

**6.2. Homology of \( SU(3) \).** The quotient map \( SU(3) \to SU(3)/SU(2) \) is a fiber bundle and hence a Serre fibration. The fibers are diffeomorphic to \( SU(2) \approx S^3 \) and the base can be identified with \( S^5 \) via the action of \( SU(3) \) on \( S^5 \subset \mathbb{C}^3 \). The Serre spectral sequence now has

\[
E^2_{p,q} = H_p(S^5) \otimes H_q(S^3) \Rightarrow H_{p+q}(SU(3)).
\]

For degree reasons, all differentials vanish, and we see that

\[
H_k(SU(3)) \cong \begin{cases} 
\mathbb{Z} & k = 0, 3, 5, 8 \\
0 & \text{otherwise.}
\end{cases}
\]

**6.3. Trivial fibrations.** The projection \( p : B \times F \to B \) is a Serre fibration. The spectral sequence has \( E^2_{p,q} = H_p(B; H_q(F)) \) and no differentials. In this case the spectral sequence contains essentially the same information as the Künneth theorem.

For an arbitrary Serre fibration \( p : E \to B \) with fibers weakly equivalent to \( F \) and \( \pi_q(B,b) \) acting trivially on \( H_p(F) \) for all \( b \in B \), we see that the \( E^2 \) page is the same as for the trivial fibration. From this point of view, the differentials in the spectral sequence measure the failure of \( p : E \to B \) being weakly equivalent to a trivial fibration.

**6.4. The Hopf fibration.** For \( p : S^3 \to S^2 \) with fibers \( S^1 \), we already know the homology of all three spaces. The spectral sequence has \( E^2_{p,q} = H_p(S^2) \otimes H_q(S^1) \) and converges to \( H_{p+q}(S^3) \). The only possible differential is \( d^2 : E^2_{2,0} \to E^2_{0,1} \), which must be an isomorphism, since \( E^3_{0,1} = E^\infty_{0,1} \) vanishes.

**6.5. Euler characteristic.** Let \( p : E \to B \) be a Serre fibration with \( B \) path connected, and write \( F = p^{-1}(b) \) for some \( b \in B \). Let \( k \) a field and assume that \( \pi_1(B) \) acts trivially on \( H_*(F;k) \). If \( H_*(B;k) \) and \( H_*(F;k) \) are both finite dimensional vector spaces (where we write \( H_* = \oplus_n H_n \)), then \( H_*(E;k) \) is also finite dimensional and

\[
\chi_k(E) = \chi_k(B)\chi_k(F)
\]

where \( \chi_k(-) \) is the usual alternating sum of the dimensions of \( H_p(-;k) \).
Proof. Under the assumption, the universal coefficient theorem applied to the $E^2$ page of the spectral sequence gives $E^2_{p,q} = H_p(B; k) \otimes H_q(E; k)$, and hence

$$\chi_k(B)\chi_k(F) = (\sum_p (-1)^p \dim H_p(B; k))(\sum_q (-1)^q \dim H_p(F; k))$$

$$= \sum_{p,q} (-1)^{p+q} \dim E^2_{p,q}$$

Since $d''$ changes $p+q$ by $-1$, passing from $E^r$ to $E^{r+1}$ does not change the number $\sum_{p,q}(-1)^{p+q} \dim E^r_{p,q}$, and after finitely many steps all differentials must vanish, so $\chi_k(B)\chi_k(F) = \sum_{p,q}(-1)^{p+q} \dim E^\infty_{p,q}$.

On the other hand, there is a (non-canonical) isomorphism $H_n(E; k) \cong \bigoplus_{p+q=n} E^\infty_{p,q}$, proving that $H_*(E; k)$ is finite dimensional and the Euler characteristic is as claimed. \(\square\)

