1. Exact couples

Following Hatcher's notes, we considered an *exact couple*, i.e. two abelian groups A and E, and maps

$$\begin{tikzcd}
A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\
& E \arrow[ul, "k"] &
\end{tikzcd}$$

making the triangle exact at each vertex. Recall that we set $d = jk : E \to E$ and defined $E' = \mathrm{Ker}(d)/\mathrm{Im}(d)$, $A' = iA$, $i'(a) = ia$, $j'(ia) = [ja]$, $k[e] = ke$ and proved that this gives the *derived couple*

$$\begin{tikzcd}
A' \arrow[rr, "i'"] & & A' \arrow[dl, "j'"] \\
& E'. \arrow[ul, "k'"] &
\end{tikzcd}$$

Iterating this construction we get a sequence of exact sequences

$$\begin{tikzcd}
A^r \arrow[rr, "i_r"] & & A^r \arrow[dl, "j_r"] \\
& E^r. \arrow[ul, "k_r"] &
\end{tikzcd}$$

which for $r = 1$ is the one we started with, and where each is the derived couple of the previous exact couple.

In applications, the groups A and E are usually (bi-)graded, and the maps i, j, k have some fixed degree. We saw that the differential $d^r : E^r \to E^r$ has degree given by

$$\deg(d^r) = \deg(j) + \deg(k) - (r-1)\deg(i).$$

1.1. The spectral sequence of a filtered space.

We can now construct our first spectral sequence. Let X be a space and $X_p \subseteq X$, $p \in \mathbb{Z}$ spaces, such that $X_p \subseteq X_{p+1}$ for all p. Now set

$$\begin{aligned}
A_{p,q} &= H_{p+q}(X_p) \\
E_{p,q} &= H_{p+q}(X_p, X_{p-1}),
\end{aligned}$$

let $i : A \to A$ be induced by the inclusions $X_{p-1} \to X_p$, let $j : A \to E$ be induced by the maps of pairs $(X_p, \emptyset) \to (X_p, X_{p-1})$, and let $k : E \to A$ be the connecting homomorphisms $H_{p+q}(X_p, X_{p-1}) \to H_{p+q-1}(X_{p-1})$.

We see that in this case

$$\begin{aligned}\deg(i) &= (1,-1)\\ \deg(j) &= (0,0)\\ \deg(k) &= (-1,0).\end{aligned}$$

We get a bigraded spectral sequence with

$$\deg(d^r) = (-r, r-1).$$

The E^1 term is given by $E^1_{p,q} = H_{p+q}(X_p, X_{p-1})$, and the differential d^1 is the composition

$$H_{p+q}(X_p, X_{p-1}) \to H_{p+q-1}(X_{p-1}) \to H_{p+q-1}(X_{p-1}, X_{p-2}).$$

Remark 1.1. This need not be a "first quadrant" spectral sequence. In general, all we can say is that $E^1_{p,q} = 0$ when $p+q < 0$.

1.2. **The terms of the spectral sequence.** Let us study the meaning of the terms E^r in the spectral sequence. It easier to do this more generally in the context of exact couples. Recall that a *subquotient* of an abelian group G is a quotient of a subgroup.

"A subquotient of a subquotient is a subquotient": If $A \subseteq B \subseteq G$, and $A' \subseteq B' \subseteq B/A$, then B'/A' is canonically isomorphic to B''/A'', where $A'' \subseteq G$ and $B'' \subseteq G$ are the unique subgroups containing A such that $(A''/A) = A'$ and $B''/A = B'$.

In the derived couple of an exact couple, $E' = \mathrm{Ker}(d)/\mathrm{Im}(d)$ is a subquotient of E, so iteratively E^r is a subquotient of E. Thus, there are canonical subgroups and isomorphisms

$$\begin{aligned}B^r \subseteq Z^r &\subseteq E\\ E^r &\cong Z^r/B^r\end{aligned}$$

The meaning of Z^r can best be explained if we introduce some (slight) abuse of notation: An element $e \in E$ with $de = 0$ represents an element $[e] \in E'$. If also $d'[e] = 0 \in E'$, then $[e]$ represents an element which should properly be written $[[e]] \in E''$. In this case we will write just $[e]$ instead of $[[e]]$. Confusion between the two meanings of $[e]$ can be avoided by emphasizing the group: $[e] \in E'$ or $[e] \in E''$. Similarly, starting with an element $e \in E$, we could potentially get an element of E^r which should be written with $(r-1)$ pairs of brackets. We will again write just $[e]$.

Then, start with $e \in E^1$, we are allowed to write $[e] \in E^2$ if $d^1 e = 0$. If also $d^2[e] = 0$ we are allowed to write $[e] \in E^3$. Etc. Then Z^r is the subset consisting of $e \in E$ for which we are allowed to write $[e] \in E^r$.

In that case $e \in B^r$ precisely if $[e] = 0 \in E^r$. Equivalently, $[e] = d^{r-1}[x]$ for some $x \in Z^{r-1}$.

Let us find more explicit formulas for Z^r and B^r, and let us also find formulas for the maps $j_r : A^r \to E^r$ and $k_r : E^r \to A^r$ in terms of the identification $E^r = Z^r/B^r$. We can set $Z^1 = E$ and $B^1 = 0$. Also we have

$$\begin{aligned} Z^2 &= \mathrm{Ker}(d) = \mathrm{Ker}(jk) = k^{-1}(\mathrm{Ker}(j)) = k^{-1}(\mathrm{Im}(i)) \\ B^2 &= \mathrm{Im}(d) = \mathrm{Im}(jk) = j(\mathrm{Im}(k)) = j(\mathrm{Ker}(i)). \end{aligned}$$

Fiddling around with the formula for d^2, you will find $Z^3 = k^{-1}(\mathrm{Im}(r^2))$ and $B^3 = j(\mathrm{Ker}(i^2))$ where we write $i^2 = i \circ i$. Fiddling some more leads us to guess a closed (non-recursive) formula for the $(r-1)$st derived couple of an exact couple.

Lemma 1.2. *We have*

$$\begin{aligned} A^r &= \mathrm{Im}(i^{r-1}) \\ Z^r &= k^{-1}(\mathrm{Im}(i^{r-1})) \\ B^r &= j(\mathrm{Ker}(i^{r-1})). \end{aligned}$$

With respect to the canonical isomorphism $E^r = Z^r/B^r$, we have

$$\begin{aligned} i_r(a) &= ia \\ j_r(i^{r-1}a) &= [ja] \\ k_r[e] &= ke. \end{aligned}$$

Proof. It is easy to see that i_r, j_r and k_r are well-defined. The statement is tautological for $r = 1$, and we proceed by induction. Assume the lemma is proved for r and let $d^r = j_r k_r$. Given an element $e \in E$ with $ke = i^{r-1}a$ (i.e. $e \in Z^r$), we have

$$d^r[e] = j_r k_r[e] = j_r i^{r-1} a = [ja] \in Z^r/B^r$$

If $ke = i^r b$, then $d^r[e] = [jib] = 0$. Conversely, assume $ke = i^{r-1}a$ and that $d^r[e] = [ja] = 0$. Using the induction hypothesis we get $ja = jb$ for some $b \in \mathrm{Ker}(i^{r-1})$. Then $a - b \in \mathrm{Ker}(j) = \mathrm{Im}(i)$, so $a = b + ix$, and we have

$$ke = i^{r-1}a = i^{r-1}(b + ix) = i^r x.$$

and we have proved the formula for Z^{r+1}.

B^{r+1} is the set of $b \in E$ such that $[b] = d^r[e]$ for some $[e] \in E^r = Z^r/B^r$. Writing again $ke = i^{r-1}a$ we have

$$b - ja \in B^r = j(\mathrm{Ker}(i^{r-1})),$$

so $b = j(a+x)$ with $i^{r-1}x = 0$. Then

$$i^r(a+x) = i^r a = ike = 0,$$

so $b \in j(\mathrm{Ker}(i^r))$. Conversely, if $b = ja$ with $i^r a = 0$, then $i^{r-1}a \in \mathrm{Ker}(i) = \mathrm{Im}(k)$, so we can write $i^{r-1}a = ke$ and then $d^r[e] = [ja] = [b]$. This proves the induction formula for B^r.

The formula for i_{r+1} is obvious, and j_{r+1} is easy: $j_{r+1}(i(i^{r-1}a)) = [j_r i^{r-1}a] = [ja]$, where the first equality is the definition of derived couple and the second is the induction hypothesis. The formula for k_r is also easy: $k_{r+1}[e] = k_r[e] = ke$ where the first equality is the definition of derived couple and the second is the induction hypothesis. $\square$

For example in the spectral sequence of a filtered space, the element

$$e \in E^1_{p,q} = H_{p+q}(X_p, X_{p-1})$$

is in Z^r if its image in $H_{p+q-1}(X_{p-1})$ comes from $H_{p+q-1}(X_{p-r})$. The resulting class $[e] \in E^r_{p,q}$ is zero if e comes from an element in $H_{p+q}(X_p)$ which vanishes in $H_{p+q}(X_{p+r-1})$.

Our next goal will be to prove that the spectral sequence "converges" to $H^*(X)$ under suitable assumptions, but first we need to study convergence in more generality.

2. CONVERGENCE

Last lecture I discussed how the Serre spectral sequence eventually stabilizes: For fixed (p,q) we have $E^r_{p,q} = E^{r+1}_{p,q} = \dots$ for large r, and we called this value $E^\infty_{p,q}$. In fact it makes sense to talk about E^∞ even without the stabilization phenomenon, and even without gradings, by taking an appropriate "limit" of the groups E^r.

The abelian group E has subgroups Z^r, B^r arranged as follows

$$0 = B^1 \subseteq B^2 \subseteq \cdots \subseteq B^r \subseteq \cdots \subseteq Z^r \subseteq \cdots \subseteq Z^2 \subseteq Z^1 = E,$$

and $E^r = Z^r/B^r$ is the quotient of a group that becomes smaller and smaller, by a groups that becomes bigger and bigger. Thus the natural definition to make is

$$Z^\infty = \bigcap_r Z^r, \quad B^\infty = \bigcup_r B^r, \quad E^\infty = Z^\infty/B^\infty.$$

This agrees with the previous definition in the graded case where the sequences $Z^r_{p,q}$ and $B^r_{p,q}$ stabilize for fixed p, q.

We can immeditately write a completely general formula for E^∞:

$$E^\infty = \frac{k^{-1}(\cap_r \mathrm{Im}(i^r))}{j(\cup_r \mathrm{Ker}(i^r))}. \tag{2.1}$$

In this generality this formula is of little use, but under additional assumptions we can rewrite it in a more useful way. Viz. let us study the additional assumption on the map $i : A \to A$ that

$$\bigcap_r \mathrm{Im}(i^r) = 0 \tag{2.2}$$

i.e. if $a \in A$ can be written as $i^r b$ for arbitrarily large r, then $a = 0$. For the exact couple associated to a filtered spaces, this holds if $X_{-1} = \emptyset$. In that case we have

$$E^\infty = \frac{\mathrm{Ker}(k)}{j(\cup_r \mathrm{Ker}(i^r))} = \frac{j(A)}{j(\cup_r \mathrm{Ker}(i^r))}.$$

$j : A \to E$ induces an isomorphism $jA \cong A/iA$, and we can write

$$E^\infty = \frac{A}{iA + \cup_r \mathrm{Ker}(i^r)}.$$

Writing $F = A/(\cup_r \mathrm{Ker}(i^r))$, there is an induced map $i : F \to F$, and we have $E^\infty = F/iF$.

Finally, notice that F is the same as the image of A in the direct limit $A^\infty = \varinjlim_i A$ of the direct system

$$\cdots \to A \xrightarrow{i} A \xrightarrow{i} A \to \dots, \tag{2.3}$$

so the E^∞ term is given very concisely by two pieces of data:

- The limit group $A^\infty = \varinjlim_i A$, together with the isomorphism $i : A^\infty \to A^\infty$ induced by $i : A \to A$.
- The subgroup $F \subseteq A^\infty$ which is the image of A in the direct limit.

Then $E^\infty \cong F/iF$. The isomorphism is also very explicit: It is the composition

$$E \xleftarrow{j} A \to A^\infty.$$

of j^{-1} (restricted to Z^∞) with the canonical map $A \to \varinjlim_i A = A^\infty$.

2.1. The spectral sequence of a filtered space. Convergence.

Let us calculate the E^∞ term of the spectral sequence of a filtered space. Recall that we have

$$A_{p,q} = H_{p+q}(X_p)$$

and that $i : A_{p,q} \to A_{p+1,q-1}$ is the map induced from the inclusion $X_p \to X_{p+1}$. Let us assume that assumption (2.2) holds, e.g. that $X_{-1} = \emptyset$. Let us also assume that the map

$$\varinjlim_{p\to\infty} H_n(X_p) \to H_n(X)$$

is an isomorphism. This holds e.g. if $X = \cup X_p$ and X has the weak topology (e.g. X is a CW-complex and the X_p are subcomplexes).

Then $A^\infty_{p,q}$ is the direct limit of the system

$$H_{p+q}(X_p) \to H_{p+q}(X_{p+1}) \to \dots,$$

i.e. $A^\infty_{p,q} = H_{p+q}(X)$ by our assumptions. The subspace $F_{p,q}$ is the image of $H_{p+q}(X_p) \to H_{p+q}(X)$, and $iF_{p-1,q+1}$ is the image of $H_{p+q}(X_{p-1})$ in $H_{p+q}(X)$. Changing notation to $F^p_{p+q} = F_{p,q}$, we see that

$$E^\infty_{p,q} = F^p_{p+q}/F^{p-1}_{p+q}.$$

In symbols we often write

$$E^1_{p,q} = H_{p+q}(X_p, X_{p-1}) \Rightarrow H_{p+q}(X),$$

meaning that we are considering a spectral sequence with the specified E^1 term, and where the E^∞ term is the filtration quotients in a filtration of $H_*(X)$. In words, we say that the spectral sequence "converges" to $H_*(X)$. Note that this spectral sequence need still not be first quadrant. $E^1_{p,q}$ vanishes unless $p \geq 0$ and $q \geq -p$, so it can occupy $\frac{3}{8}$ of the plane in general.

Let us briefly discuss more explicitly what the spectral sequence does. That an element $x \in E^1_{p,q}$ is in Z^r means that $d^1x = 0$, $d^2[x] = 0$, $\dots$, $d^{r-1}[x] = 0$, so x defines an element in E^r. The formula $Z^r = k^{-1}(\mathrm{Im}(i^{r-1}))$ can be interpreted in the diagram

$$\begin{tikzcd}
H_{p+q}(X_p, X_{p-r}) \arrow[r] \arrow[d] & H_{p+q}(X_p, X_{p-1}) \arrow[r] \arrow[d, "k"] & H_{p+q-1}(X_{p-1}, X_{p-r}) \arrow[d, equal] \\
H_{p+q-1}(X_{p-r}) \arrow[r, "i^{r-1}"] & H_{p+q-1}(X_{p-1}) \arrow[r] & H_{p+q-1}(X_{p-1}, X_{p-r})
\end{tikzcd}$$

with exact rows. It follows from the diagram that $Z^r_{p,q} \subseteq E^1_{p,q}$ consists of the elements $x \in H_{p+q}(X_p, X_{p-1})$ that come from $H_{p+q}(X_p, X_{p-r})$. By a similar argument, $B^r_{p,q}$ is the group of elements $x \in H_{p+q}(X_p, X_{p-1})$ that vanish when mapped to $H_{p+q}(X_{p+r-1}, X_{p-1})$.