7. Cohomology spectral sequences

7.1. Cohomology spectral sequence of a filtered space. In the same sense as we can use a spectral sequence to “calculate” the homology of a filtered space $X$ from the groups $H_*(X_p, X_{p-1})$, there is a spectral sequence starting with $H^*(X_p, X_{p-1})$ and calculating $H^*(X)$, at least in good cases. Namely, we may form an exact couple with $A = \bigoplus_p H^{p+q}(X, X_p)$ and $E = \bigoplus_p H_{p+q}(X, X_{p-1})$, by summing the the long exact sequences in cohomology from the triples $(X, X_p, X_{p-1})$, over all $p$. The degrees in the resulting spectral sequence work out a bit differently, which we will indicate by using upper indices, such as $E^1_{p,q}$, etc. The spectral sequence has

$$E^r_{p,q} = H^{p+q}(X_p, X_{p-1}),$$

the differential $d^1 : E^1_{p,q} \to E^1_{p+1,q}$ is the composition $H^{p+1}(X_p, X_{p-1}) \to H^{p+q-1}(X, X_p) \to H^{p+q+1}(X_{p-1}, X_p)$, and the map $i : H^{p+q}(X, X_p) \to H^{p+q}(X, X_{p-1})$ is induced by the inclusion $X_{p-1} \to X_p$. In general the differential $d'' : E^p_{p,q} \to E^p_{p+r,q-r-1}$.

For this spectral sequence to converge to $H^*(X)$, we must impose slightly stronger conditions on the filtration than in the homological case (arising from the fact that inverse limits are not as well behaved as direct limits), viz. we shall make the following two assumptions

(i) The map

$$\lim_{p \to -\infty} H^*(X, X_p) \to H^*(X)$$

is an isomorphism. This is automatic if $X_{-1} = \emptyset$.

(ii) The maps $i^r : H^n(X, X_{p+r}) \to H^n(X, X_p)$ have $\cap \text{Im}(i^r) = 0$. This holds if $X = \bigcup X_p$ with the direct limit topology, and $H^n(X, X_{p+r})$ is independent of $r$ for large $r$.

In that case the exact couple satisfies (3.2), and the spectral sequence converges to $\lim H^*(X, X_p) = H^*(X)$. This means that each group $H^*(X)$ comes with a filtration $F^p = \text{Im}(H^*(X, X_p) \to H^*(X))$ which is now increasing: $F^n \supset F^{n+1} \supset \ldots$, and there is an isomorphism $E^\infty_{p,q} = F^p/F^{p+1}$.
7.2. Cohomology with local coefficients. We will use the cohomology spectral sequence of a filtered space to deduce a Serre spectral sequence in cohomology. Let us briefly discuss cochains of a space $X$ with coefficients in a coefficient system $\mathcal{A}$: $\pi(X) \to \text{Ab}$. Cohains are defined as

$$C^p(X; \mathcal{A}) = \prod_{\sigma: \Delta^p \to X} \mathcal{A}_\sigma,$$

where the product is over all continuous maps and $\mathcal{A}_\sigma$ is the group from § 4.4. (It would perhaps be more natural to define $\mathcal{A}^p$ as the colimit of the functor $\sigma^*\mathcal{A} : \pi(\Delta^p) \to \text{Ab}$ and work with that instead. The natural map from the colimit to the limit is an isomorphism in this case, so we shall not emphasize this distinction.)

To each face map $F_i : \Delta^{p-1} \to \Delta^p$ we associated an isomorphism $d_i : \mathcal{A}_\sigma \cong \mathcal{A}_{\sigma F_i}$, and we shall denote its inverse by $d^i : \mathcal{A}_{\sigma F_i} \to \mathcal{A}_\sigma$. These assemble to homomorphisms $d^i : C^{p-1}(X; \mathcal{A}) \to C^p(X; \mathcal{A})$ for $i = 0, \ldots, p$, and we shall define a coboundary as

$$\delta = \sum_{i=0}^p (-1)^i d^i : C^{p-1}(X; \mathcal{A}) \to C^p(X; \mathcal{A})$$

7.3. The cohomology Serre spectral sequence. The cohomological Serre spectral sequence associated to a Serre fibration $p : E \to B$ calculates the cohomology of $E$ starting from the cohomology of $B$ with coefficients in the fibers. It is constructed similarly to the homology spectral sequence: When $B$ is a $\Delta$-complex we consider the spectral sequence associated to the filtration of $E$ by inverse images of skeleta of $B$, and for general $B$ we first pull back along $c : \Gamma B \to B$.