Z^∞ is the elements that come from $H_{p+q}(X_p)$, and the isomorphism $E^\infty_{p,q} \to F^p_{p+q}/F^{p-1}_{p+q}$ is given by lifting to $H_{p+q}(X_p)$ and mapping to $H_{p+q}(X)$.

3. THE SERRE SPECTRAL SEQUENCE

The Serre spectral sequence is associated to a (Serre) fibration $p : E \to B$. There is both a homology and a cohomology version. The homology version has

$$E^2_{p,q} = H_p(B; H_q(F))$$

and converges to $H_*(E)$. First we need a few preliminaries about the fundamental group of B.

Let $\lambda : [0,1] \to B$ be a path, and let $E_\lambda \to [0,1]$ be the pullback of the fibration p. Let $F_\nu = p^{-1}(\lambda(\nu))$ for $\nu = 0, 1$, and let $i_\nu : F_\nu \to E_\sigma$ be the inclusions. The long exact sequence in homotopy groups shows that i_ν are both weak equivalences, so they induce isomorphisms in homology. We get an induced isomorphism

$$\lambda_* = (i_1)_* \circ (i_0)_*^{-1} : H_*(F_0) \to H_*(F_1).$$

Thus, if B is path connected, all fibers have isomorphic homology, although in general the isomorphism will depend on a choice of λ. We have $(\lambda * \sigma)_* = \lambda_* \circ \sigma_*$, and hence for each $b \in B$ we get an action of $\pi_1(B, b)$ on $H_*(p^{-1}(b))$. If this action is trivial, we have a *canonical* isomorphism between the homologies of two fibers. Therefore we can write $H_*(F)$ for the homology of a fiber: up to canonical isomorphism it doesn't depend on which fiber we pick.

Theorem 3.1. *Let $p : E \to B$ be a Serre fibration with B path connected and assume that $\pi_1(B)$ acts trivially on the homology of the fiber. Then there is a spectral sequence, functorial with respect to all maps of fibrations*

$$E^2_{p,q} = H_p(B; H_q(F)) \Rightarrow H_{p+q}(E).$$

Proof. We first consider the case where B is a path connected CW complex. Let B^p denote the p-skeleton of B, and let us make the following assumptions on B, to be used later.

(i) B has only one 0-cell b_0, which we consider the basepoint. Let $F = p^{-1}(b_0)$.
(ii) The attaching maps $\partial\sigma : \partial D^p \to B^{p-1}$ are based maps, they map finitely many disjoint open $(p-1)$-disks in ∂D^p homeomorphically onto open $p-1$-cells of B^{p-1}, and maps the complement of these disks to B^{p-2}.

At the end, we will reduce the general case to this special case (easy).

Define a filtration on E by $E_p = p^{-1}(B^p)$. We get a spectral sequence with

$$E^1_{p,q} = H_{p+q}(E_p, E_{p-1}). \tag{3.1}$$

We obviously have $E_{-1} = \emptyset$ and $\varinjlim H_*(E_p) = H_*(E)$, so the spectral sequence converges to $H_*(E)$. We need to prove that the E^2 term is what we claim. When B is a CW complex, the groups $H_p(B; H_q(F))$ can be calculated by the cellular chain complex

$$C^{\mathrm{CW}}_p(B; H_q(F)) = H_p(B^p, B^{p-1}) \otimes H_q(F). \tag{3.2}$$

The proof in the case B is a CW complex will consist of identifying the groups (3.2) with the E^1 term (3.1), and identifying the cellular boundary map with the d^1 differential in the spectral sequence.

Let $\sigma : D^p \to B^p$ be the characteristic map of a cell in B. Let $p_\sigma : E_\sigma \to D^p$ be the pullback fibration, and let $E_{\partial\sigma} \to \partial D^p$ be the restriction. Let $V_p \subseteq B_p$ denote the complement of the center points of the p-cells and let $U_p = p^{-1}(V_p)$. Then $B_{p-1} \to V_p$ is a homotopy equivalence, and the five-lemma gives that $E_{p-1} \to U_p$ is a weak equivalence. Similarly if we let $U_\sigma = p_\sigma^{-1}(D^p - \{0\})$, then $E_{\partial\sigma} \to U_\sigma$ is a weak equivalence. Using these open sets, an excision argument gives that the characteristic maps induce isomorphisms

$$\bigoplus_\sigma H_{p+q}(E_\sigma, E_{\partial\sigma}) \to H_{p+q}(E_p, E_{p-1}) = E^1_{p,q}.$$

Let us calculate the groups $H_{p+q}(E_\sigma, E_{\partial\sigma})$. Pick once and for all a CW approximation $K \to F$. Then, the constant map $K \times D^p \to D^p$ to the basepoint $* \in \partial D^p$ is homotopic to the projection map. Lift this homotopy to a homotopy of maps $K \times D^p \to E_\sigma$, which starts at the projection $K \times D^p \to K \to F \subseteq E_\sigma$. We can also assume that the lifted homotopy is relative to $K \times \{*\}$. That gives a diagram of fibrations

$$\begin{tikzcd}
K \times D^p \arrow[r, "g_\sigma"] \arrow[d, "\pi"'] & E_\sigma \arrow[d, "p_\sigma"] \\
D^p \arrow[r, equal] & D^p
\end{tikzcd}$$

where π denotes the projection to the second coordinate. The vertical maps are Serre fibrations, and over $* \in D^p$, the map of fibers is the fixed CW approximation $K \to F$. Then the LES of homotopy groups and the five-lemma shows that the map $g_\sigma : K \times D^p \to E_\sigma$ is a weak equivalence. Restricting to ∂D^p does not change the fibers, so the five-lemma gives that $g_{\partial\sigma} : K \times \partial D^p \to E_{\partial\sigma}$ is also a weak equivalence. Thus g_σ gives an equivalence of pairs $K \times (D^p, \partial D^p) \to (E_\sigma, E_{\partial\sigma})$ which induces an isomorphism

$$H_{p+q}(E_\sigma, E_{\partial\sigma}) \cong H_q(K) \otimes H_p(D^p, \partial D^p) = H_q(K) \tag{3.3}$$

We have constructed the desired isomorphism

$$E^1_{p,q} = \bigoplus_\sigma H_q(K) \cong C_p^{\mathrm{CW}}(B; H_q(K)) \cong C_p^{\mathrm{CW}}(B; H_q(F))$$

and it remains to see that the d^1 differential and the cellular differential correspond under this isomorphism.

Let us first recall the definition of the cellular boundary map in a convenient form. It is the composition of the connecting homomorphism $H_p(B^p, B^{p-1}) \to H_{p-1}(B_{p-1})$ with the map $H_{p-1}(B^{p-1}) \to H_{p-1}(B^{p-1}, B^{p-2})$. The characteristic maps $\sigma : D^p_\sigma \to B^p$ gives an isomorphism $\oplus_\sigma H_p(D^p_\sigma, \partial D^p) \to H_p(B^p, B^{p-1})$, and we have a commutative diagram

(3.4)
$$\begin{tikzcd}
\bigoplus_\sigma H_p(D^p_\sigma, \partial D^p) \arrow[r] \arrow[d, "\cong"'] & \bigoplus_\sigma H_{p-1}(\partial D^p_\sigma) \arrow[d] \\
H_p(B^p, B^{p-1}) \arrow[r] & H_{p-1}(B^{p-1}).
\end{tikzcd}$$

Let us use our special assumption on the attaching map $\phi : \partial D^p_\sigma \to B^{p-1}$. It implies that

$$\phi^{-1}(B^{p-1} - B^{p-2}) = \bigcup_i A_i \subseteq \partial D^p_\sigma$$

where the A_i are (finitely many) disjoint open disks, and ϕ restricts to a homeomorphism from each A_i to an open $(p-1)$-cell in B. Let $a_i \in A_i$ be the point that maps to the center the $(p-1)$-cell; then $\phi^{-1}(V_{p-1} - B^{p-2}) = \coprod_i A_i - \{a_i\}$. To understand the cellular boundary map on the summand $H_p(D^p_\sigma, \partial D^p_\sigma)$, we must study the diagram $\partial D^p_\sigma \to (B^{p-1}, B^{p-2}) \leftarrow \coprod_\tau (D^{p-1}_\tau, \partial D^{p-1}_\tau)$, where the first map is the attaching map of σ, and the second induces an isomorphism in homology. This is the left column in the following diagram.

(3.5)
$$\begin{tikzcd}
\partial D^p_\sigma \arrow[r] \arrow[d, "\phi"] & (\partial D^p_\sigma, \partial D^p_\sigma - \cup_i\{a_i\}) \arrow[d, "\phi"] & \cup_i(A_i, A_i - \{a_i\}) \arrow[l, "\sim"'] \arrow[d, "\phi"] \\
(B^{p-1}, B^{p-2}) \arrow[r] & (B^{p-1}, V_{p-1}) & (B^{p-1} - B^{p-2}, V_{p-1} - B^{p-2}) \arrow[l] \\
\coprod_\tau (D^{p-1}_\tau, \partial D^{p-1}_\tau) \arrow[r, "\sim"] \arrow[u, "\sim"] & \coprod_\tau (D^{p-1}_\tau, V_\tau) \arrow[u] & \coprod_\tau (D^{p-1}_\tau - \partial D^{p-1}_\tau, V_\tau - \partial D^{p-1}_\tau) \arrow[l] \arrow[u, "\sim"]
\end{tikzcd}$$

The maps labeled $\xrightarrow{\sim}$ become isomorphisms in homology, either because they are homotopy equivalences, or because they are excision maps. The diagram is commutative, so instead of following the left column from the top left corner to the bottom left corner, we can go along the other three edges of the diagram (6 maps forming a $\supset$ shape). In any case, the cellular boundary map is given by the resulting maps $H_{p-1}(\partial D^{p-1}_\sigma) \to H_{p-1}(D^{p-1}_\tau, \partial D^{p-1}_\tau)$, one for each (σ, τ). The cellular boundary map in $C_*(B; H_q(K))$ is obtained by tensoring these maps with (the identity map of) $H_q(K)$ or equivalently, by multiplying all spaces in the above diagrams with (the identity map of) K and then applying homology.

We now compare this to the differential $d^1 = jk$ in the spectral sequence. This is the composition connecting homomorphism $H_{p+q}(E_p, E_{p-1}) \to H_{p+q-1}(E_{p-1})$ with the map $H_{p+q-1}(E_{p-1}) \to H_{p+q-1}(E_{p-1}, E_{p-2})$. This is the exact same construction as the cellular boundary, except that all spaces are replaced by their inverse image under $p : E \to B$, and thus it can be analyzed by diagrams similar to (3.4) and (3.5). For the diagram (3.4), we get

$$\begin{tikzcd}
\bigoplus_\sigma H_{p+q}(E_\sigma, E_{\partial\sigma}) \arrow[r] \arrow[d] & \bigoplus_\sigma H_{p+q-1}(E_{\partial\sigma}) \arrow[d] \\
H_{p+q}(E_p, E_{p-1}) \arrow[r, "j"] & H_{p+q-1}(E_{p-1}),
\end{tikzcd}$$

We can use the weak equivalence $g_\sigma : K \times D^p_\sigma \to E_\sigma$ to replace the top row by the top row of (3.4) tensored with (the identity map of) $H_q(K)$:

$$\begin{tikzcd}
\bigoplus_\sigma H_q(K) \otimes H_p(D^p_\sigma, \partial D^p_\sigma) \arrow[r] \arrow[d, "(g_\sigma)_*"] & \bigoplus_\sigma H_q(K) \otimes H_{q-1}(\partial D^p_\sigma) \arrow[d, "(g_{\partial\sigma})_*"] \\
\bigoplus_\sigma H_{p+q}(E_\sigma, E_{\partial\sigma}) \arrow[r] \arrow[d] & \bigoplus_\sigma H_{p+q-1}(E_{\partial\sigma}) \arrow[d] \\
H_{p+q}(E_p, E_{p-1}) \arrow[r, "j"] & H_{p+q-1}(E_{p-1}),
\end{tikzcd}$$

A similar procedure works to analyze the map $E_{p-1} \to (E_{p-1}, E_{p-2}) \leftarrow \coprod_\tau (E_\tau, E_{\partial\tau})$: All the spaces in diagram (3.5) have a preferred map to B, so we pull back the fibration $p : E \to B$ to each space in the diagram and write the resulting total space instead. For example A_i becomes $(p_{\partial\sigma})^{-1}(A_i)$ and the map ϕ becomes

$$\cup_i (p^{-1}_{\partial\sigma}(A_i), p^{-1}_{\partial\sigma}(A_i - \{a_i\}) \xrightarrow{\tilde{\phi}} (E_{p-1} - E_{p-2}, U_{p-1} - E_{p-2}).$$

We will not write out the resulting diagram, but let us call it the "lifted diagram". We can use the weak equivalences $g_{\partial\sigma} : K \times \partial D^p_\sigma \to E_\sigma$ and $g_\tau : K \times D^{p-1} \to E_\tau$ to compare the diagram (3.5) and the lifted diagram. It suffices to compare the spaces in the outer 6-map $\supset$-shaped part of the diagram. If we multiply all these spaces in diagram (3.5) by K, then either a restriction of $g_{\partial\sigma}$ or of some g_τ will define a map from $K\times$(3.5) to the lifted diagram, and the resulting ladder will be commutative, *except* at one point, namely the diagram

$$\begin{tikzcd}
K \times \big(\cup_i (A_i, A_i - \{a_i\}) \big) \arrow[r, "\sim"] \arrow[d, "K\times\phi"] & \cup_i(p^{-1}_{\partial\sigma}(A_i), p^{-1}_{\partial\sigma}(A_i - \{a_i\})) \arrow[d] \\
K \times (B^{p-1} - B^{p-2}, V_{p-1} - B^{p-2}) \arrow[r, "\sim"] & (E_{p-1} - E_{p-2}, U_{p-1} - E_{p-2}).
\end{tikzcd}$$

In this diagram, the horizontal maps are weak equivalences, but the top map is induced by g_σ, and the bottom map by various g_τ's, and they need not agree: if $b \in B$ is in the image of the characteristic maps of both σ and τ, there results two weak equivalences of K with $p^{-1}(b)$. Thus, to make the diagram commutative, we should use a different map $K \to K$ for each disk A_i. However, our assumptions on $\pi_1(B)$ imply that these maps $K \to K$ all induce the identity map on homology, and hence the diagram *is* commutative after applying homology.

We have proved that $d^1 = H_q(K) \otimes \partial_*^{\mathrm{CW}}$ by splitting both d^1 as a (fairly long) composition of maps, and proving that each map in the composition is $H_q(K)$ tensor the corresponding map in a decomposition of ∂_*^{CW}. $\square$

4. Examples

4.1. **Loop space of a sphere.** Let's calculate the homology of ΩS^n using the Serre spectral sequence of the fibration $\Omega S^n \to PS^n \to S^n$. We have

$$E^2_{p,q} = H_p(S^n; H_q(\Omega S^n)) \Rightarrow H_{p+q}(PS^n)$$

We can calculate the E^2 term using the universal coefficient theorem. Since $H_p(S^n)$ has not torsion, the Tor term vanishes, and we have an isomorphism

$$E^2_{p,q} = H_p(S^n) \otimes H_q(\Omega S^n) = \begin{cases} H_q(\Omega S^n) & p = 0, n \\ 0 & \text{otherwise.} \end{cases}$$

We can immediately see that $E^2_{n,0} = \mathbb{Z}$, generated by the fundamental class $\sigma = [S^n] \in H_n(S^n)$. Since d^r has bidegree $(-p, p-1)$, the only possible non-zero differential is

$$d^n : E^n_{n,q} \to E^n_{0,q+n-1}.$$

This map has to be an isomorphism (except when $q = -(n-1)$), since otherwise we would have non-zero classes in $E^{n+1} = E^\infty$, in contradiction with $H_n(PX) = 0$ for $n \neq 0$. Combining this with the formula for $E^2 = E^n$, we can deduce (this part is better explained on a blackboard) that

$$H_k(\Omega S^n) = \begin{cases} \mathbb{Z} & (n-1)|k \\ 0 & \text{otherwise.} \end{cases}$$

If the generator in $E^2_{0,k(n-1)} = H_{k(n-1)}(\Omega S^n)$ is denoted x_k, then the generator of $E^2_{n,k(n-1)}$ is $\sigma \otimes x_k \in H_n(S^n) \otimes H_{k(n-1)}(\Omega S^n)$, and the differential is given by $d(\sigma \otimes x_k) = x_{k+1}$.