Theorem 7.1. For any Serre fibration $p : E \to B$ there is a spectral sequence with

$$E_2^{p,q} = H^p(B; \mathcal{H}^q),$$

where $\mathcal{H}^q : \pi(B) \to \text{Ab}$ is the coefficient system $b \mapsto H^q(p^{-1}(b))$. The $d_r$ differential is a homomorphism $d_r : E_r^{p,q} \to E_r^{p+r,q-r}$. The spectral sequence has

$$E_\infty^{p,q} = F_0^p H^{p+q}(E) / F_0^{p+1} H^{p+q}(E),$$

where $H^*(E) = F_0^0 H^0(E) \supset F_0^1 H^0(E) \supset \cdots \supset F_0^p H^n(E) \supset F_0^{p+1} H^n(E) = 0$ is a filtration of $H^0(E)$.

The cohomology version of the Serre spectral sequence has several advantages over the homology version. As we shall see, the whole spectral sequence can be endowed with a product structure, which relates the cup products on $H^*(E)$, $H^*(B)$ and each $H^*(p^{-1}(b))$.

The fact that cohomology is contravariant and that the differentials go in the opposite direction is useful in itself, as we shall see.

8. Examples

8.1. The Euler class of a spherical fibration. A spherical fibration is a Serre fibration $p : E \to B$ such that all fibres $p^{-1}(b)$ are weakly equivalent to $S^n$, and thus each $H^n(p^{-1}(b)) \approx \mathbb{Z}$. An orientation of $p$ is a “continuous” choice of isomorphism $H^n(p^{-1}(b)) \approx \mathbb{Z}$, i.e. an isomorphism from $\mathcal{H}^n$ to the constant coefficient system $\mathbb{Z}$. These may or may not exist. An orientation gives an isomorphism $E_2^{0,n} = H^0(B)$ and in particular $E_2^{0,n} = H^0(B)$. 
We may then define \( H \) that the filtration on the group \( E \) in the following way. The fact that the groups \( E \) define the Chern classes of a complex vector bundle (at least up to a sign).

**8.3. Pushforward in cohomology.** Let \( p : E \to B \) be a fibration and assume that each \( p^{−1}(b) \) is weakly equivalent to an oriented closed \( n \)-manifold for each \( b \). Then \( H^p(p^{−1}(b)) \approx \mathbb{Z} \) for all \( b \), and we shall in addition assume given a “continuous choice” of such isomorphisms: that is, an isomorphism of coefficient systems \( \mathcal{H}^n \cong \mathbb{Z} \). We can then use the Serre spectral sequence to define a map

\[
p_n : H^{n+k}(E) \to H^k(B)
\]

in the following way. The fact that the groups \( E^{0,q} \) vanish for \( q > n \) imply firstly that the filtration on the group \( H^{k+n}(E) \) has \( F^0 H^{n+k}(E) = \cdots = F^k H^{n+k}(E) = H^{n+k}(E) \) and secondly that no differentials go into \( E^{k,n} \) and hence \( E_{k,n}^\infty \subset E_{k,n}^2 \).

We may then define \( p_n \) as the composition

\[
H^{n+k}(E) \to H^{n+k}(E)/F^{k+1}H^{n+k}(E) = E_{k,n}^{k,n} \Rightarrow E_{k,n}^{2} = H_k(B; \mathbb{Z}).
\]

**8.3. Chern classes.** We shall use the cohomological Serre spectral sequence to define the Chern classes of a complex vector bundle (at least up to a sign).

The homeomorphism \( U(k)/U(k−1) \approx S^{2k−1} \) proves that the inclusion \( U(k−1) \to U(k) \) is \( (2k−2) \)-connected. For \( n > k \), the inclusion \( S^{2k−1} = U(k)/U(k−1) \to U(n)/U(k−1) \) is then \( 2k \)-connected. It follows that \( U(n)/U(k−1) \) is \( (2k−2) \)-connected and \( H_{2k−1}(U(n)/U(k−1)) = \pi_{2k−1}(U(n)/U(k−1)) = \mathbb{Z} \) and hence \( H^{2k−1}(U(n)/U(k−1)) = \mathbb{Z} \).