4.2. **Euler characteristic.** Let $F \to E \to B$ be a Serre fibration and k a field such that $\pi_1(B)$ acts trivially on $H_*(F; k)$. If $H_*(B; k)$ and $H_*(F; k)$ are finite dimensional vector spaces (where we write $H_* = \oplus_n H_n$), then $H_*(E)$ is also finite dimensional and

$$\chi(E; k) = \chi(B; k)\chi(F; k)$$

where $\chi(-; k)$ is the usual alternating sum of the dimensions of $H_p(-; k)$.

4.3. **Thom isomorphism.** Let $p : E \to B$ be a *spherical fibration*, i.e. a fibration such that all fibers are weakly equivalent to S^n. Let us say that the fibration is *oriented* if $\pi_1(B)$ acts trivially on $H_*(S^n)$.

Theorem 4.1. *If $p : E \to B$ is an oriented spherical fibration, with a section $s : B \to E$ (i.e. $ps = \mathrm{id}$), then there is an isomorphism*

$$H_{p+n}(E, sB) \cong H_p(B)$$

4.4. **The Atiyah-Hirzebruch spectral sequence.** If h_* is a homology theory (satisfying the axioms in Hatcher's book), then any filtered space also gives an exact pair with $A_{p,q} = h_{p+q}(X_p)$ and $E_{p+q}(X_p, X_{p-1})$. If the filtration has $X_{-1} = \emptyset$, then the spectral sequence converges to $\varinjlim_n h_{p+q}(X_n)$, which is $h_{p+q}(X)$ when e.g. $X = \cup X_p$ is a union of CW subcomplexes. If X is a CW complex and we filter it by skeleta, we get $E^1 = h_{p+q}(X^p, X^{p-1}) = C_p^{\mathrm{CW}}(X; h_q(*))$. In the same way as for the Serre spectral sequence it can be checked that d^1 is the cellular boundary, so we get a spectral sequence with

$$E^2_{p,q} = H_p(X; h_q(*)) \Rightarrow h_{p+q}(X).$$

It need not be first quadrant, but it is always in the right halfplane. Thus, any given spot (p, q) with $p > 0$ has only finitely many non-zero differentials going out of it, but could have infinitely many non-zero differentials hitting it. Still, it *always* converges to $h_*(X)$. (There is a cohomology version whose convergence is much more delicate.)

The Atiyah-Hirzebruch spectral sequence and the Serre spectral sequence can be mixed to a spectral sequence of a fibration $F \to E \to B$ where $\pi_1(B)$ acts trivially on $h_*(F)$. The spectral sequence has

$$E^2_{p,q} = H_p(B; h_q(F)) \Rightarrow h_q(F).$$

As a special case we could have $h_q = H_q(-;k)$ for some field. In this case, the universal coefficient theorem gives an isomorphism $E^2_{p,q} = H_p(B;k) \otimes H_q(F;k)$.

5. MULTIPLICATIVE STRUCTURE

5.1. **Filtered chain complexes.** The spectral sequence of a filtered space is a special case of the spectral sequence of a filtered complex, which we briefly describe. Let (C_n, ∂) be a chain complex of abelian groups. (We assume $\partial : C_n \to C_{n-1}$.) Let $C^p_n \subseteq C_n$ be subgroups with $C^{p-1}_n \subseteq C^p_n$ and $\partial(C^p_n) \subseteq C^p_{n-1}$. Thus ∂ restricts to a differential on each C^p_*, and we have short exact sequences of chain complexes

$$0 \to C^{p-1}_n \xrightarrow{i} C^p_n \xrightarrow{j} C^p_n / C^{p-1}_n \to 0,$$

inducing long exact sequences in homology. We get an exact couple by setting

$$\begin{aligned} A_{p,q} &= H_{p+q}(C^p_*) \\ E_{p,q} &= H_{p+q}(C^p / C^{p-1}). \end{aligned}$$

The exact couple of a filtered space is the special case $C_* = C_*(X)$ and $C^p_* = C_*(X_p)$. The discussion of convergence in that case applies in the exact same way here. More precisely, let us make the following two assumptions.

1. The map $\varinjlim H_*(C^p_*) \to H_*(C_*)$ is an isomorphism. This holds if the filtration is *exhausting*, i.e. that $C_* = \cup_p C^p$.
2. The map $i^r : H_*(C^{p-r}) \to H_*(C^p)$ has $\cap \mathrm{Im}(i^r) = 0$.

Then the spectral sequence converges to $H_*(C_*)$: There is a filtration on $H_n(C_*)$ and an isomorphism

$$E^\infty_{p,q} \cong F^p_{p+q} / F^{p-1}_{p+q}.$$

Again, we write

$$E^1_{p,q} = H_{p+q}(C^p_* / C^{p-q}_*) \Rightarrow H_{p+q}(C_*)$$

5.2. **Chain level products.** Let ${}^1C_*$, ${}^2C_*$ be two filtered chain complexes, filtered by ${}^1C^p_n$ and ${}^2C^p_n$. Then the tensor product

$${}^3C_n = ({}^1C \otimes^2 C)_n = \bigoplus_{s+t=n} {}^1C_s \otimes {}^2C_t$$

has a differential given by $\partial(x \otimes y) = (\partial x) \otimes y + (-1)^{|x|} x \otimes (\partial y)$, and there is a compatible filtration in which $({}^1C \otimes {}^2C)^p_n$ is the sum of the image of ${}^1C^q_s \otimes {}^2C^r_t$ over all indices with $s + t = n$, $q + r = p$. This situation gives rise to three spectral sequences, and there is an induced product on filtration quotients

$$({}^1C^p_n/{}^1C^{p-1}_n) \times ({}^2C^{p'}_{n'}/{}^2C^{p'-1}_{n'}) \xrightarrow{\times} ({}^3C^{p+p'}_{n+n'}/{}^3C^{p+p'-1}_{n+n'}),$$

given by $[x]\times[y] = [x\otimes y]$, which in turn induces a product on homology

$${}^1E^1_{p,q} \times {}^2E^1_{p',q'} \xrightarrow{\times} {}^3E^1_{p+p',q+q'}.$$

If we can show that d^1 satisfies a Leibniz rule, then there is an induced product on the E^2 terms; if d^2 satisfies a Leibnitz rule, there is an induced product on E^3, etc.

Proposition 5.1. *The product* ${}^1E^1_{p,q} \times {}^2E^1_{p',q'} \to {}^3E^1_{p+p',q+q'}$ *defined above induces well defined products on subquotients*

$${}^1E^r_{p,q} \times {}^2E^r_{p',q'} \to {}^3E^r_{p+p',q+q'}$$

for all r*, and* d^r *satisfies the Leibnitz rule* $d^r(e \times f) = (d^r e) \times f + (-1)^e e \times (d^r f)$ *with respect to these. (Here* $|e| = p + q$ *if* $e \in E^r_{p,q}$*.)*

Finally, we would like the relation between E^∞ and $H_*(C_*)$ to reflect something about the products. This is indeed the case.