Let \( V \to X \) be a complex \( n \)-dimensional vector bundle, and let \( P \to X \) be the corresponding principal \( U(n) \)-bundle. (I.e. \( P \) is the space of orthonormal frames in \( V \) with respect to some chosen Hermitian metric.) This is a fiber bundle with fiber \( U(n) \), and \( p_k : P/U(k−1) \to X \) is a fiber bundle with fiber \( U(n)/U(k−1) \). The spectral sequence has \( E_{2,0}^{q,q} = 0 \) for \( 0 < q < 2k−1 \) and

\[
E_{2,0}^{0,2k−1} = H^0(X; H^{2k−1}(U(n)/U(k−1))) = H^0(B) \otimes H^{2k−1}(U(n)/U(k−1)).
\]

If we let \( \sigma \in H^{2k−1}(U(n)/U(k−1)) \approx \mathbb{Z} \) denote a generator, we have the canonical element \( 1 \otimes \sigma \in E_{2,0}^{0,2k−1} \). There are no possible differentials coming into this group, and the first possible differential going out of it is \( d_{2k} : E_{2k}^{0,2k−1} \to E_{2k}^{2k,0} = H^{2k}(X) \).

We then set \( \sigma_k(V) = \pm d_{2k}(1 \otimes \sigma) \). (The sign depends on the choice of generator \( \sigma \).)

**8.4. Stiefel-Whitney classes.** For a real vector bundle \( V \to X \), the Stiefel-Whitney classes \( w_k(V) \in H^k(X; \mathbb{F}_2) \) can be defined similarly to the Chern classes, using \( O(n) \) in place of \( U(n) \).

**9. Products in spectral sequences**

**9.1. Definitions.**

**Definition 9.1.** Let \( R \) be a ring. A filtration \( R \supset \cdots \supset F^n R \supset F^{n+1} R \supset \cdots \) is multiplicative if \((F^n R)(F^m R) \subset F^{n+m} R \). The associated graded

\[
\bigoplus_n F^n R/F^{n+1} R
\]
then inherits the structure of a graded ring, where \((F^n R/F^{n+1} R) \otimes (F^m R/F^{m+1} R) \to F^{n+m} R/F^{n+m+1} R\) is defined as \([a][b] = [ab]\).

**Definition 9.2.** Let \(R = \bigoplus R^{p,q}\) be a bigraded ring. A differential \(d : R \to R\) is a derivation if \(d(xy) = (dx)y + (-1)^{|x|} x(dy)\), where \(|x| = p + q\) if \(x \in R^{p,q}\). The homology groups \(H^p,q(R, d) = \text{Ker}(d : R^p,q \to R^{p+1,q})/\text{Im}(d : R^{p,q} \to R^{p,q})\) then inherit a product, given by \([a][b] = [ab]\).

### 9.2. Cup products in the Serre spectral sequence.

We have explained all ingredients in the following theorem, except that in (iv) we must explain how the cup products in the cohomology of the fibers and the base combine to a product on \(H^p(B; \mathcal{H}_q)\). We do this in a special case in Remark 9.4 below, and in full generality in Section 9.4.

**Theorem 9.3.** The cohomology Serre spectral sequence of a Serre fibration \(p : E \to B\) admits a natural “cup product” map

\[ E^r_{p,q} \otimes E^r_{p',q'} \to E^{r+p+q,q+q'}_{r} \]

with the following properties

(i) \(E^r_{p,*}\) is bigraded ring for each \(r \geq 2\).

(ii) \(d_r : E_r \to E_r\) is a derivation.

(iii) The isomorphism \(E^p,q_{r+1} \cong H^p,q(E_r, d_r)\) is an isomorphism of bigraded rings.

(iv) The isomorphism \(E^p,q_{r} \cong H^p(B; \mathcal{H}_q)\) is an isomorphism of bigraded rings.

(v) The filtration of \(H^*(E)\) is multiplicative.

(vi) The isomorphism \(E^\infty_{p,q} \cong F^p H^{p+q}(E)/F^{p+1} H^{p+q}\) defines an isomorphism of bigraded rings.

**Remark 9.4.** In the special case where the coefficient systems \(b \mapsto H^q(p^{-1}(b))\) are constant and the Ext term in the universal coefficient theorem vanishes, we have an additive isomorphism

\[(9.1) \quad H^p(B) \otimes H^q(F) \to H^p(B; \mathcal{H}_q)\]

where \(F\) is the fiber over some chosen basepoint. When added over all \(p,q\), the left hand side of this isomorphism is naturally a ring, whose product is given by \((x \otimes y)(x' \otimes y') = (-1)^{|x'||y|} (xx') \otimes (yy')\). The isomorphism (9.1) thus induces a product on \(H^p(B; \mathcal{H}_q)\), and the meaning of statement (iv) of the theorem is now that the isomorphism to \(E^\infty_{p,q}\) is multiplicative.

### 9.3. Products in homology spectral sequences.

**Theorem 9.5.** Let \(X, Y\) and \(Z\) be filtered spaces, and let \(\mu : X \times Y \to Z\) be a filtered map (i.e. \(\mu(X_p \times Y_{p'}) \subset Z_{p+p'}\)). In the resulting spectral sequences there are products

\[ X E^r_{p,q} \otimes Y E^r_{p',q'} \to Z E^r_{p+p',q+q'} \]

such that \(d^r\) satisfies the Leibniz rule and the isomorphism \(E^\infty_{p,q} \cong F^p_{p+q}/F^{p-1}_{p+q}\) preserves products.

**Corollary 9.6.** Let \(\nu F \to \nu E \xrightarrow{\nu p} \nu B\) be Serre fibrations, \(\nu = 1, 2, 3\) and let \(\nu \mathcal{H}_q : \pi(\nu B) \to \text{Ab}\) be the corresponding coefficient systems. Then a commutative
induces a product on spectral sequences

\[ E^r_{p,q} \otimes E^r_{p',q'} \rightarrow E^r_{p+p',q+q'} \]

satisfying the Leibniz rule, and such that the isomorphism \( E^\infty \cong F^p/F^{p-1} \) preserves products. On the \( E^2 \) terms, the product

\[ H_p(1; B; ^\ast \mathbb{H}_q) \otimes H_p(2; B; ^\ast \mathbb{H}_q) \rightarrow H_{p+p'}(3; B; ^\ast \mathbb{H}_{q+q'}) \]

agrees with the one induced by cross products in fibers and base.

A typical application would be to the case when \( p : E \rightarrow B \) is obtained by looping another Serre fibration. Then we can apply the corollary with all three fibrations being \( p : E \rightarrow B \), and the maps \( \mu \) being concatenation of loops.

9.4. Appendix: cup products with local coefficients in graded rings. The coefficient system \( ^\ast \mathbb{H}^p : b \mapsto H^p(\pi^{-1}(b)) \) attached to the Serre fibration \( p : E \rightarrow B \) has a product induced from cup product, i.e. the direct sum \( \oplus \) has a product induced from cup product, i.e.

\[ \mathbb{H}^p(X; R) \otimes \mathbb{H}^p(Y; R) \rightarrow \mathbb{H}^p(X \times Y; R) \]

where \( GRing \) is the category of graded rings. In this section we explain why \( H^p(X; R) \) inherits a cup product for any functor \( R^p : \pi(X) \rightarrow GRing \).

Definition 9.7. Let \( X \) be a space, and \( A : \pi(X) \rightarrow Ab \) and \( B : \pi(X) \rightarrow Ab \) be coefficient systems. Let \( A \otimes B : \pi(X) \rightarrow Ab \) be the coefficient system given on objects by \( (A \otimes B)(x) = (A(x)) \otimes (B(x)) \). Let

\[ C^p(X; A) \otimes C^p(X; B) \xrightarrow{\cup} C^{p+p'}(X; A \otimes B) = \prod_{\sigma \Delta^p \rightarrow X} (A \otimes B)_\sigma \]

be the map whose \( \sigma \)th coordinate is the natural isomorphism

\[ A_{\sigma | [e_0, \ldots, e_p]} \otimes B_{\sigma | [e_p, \ldots, e_{p+p'}]} \rightarrow (A \otimes B)_\sigma, \]

where \( (\sigma | [e_0, \ldots, e_p]) : \Delta^p \rightarrow X \) and \( (\sigma | [e_p, \ldots, e_{p+p'}]) : \Delta^{p'} \rightarrow X \) denote the restriction to the “front” and “back” faces of \( \Delta^{p+p'} \). (Both sides of the isomorphism are canonically isomorphic to \( A_\sigma(e_p) \otimes B_\sigma(e_{p+p'}) \).)

This construction is obviously natural in both \( X, A \) and \( B \). If \( R : \pi(B) \rightarrow GRing \) is a coefficient system of rings, we therefore obtain a homomorphism

\[ C^p(X; R) \otimes C^{p'}(X; R) \xrightarrow{\cup} C^{p+p'}(X; R \otimes R) \rightarrow C^{p+p'}(X; R), \]

where the last homomorphism is induced by the multiplication in \( R \).