Proposition 5.2. *In the above situation, the product on* E^1 *induces a product*

$${}^1E^\infty_{p,q} \times {}^2E^\infty_{p',q'} \xrightarrow{\times} {}^3E^\infty_{p+p',q+q'},$$

The filtrations on $H_*({}^1C_*)$*,* $H_*({}^2C_*)$ *and* $H_*({}^3C_*)$ *satisfy* $({}^1F^p)\times({}^2F^{p'}) \subseteq {}^3F^{p+p'}$*, so there is an induced multiplication on filtration quotients. With respect to these products, the isomorphisms* $(\nu = 1, 2, 3)$

$${}^\nu\varphi : {}^\nu E^\infty_{p,q} \to {}^\nu F^p_{p+q}/{}^\nu F^{p-1}_{p+q}$$

satisfy ${}^3\varphi(e \times f) = {}^1\varphi(e) \times {}^2\varphi(f)$*.*

Corollary 5.3. *If* ${}^1C_*$*,* ${}^2C_*$ *and* ${}^3C_*$ *are filtered chain complexes, and*

$$\times : {}^1C_n \otimes {}^2C_m \to {}^3C_{n+m}$$

are maps satisfying $d(x \times y) = (dx) \times y + (-1)^n x \times dy$ *and that* $({}^1C^p_n) \times ({}^2C^{p'}_{n'}) \subseteq {}^3C^{p+p'}_{n+n'}$*, then there are induced bigraded products* ${}^1E^r \otimes {}^2E^r \to {}^3E^r$ *for all* r*. For each* r *they satisfy the Leibniz rule, and the induced product on* $\mathrm{Ker}(d^r)/\mathrm{Im}(d^r)$ *is the product on* E^{r+1}*. Furthermore the isomorphism* $E^\infty \cong F^p/F^{p-1}$ *preserves products.*

Proof. If we give $\times {}^1C_* \otimes {}^2C_*$ the induced differential and filtration, the map $\times {}^1C_* \otimes {}^2C_* \to {}^3C_*$ is a filtered chain map, and then induces a map of spectral sequences. The result then follows from the proposition. □

Corollary 5.4. *Let* X*,* Y *and* Z *be filtered spaces, and let* $\mu : X \times Y \to Z$ *be a filtered map (i.e.* $\mu(X_p \times Y_{p'}) \subseteq Z_{p+p'}$*). In the resulting spectral sequences there are products*

$$^XE^r_{p,q} \otimes {}^YE^r_{p',q'} \to {}^ZE^r_{p+p',q+q'}$$

such that d^r *satisfies the Leibniz rule and the isomorphism* $E^\infty_{p,q} \cong F^p_{p+q}/F^{p-1}_{p+q}$ *preserves products.*

Proof. Apply the previous corollary to the filtered chain complexes $C_*(X)$, $C_*(Y)$ and $C_*(Z)$, and the product $C_*(X) \otimes C_*(Y) \to C_*(Z)$ which is the composition of the cross product and μ_*. □

Corollary 5.5. *Let* ${}^\nu F \to {}^\nu E \xrightarrow{{}^\nu p} {}^\nu B$ *be Serre fibrations,* $\nu = 1, 2, 3$ *where* $\pi_1({}^\nu B)$ *acts trivially on* $H_*({}^\nu F)$*. Then a commutative diagram*

$$\begin{tikzcd}
{}^1E \times {}^2E \arrow[r, "\mu"] \arrow[d, "{}^1p \times {}^2p"'] & {}^3E \arrow[d, "{}^3p"] \\
{}^1B \times {}^2B \arrow[r, "\mu"] & {}^3B
\end{tikzcd}$$

induces a product on spectral sequences

$$^1E^r_{p,q} \otimes {}^2E^r_{p',q'} \to {}^3E^r_{p+p',q+q'}$$

satisfying the Leibniz rule, and such that the isomorphism $E^\infty \cong F^p/F^{p-1}$ *preserves products. On the* E^2 *terms, the products*

$$H_p({}^1B; H_q({}^\nu F)) \otimes H_{p'}({}^2B; H_{q'}({}^2F)) \to H_{p+p'}({}^3B; H_{q+q'}({}^3F)).$$

agrees with the product induced from μ_* *and the cross products.*

5.3. **Example: Loop space of S^n.**

The map $\Omega S^n \times \Omega S^n \to \Omega S^n$ given by concatenation of loops induces a product on $H_*(\Omega S^n)$. Let's use the Serre spectral sequence to calculate the ring structure of $H_*(\Omega S^n)$. The map $\Omega S^n \to *$ can be considered a fibration with fiber ΩS^n. Concatenation of paths gives a map $\Omega S^n \times PS^n$ that fits in a commutative diagram

$$\begin{tikzcd}
\Omega S^n \times PS^n \arrow[r] \arrow[d] & PS^n \arrow[d] \\
* \times S^n \arrow[r, equal] & S^n.
\end{tikzcd}$$

The induced map on fibers is concatenation of loops. By the general theory, this diagram gives a product on spectral sequences ${}^1E^r \otimes {}^2E^r \to {}^3E^r$, where ${}^1E^r$ is the spectral sequence of the fibration $\Omega S^n \to \Omega S^n \to *$, and both ${}^2E^r = {}^3E^r$ is the spectral sequence of the path-loop fibration. We already calculated what the latter looks like, namely

$${}^2E^2_{p,q} = {}^3E^2_{p,q} = \begin{cases} \mathbb{Z} & p = 0, n \text{ and } (n-1)|q \\ 0 & \text{otherwise.} \end{cases}$$

and we had names for the generators: If $\sigma \in E^2_{n,0} = H_n(S^n)$ is the fundamental class, then we have generators

$$\begin{aligned}
x_1 &= d\sigma \in E^2_{0,n-1} & \sigma \otimes x_1 &\in E^2_{n,n-1} \\
x_2 &= d(\sigma \otimes x_1) \in E^2_{0,2n-2} & \sigma \otimes x_2 &\in E^2_{n,2n-2} \\
x_3 &= d(\sigma \otimes x_2) \in E^2_{0,3n-3} & \sigma \otimes x_3 &\in E^2_{n,3n-3} \\
&\dots
\end{aligned}$$

Now, the Serre spectral sequence of the fibration $\Omega S^n \to \Omega S^n \to *$ is very boring: It has $E^r_{0,q} = H_q(\Omega S^n)$ for all q and $E^r_{p,q} = 0$ for $p > 0$, and all differentials vanish. Thus, the induced product of spectral sequences in this case is

$$H_q(\Omega S^n) \otimes E^r_{p',q'} \xrightarrow{\times} E^r_{p',q+q'}.$$

It is not hard to see that the product $x_k \times \sigma \in E^2_{n,n-1}$ is the same element that we previously denoted $\sigma \otimes x_k$ (you have to trace through what the product on E^2 is, and check that it is the same as the one appearing in the universal coefficient theorem). Thus we have

$$\begin{aligned}
&x_2 = d(x_1 \times \sigma) = (dx_1) \times \sigma \pm x_1 \times (d\sigma) = \pm x_1^2 \\
&x_3 = \pm d(x_1^2 \times \sigma) = (dx_1^2) \times \sigma \pm x_1^2 \times (d\sigma) = \pm x_1^3 \\
&\dots \\
&x_k = \pm x_1^k,
\end{aligned}$$

where the power x_1^k is calculated with respect to the product on $H_*(\Omega S^n)$ induced form concatenation of loops. Thus, we might as well use $x_1^k \in H_{k(n-1)}(\Omega S^n)$ as a generator, and we have proved the following theorem.