If \( GRing \) denotes the category of graded rings, i.e. rings \( R \) with a decomposition \( R = \oplus_q R^q \) such that \( (R^q)(R^{q'}) \subset R^{q+q'} \), then for a coefficient system \( R : \pi(X) \rightarrow GRing \) we shall define a product on \( C^*(X; R) = \bigoplus_{p,q} C^p(X; R^q) \) as

\[ C^p(X; R^q) \otimes C^{p'}(X; R^{q'}) \xrightarrow{\cup} C^{p+p'}(X; R^q \otimes R^{q'}) \rightarrow C^{p+p'}(X; R^{q+q'}), \]

where the last homomorphism is induced by the multiplication \( R^q \otimes R^{q'} \rightarrow R^{q+q'} \) times the sign \((-1)^{p'q} \).
Remark 9.8. When \( R = \oplus R^q \) is a constant coefficient system, the sign \((-1)^{p,q}\) makes natural maps \( H^p(X) \otimes R^q \to H^p(X; R^q) \) given by \([\phi] \otimes r \mapsto [r \phi]\) assemble into a homomorphism of bigraded rings.

## References


## 10. Examples

### 10.1. Rational cohomology of \( K(\mathbb{Z}; n) \)

We know that \( K(\mathbb{Z}; 1) \simeq S^1 \) and \( K(\mathbb{Z}; 2) \simeq \mathbb{C}P^\infty \) and hence \( H^*(K(\mathbb{Z}; 1); \mathbb{Q}) \cong \Lambda_Q[\iota_1] \) with \( |\iota_1| = 1 \) and \( H^*(K(\mathbb{Z}; 2); \mathbb{Q}) \cong \mathbb{Q}[\iota_2] \) with \( |\iota_2| = 2 \). Here \( \Lambda_Q[x] \) denotes the *exterior algebra* on a generator \( x \) of odd degree.

Let us use the path-loop fibration to calculate \( H^*(K(\mathbb{Z}; 3); \mathbb{Q}) \). By the Hurewicz theorem and the universal coefficient theorem, we see that

\[
H^k(K(\mathbb{Z}; 3); \mathbb{Q}) = \begin{cases} \mathbb{Q} & k = 0, 3 \\ 0 & k = 1, 2 \end{cases}
\]

and we claim that \( H^k(K(\mathbb{Z}; 3); \mathbb{Q}) = 0 \) for \( k > 3 \). If this were not the case, there would exist some \( z \neq 0 \in H^{|z|}(K(\mathbb{Z}; 3); \mathbb{Q}) \) for \( |z| > 3 \). Let us pick such a \( z \) with \( |z| \) minimal.

The Serre spectral sequence in rational cohomology for the path-loop fibration has

\[
E^{p,q}_2 = H^p(K(\mathbb{Z}; 3); H^q(\mathbb{C}P^\infty; \mathbb{Q})) \cong H^p(K(\mathbb{Z}; 3); \mathbb{Q}) \otimes H^q(K(\mathbb{Z}; 2); \mathbb{Q})
\]

For degree reasons, we must have \( d_2(1 \otimes \iota_2) = 0 \) and \( E^{3,0}_2 = E^{3,0}_3 = \mathbb{Q}, \iota_3 \otimes 1 \) and \( E^{0,2}_2 = E^{0,2}_3 = \mathbb{Q}, 1 \otimes \iota_2 \). The only option consistent with the vanishing of \( E^\infty_3 \) is that \( d_3(\iota_2 \otimes 1) = 1 \lambda \otimes \iota_3 \) for a non-zero \( \lambda \in \mathbb{Q} \). Writing \( x = \iota_3 \otimes 1 \) and \( y = 1 \otimes \iota_2 \), the fact that \( d_r \) is a derivation then implies that \( d_2(1 \otimes \iota_2^k) = d_2(y^k) = 2ky^{k-1}d_2(y) = 0 \), \( d_2(\iota_3 \otimes \iota_2^k) = d_2(xy^k) = 0 \) and \( d_3(1 \otimes \iota_2^k) = d_3(y^k) = ky^{k-1}d_3y = ky^{k-1} \),

and hence \( d_3 : E^{3,2k+2}_3 \to E^{3,2k}_3 \) is an isomorphism for all \( k \geq 0 \). We have now completely understood the structure of \( (E^r_{pq}, d_r) \) for all \( r \geq 2 \), all \( q \) and all \( p < |z| \).

The element \( z \otimes 1 \neq 0 \in E^{p,0}_2 \) would have to be hit by a non-zero differential \( d_r \) for some \( r \), but that would violate the known structure of \( E^{p,q}_2 \).

By a similar argument, we can show that \( H^*(K(\mathbb{Z}; n); \mathbb{Q}) \cong \Lambda_Q[\iota_n] \) for all odd \( n \) and \( H^*(K(\mathbb{Z}; n); \mathbb{Q}) \cong \mathbb{Q}[\iota_n] \) for all even \( n > 0 \).

### 10.2. Pushforward in cohomology

Elaborating on the example in Section 8.2, the multiplicativity of the spectral sequence implies that if the fibers of \( p : E \to B \) are closed oriented manifolds (with orientations varying “continuously” as in Section 8.2), then

\[
p(x \cup p^*(y)) = (p(x)) \cup y.
\]
10.3. The Gysin sequence. Elaborating on the example in Section 8.1, we can see that under the assumptions in that section, the entire structure of the spectral sequence follows from the value of \(e(p) \in H^{n+1}(B)\). If we let \(\sigma \in H^n(p^{-1}(b)) \cong \mathbb{Z}\) denote the generator corresponding to 1 \(\in \mathbb{Z}\) under the chosen isomorphism, we have defined \(e(p)\) by \(d_{n+1}(1 \otimes \sigma) = (e(p)) \otimes 1\). It then follows that \(d_{n+1}(x \otimes \sigma) = ((e(p)) \otimes 1)(x \otimes 1) = xe(p) \otimes 1\). The filtration on \(H^*(E)\) has precisely two non-zero filtration quotients, and the convergence of the spectral sequence translates to the long exact sequence

\[
\ldots H^p(B) \xrightarrow{\partial} H^p(E) \xrightarrow{d_{n+1}} H^{p+n+1}(E) \xrightarrow{p_n} H^{p+1}(B) \rightarrow \ldots
\]

10.4. Homology of the loop space of \(S^n\). This example uses products in the homology spectral sequence.

The map \(\Omega S^n \times \Omega S^n \to \Omega S^n\) given by concatenation of loops induces a product on \(H_*(\Omega S^n)\). Let’s use the Serre spectral sequence to calculate the ring structure of \(H_*(\Omega S^n)\). The map \(\Omega S^n \to \ast\) can be considered a fibration with fiber \(\Omega S^n\). Further, the homology of the loop space of \(S^n\) can be considered a fibration with fiber \(\Omega S^n\). Concatenation of paths gives a map \(\Omega S^n \times PS^n\) that fits in a commutative diagram

\[
\begin{array}{ccc}
\Omega S^n \times PS^n & \xrightarrow{\partial} & PS^n \\
\downarrow & & \downarrow \\
\ast \times S^n & \xrightarrow{\partial} & S^n.
\end{array}
\]

The induced map on fibers is concatenation of loops. By the general theory, this diagram gives a product on spectral sequences \(\E^{1}_{r} \otimes \E^{2}_{r} \rightarrow \E^{3}_{r}\), where \(\E^{1}_{r}\) is the spectral sequence of the fibration \(\Omega S^n \to \Omega S^n \to \ast\), and both \(\E^{2}_{r} = \E^{3}_{r}\) is the spectral sequence of the path-loop fibration. We already calculated what the latter looks like, namely

\[
\E^{2}_{p,q} = \E^{3}_{p,q} = \begin{cases} 
\mathbb{Z} & p = 0, n \text{ and } (n-1)|q \\
0 & \text{otherwise.}
\end{cases}
\]

and we had names for the generators: If \(\sigma \in E^{2}_{n,0} = H_n(S^n)\) is the fundamental class, then we have generators

\[
\begin{align*}
x_1 &= d\sigma \in E^{2}_{0,n-1} & \sigma \otimes x_1 & \in E^{2}_{n,n-1} \\
x_2 &= d(\sigma \otimes x_1) \in E^{2}_{0,2n-2} & \sigma \otimes x_2 & \in E^{2}_{n,2n-2} \\
x_3 &= d(\sigma \otimes x_2) \in E^{2}_{0,3n-3} & \sigma \otimes x_3 & \in E^{2}_{n,3n-3} \\
\ldots
\end{align*}
\]