Theorem 5.6. *There is an isomorphism of rings*

$$\mathbb{Z}[x] \to H_*(\Omega S^n),$$

where $x \in H_{n-1}(\Omega S^n)$ *is the image of the identity map under the Hurewicz homomorphism* $\pi_n(S^n) = \pi_{n-1}(\Omega S^n) \to H_{n-1}(\Omega S^n)$. ☐

6. COHOMOLOGY SPECTRAL SEQUENCES

6.1. **Cohomology spectral sequence of a filtered space.** Suppose we want to use a spectral sequence to "calculate" the cohomology $H^*(X)$, starting with $H^*(X_p, X_{p-1})$. This can be done, using a special case of the spectral sequence of a filtered complex. To make the degrees work out, we need to change the signs of many indices. Let

$$\begin{aligned} C_{-n} &= C^n(X) \\ C_{-n}^{-p} &= C^n(X, X_{p-1}). \end{aligned}$$

Then the quotient $C_{-n}^{-p}/C_{-n}^{-p-1}$ is canonically isomorphic to $C^n(X_p, X_{p-1})$ and we get a spectral sequence with $E^1_{-p,-q} = H^{p+q}(X_p, X_{p-1})$. It converges to $H^*(X)$ provided

(i) The map $\varinjlim H^*(X, X_p) \to H^*(X)$ is an isomorphism. This holds if $X_{-1} = \emptyset$.

(ii) The map $i^r : H^n(X, X_{p+r}) \to H^n(X, X_p)$ has $\cap \mathrm{Im}(i^r) = 0$. This holds if $X = \cup X_p$ with the direct limit topology, *and* $H^n(X, X_{p+r})$ is independent of r for large r.

Written this way, the differentials in the spectral sequence have the same degrees as in the homology case, but the spectral sequence is concentrated in a different $\frac{3}{8}$ of the plane. It is customary to change signs of p and q, and indicate this by writing them in the superscript instead of subscript. Thus the cohomology spectral sequence has

$$E_1^{p,q} = H^{p+q}(X_p, X_{p-1}),$$

the filtration on $H^*(X)$ becomes a *decreasing* fitration $H^*(X) \supseteq F^0 \supseteq F^1 \supseteq \dots$, and we have isomorphisms $E_\infty^{p,q} = F^p_{p+q}/F^{p+1}_{p+q}$.

6.2. **Cup product.** The following theorem can be deduced from theorem 5.1.

Theorem 6.1. *The cohomology Serre spectral sequence of a fibration* $F \to E \to B$ *where* $\pi_1(B)$ *acts trivially on* $H^*(F)$ *is a spectral sequence of algebras, i.e. each* E_r *has a bigraded product, with respect to which*

d_r is a derivation and induces the product on E_{r+1}. The product is graded-commutative on each E_r and on

$$E_2 = H^*(B; H^*(F))$$

it agrees with the product induced by the cup-products. The isomorphism $E_\infty = F^p/F^{p+1}$ is multiplicative.

6.3. **Example: cohomology of the loop space of a sphere.** Let us calculate the ring $H^*(\Omega S^n; Z)$ using the Serre spectral sequence of the path-loop fibration. This is very similar to the calculation of the homology algebra $H_*(\Omega S^n)$.

The E_2 term is

$$E_2^{p,q} = H^p(S^n; H^q(\Omega S^n)) = H^p(S^n) \otimes H^q(\Omega S^n) = \begin{cases} H^q(\Omega S^n) & p = 0, n \\ 0 & \text{otherwise,} \end{cases}$$

so the only possible differential is $d_n : E_2^{0,q} \to E_2^{n,q-n+1}$. Since the E_∞ term vanishes except in degree $(0,0)$, d^n must give an isomorphism $H^q(\Omega S^n) \to H^{q-n+1}(\Omega S^n)$. By the same induction argument as for homology, we get

$$H^q(\Omega S^n) = \begin{cases} \mathbb{Z} & (n-1)|q \\ 0 & \text{otherwise.} \end{cases}$$

Let $y_k \in H^{k(n-1)}(\Omega S^n)$ and $\sigma \in H^n(S^n)$ be generators. Then the product $\sigma y_k \in E_2^{n,k(n-1)} = \mathbb{Z}$ is a generator. Since $d_n : E_n^{0,k(n-1)} \to E_n^{n,(k-1)(n-1)}$ is an isomorphism, we can assume that $d_n(y_k) = y_{k-1}\sigma$ (otherwise, change the signs of the y_k's).

There are now two cases, depending on whether n is even or odd. Let us first treat the case where n is odd. In this case, all the elements y_k have even degree, so they commute strictly with all elements in the spectral sequence. In particular we get

$$d_n(y_1^k) = k y_1^{k-1} d(y_1) = k(\sigma y_1^{k-1}).$$

Comparing this formula with $d_n y_k = \sigma y_{k-1}$ it follows by induction that $y_1^k = k! y_k$, and hence the ring structure is

$$y_l y_k = \binom{k+l}{k} y_{k+l}.$$

The case where n is even is a little different. In this case $y_1^2 = 0$ by graded commutativity. Letting σ and y_k be additive generators as before, we first investigate multiplication by y_1. We can write $y_1 y_k = \alpha(k) y_{k+1}$ for some function α. Differentiating gives

$$\alpha(k)\sigma y_k = \alpha(k) d(y_{k+1}) = d(y_1 y_k) = \sigma y_k - y_1 \sigma y_{k-1} = (1 - \alpha(k-1))\sigma y_k,$$

and hence $\alpha(k) = 1 - \alpha(k-1)$. Therefore

$$y_1 y_k = \begin{cases} y_{k+1} & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

We now investigate the power y_2^k. Since y_2 is even dimensional, it strictly commutes with everything.

$$d(y_2^k) = k y_2^{k-1} dy_2 = k y_2^{k-1} y_1 \sigma.$$

If we write $y_2^k = \alpha(k) y_{2k}$, this gives $\alpha(k) dy_{2k} = k\alpha(k-1) y_{2(k-1)} y_1 \sigma$. On the other hand $dy_{2k} = \sigma y_{2k-1} = \sigma y_1 y_{2(k-1)}$, so we get $\alpha(k) = k\alpha(k-1)$ and hence $\alpha(k) = k!$. From this it follows that the multiplication is given by

$$\begin{aligned} y_{2k} y_{2l} &= \binom{k+l}{k} y_{2k+21} \\ y_{2k+1} y_{2l} = y_{2k} y_{2l+1} &= \binom{k+l}{k} y_{2k+2l+1} \\ y_{2k+1} y_{2k+2} &= 0 \end{aligned}$$

Appendix A. Extra stuff

Things I haven't fully covered yet.

A.1. **Example: Double loop space of S^n.** Let us now consider the Serre fibration $\Omega^2 S^n \to \Omega P S^n \to \Omega S^n$ obtained by applying Ω to the fibration from the last section. In this case we have more structure than before: We now have a concatenation map on total spaces $\Omega P S^n \times \Omega P S^n$ which fits in a commutative diagram

$$\begin{tikzcd} \Omega P S^n \times \Omega P S^n \arrow[r] \arrow[d] & \Omega P S^n \arrow[d] \\ \Omega S^n \times \Omega S^n \arrow[r] & \Omega S^n \end{tikzcd}$$

where all horizontal maps are concatenation of loops. The general theory of products in the homology Serre spectral sequence can then be applied with all three fibrations being $\Omega^2 S^n \to \Omega P S^n \to \Omega S^n$. If we denote the Serre spectral sequence of this fibration by E^r, we thus have products

$$E^r_{p,q} \otimes E^r_{p',q'} \to E^r_{p+p',q+q'}$$

for all r. This gives a *bigraded product* on E^r which is associative since concatenation is homotopy associative. In this case, the spectral sequence becomes a *spectral sequence of algebras*, meaning that each

E^r is a bigraded algebra in which the differential satisfies the Leibniz rule.