Now, the Serre spectral sequence of the fibration \(\Omega S^n \to \Omega S^n \to \ast\) is very boring: It has \(E^{r}_{0,q} = H_q(\Omega S^n)\) for all \(q\) and \(E^{r}_{p,q} = 0\) for \(p > 0\), and all differentials vanish. Thus, the induced product of spectral sequences in this case is

\[
H_q(\Omega S^n) \otimes \E^{r}_{p',q'} \xrightarrow{\times} \E^{r}_{p'+q'+q'}.\]

It is not hard to see that the product \(x_k \times \sigma \in E^{2}_{n,n-1}\) is the same element that we previously denoted \(\sigma \otimes x_k\) (you have to trace through what the product on \(E^{2}\) is, and check that it is the same as the one appearing in the universal coefficient
By the same induction argument as for homology, we get

\[ y \in H^n(S^n) \] 
for \( y \) of the form \( y \) with even degree, so the only possible differential is \( d \). Since the \( E_\infty \) term vanishes except in degree \((0,0)\), \( d \) must give an isomorphism \( H^q(S^n) \to H^q(S^n) \). By the same induction argument as for homology, we get

\[ H^q(S^n) = \begin{cases} 
\mathbb{Z} & (n-1)|q \\
0 & \text{otherwise}.
\end{cases} \]

Let \( y_k \in H^{k(n-1)}(S^n) \) and \( \sigma \in H^n(S^n) \) be generators. Then the product \( \sigma y_k \in E_2^{n,k(n-1)} = \mathbb{Z} \) is a generator. Since \( d_n : E_n^{n,k(n-1)} \to E_n^{n,(k-1)(n-1)} \) is an isomorphism, we can assume that \( d_n(y_k) = y_{k-1}\sigma \) (otherwise, change the signs of the \( y_k \)'s).

There are now two cases, depending on whether \( n \) is even or odd. Let us first treat the case where \( n \) is odd. In this case, all the elements \( y_k \) have even degree, so they commute strictly with all elements in the spectral sequence. In particular we get

\[ d_n(y_k) = ky_{k-1}d(y_1) = k(\sigma y_1^{k-1}). \]

Comparing this formula with \( d_n y_k = \sigma y_{k-1} \) it follows by induction that \( y_k^k = k! y_k \), and hence the ring structure is

\[ y_i y_k = \binom{k + l}{k} y_{k+i}. \]

The case where \( n \) is even is a little different. In this case \( y_1^2 = 0 \) by graded commutativity. Letting \( \sigma \) and \( y_k \) be additive generators as before, we first investigate multiplication by \( y_1 \). We can write \( y_1 y_k = \alpha(k) y_{k+1} \) for some function \( \alpha \). Differentiating gives

\[ \alpha(k)\sigma y_k = \alpha(k)d(y_{k+1}) = d(y_1 y_k) = \sigma y_k - y_1 \sigma y_{k-1} = (1 - \alpha(k - 1))\sigma y_k, \]
and hence $\alpha(k) = 1 - \alpha(k - 1)$. Therefore

$$y_1 y_k = \begin{cases} y_{k+1} & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

We now investigate the power $y_2^k$. Since $y_2$ is even dimensional, it strictly commutes with everything.

$$d(y_2^k) = ky_2^{k-1}dy_2 = ky_2^{k-1}y_1\sigma.$$ If we write $y_2^k = \alpha(k)y_{2k}$, this gives $\alpha(k)dy_{2k} = k\alpha(k-1)y_{2(k-1)}y_1\sigma$. On the other hand $dy_{2k} = \sigma y_{2k-1} = \sigma y_1 y_{2(k-1)}$, so we get $\alpha(k) = k\alpha(k-1)$ and hence $\alpha(k) = k!$. From this it follows that the multiplication is given by

$$y_{2k} y_{2l} = \binom{k+l}{k} y_{2k+2l}$$

$$y_{2k+1} y_{2l} = y_{2k} y_{2l+1} = \binom{k+l}{k} y_{2k+2l+1}$$

$$y_{2k+1} y_{2k+2} = 0$$

11. Construction of multiplicative structures

[coming soon]