Let us consider everything in mod 2 homology (and for readability we omit this from the notation, and write $H_p(X)$ for $H_p(X; \mathbb{F}_2)$). Then the spectral sequence has

$$E^2_{p,q} = H_p(\Omega S^n; H_q(\Omega^2 S^n)) = H_p(\Omega S^n) \otimes H_q(\Omega^2 S^n),$$

and to begin with we only know the line $E^2_{*,0} = H_*(\Omega S^n) = \mathbb{F}_2[x]$, for $x \in H_{n-1}(\Omega S^n)$. Since the spectral sequence converges to $H_*(\Omega P S^n) = H_*(*)$, we have $E^\infty_{p,q} = 0$ except in bidegree $(0,0)$. Thus, there must be an element $y_1 \in H_{n-2}(\Omega^2 S^n)$ such that $d^{n-1}x = y$. We can now use the Leibniz rule to find a lot more differentials in E^{n-1}. In general the Leibniz rule gives for any element z that

$$d(z^2) = (dz)z \pm z(dz) = 0$$

(using that we're working mod 2), and hence we get all differentials on $E^{n-1}_{*,0}$ by the formulas

$$d^{n-1}(x^{2k}) = 0, \quad d^{n-1}(x^{2k+1}) = yx^{2k} \in E^{n-1}_{2k(n-1),n-2}.$$

Then using the Leibniz rule again, this allows us to find all differentials in $E^{n-1}_{p,q} = H_p(\Omega S^n) \otimes H_q(\Omega^2 S^n)$. Namely if $z \in H_q(\Omega^2 S^n)$, then

$$\text{(A.1)} \qquad d^{n-1}(zx^{2k}) = 0, \quad d^{n-1}(zx^{2k+1}) = zyx^{2k}.$$

This tells us firstly that $E^n_{0,*}$ is the quotient ring $H_*(\Omega^2 S^n)/(y)$. The next line $E^n_{n-1,*}$ could potentially be non-zero if y is a zero-divisor in $H_*(\Omega^2 S^n)$, although we will show that this doesn't happen. The line $E^n_{2(n-1),*}$ has $x^2 \in E^n_{2(n-1),0}$, and multiplication by this element gives an isomorphism from the line $E^n_{0,*}$ to this line.

The next possible differential is $d^{2(n-1)}$, which must map $x^2 \in E^{2(n-1)}_{2(n-1),0}$ to some non-zero element $y_1 \in E^{2(n-1)}_{0,2n-3}$, and then the differential is completely determined by the formula

$$\text{(A.2)} \qquad d^{2(n-1)}(zx^{4k}) = 0, \quad d^{2(n-1)}(zx^{4k+2}) = zy_1x^{4k}.$$

We see that $E^{2n-1}_{0,*}$ is the quotient ring $H_*(\Omega^2 S^n)/(y, y_1)$. The line $E^{2(n-1)}_{n-1,*}$ is unchanged. The line $E^{2(n-1)}_{2(n-1),*}$ could potentially be non-zero if $y_1 \in H_*(\Omega^2 S^n)/(y)$ is a zero-divisor. Etc. On the $(4(n-1),*)$ line, the element x^4 survives.

The next possible differential seems to be $d^{3(n-1)}$, and as far as I can see, there is no obvious reason that e.g. $d^{3(n-1)}x^4 = 0$. This is not what happens however, but that's something one has to prove. Let's make a short digression.

Definition A.1. In the Serre spectral sequence of a fibration $F \to E \to B$, an element $x \in H_k(B)$ is called *transgressive* if $d^p x = 0$ for $p < k$. If $y \in H_{k-1}(F)$ is an element whose class in $E^k_{k-1,0}$ satisfies $d^k x = y$, then we say that x *transgresses* to y.

The following theorem is part of "Kudo's transgression theorem". We will use it as a black box.

Theorem A.2. *Consider the Serre spectral sequence of a fibration of the form* $\Omega^2 X \to \Omega PX \to \Omega X$, *and let* $x \in H_p(\Omega X) = E^2_{p,0}$. *If* x *transgresses to an element* $y \in H_{p-1}(\Omega^2 S^n)$, *then* x^2 *transgresses to an element* $Q_1(y) \in H_{2p-1}$.

Returning to our spectral sequence we can then rewrite our element y_1 as $Q_1(y)$, and we also know that $d^{3(n-1)}(x^4) = 0$ and that $d^{4(n-1)}x^4 = y_2 = Q_1 Q_1(y)$. In fact, this is enough information to completely determine what the spectral sequence looks like. Some fiddling around will convince you that

$$H_*(\Omega^2 S^n) = \mathbb{F}_2[y, y_1, y_2, \ldots],$$

where $y_k \in H_*(\Omega^2 S^n)$ are defined by $y_k = Q_1(y_{k-1})$. The best way to write give a formal proof of this is to use the following theorem, known as the "comparison theorem". You can think of it as an analogue of the 5-lemma. Again, we will use it as a black box, although the proof is straightforward (it's in Hatcher's notes).

Theorem A.3. *Let* ${}^{I}E^r_{p,q} \to {}^{II}E^r_{p,q}$ *be a map of first quadrant spectral sequences. If two of the three maps*

(i) ${}^{I}E^r_{*,0} \to {}^{II}E^r_{*,0}$
(ii) ${}^{I}E^r_{0,*} \to {}^{II}E^r_{0,*}$
(iii) ${}^{I}E^\infty_{*,*} \to {}^{II}E^\infty_{*,*}$

are isomorphisms, then so is the third.

Knowing the differentials on all x^k in the Serre spectral sequence for $\Omega^2 S^n \to \Omega PS^n \to \Omega S^n$ means that we have a map *from* the spectral sequence defined completely algebraically by

$${}^{I}E^2_{*,*} = \mathbb{F}_2[x, y_0, y_1, y_2, \ldots],$$

where x has bidegree $(n-1, 0)$, y has bidegree $(0, n-2)$, and y_k has bidegree $(0, 2^k(n-1))$. The differentials are so that x^{2^k} transgresses to y_k and that the Leibniz rule is satisfied (this determines the differentials uniquely). The E^∞ term of this spectral sequence vanishes except in degree $(0,0)$. Thus the map of spectral sequences induces an isomorphism in E^∞ and in $E^2_{*,0}$, so it must also be an isomorphism on $E^2_{0,*}$.

Proof of Proposition 5.1. This is really just a matter of chasing through a lot of definitions while avoiding freaking out about indices. Let's do it. The proposition has two claims: the products are well defined for all r, and the differential satisfies the Leibnitz rule. If we formulate the proof as an induction proof, we only need to prove the Leibnitz rule on E^r, since then the product on E^{r+1} is well defined. We will use the explicit formulas for d^r and for the E^r terms as subquotients of the E^1 terms.

Let $e \in {}^1E^1_{p,q}$ be in ${}^1Z^r$ i.e. $[e]$ represents an element in the subquotient ${}^1E^r$. In the exact couple, that means we can write $ke = i^{r-1}a$ for some $a \in H_{p+q-1}(C^{p-r}_*)$ and in that case $d^r[e] = [ja]$. Let us write $e = [x]$ for a representative

$$x \in {}^1C^p_{p+q}/({}^1C^{p-1}_{p+q})$$

and spell out how to find a chain level representative for $d^r[e]$. Here is the recipe.

(1) Lift x to $\overline{x} \in {}^1C^p_{p+q}$. Then $\partial\overline{x} \in {}^1C^{p-1}_{p+q-1}$, and in here, it is a cycle which represents $k[e]$.
(2) Find elements $\overline{\alpha} \in {}^1C^{p-r}_{p+q-1}$ and $\sigma \in {}^1C^{p-1}_{p+q}$ such that

$$\partial\overline{x} = \overline{\alpha} + \partial\sigma.$$

Then $\overline{\alpha}$ is a cycle which represents an element a with $i^{r-1}(a) = ke$.
(3) Let $\alpha \in {}^1C^{p-r}_{p+q-1}/{}^1C^{p-r-1}_{p+q-1}$ be the reduction of $\overline{\alpha}$. This is a cycle which represents an element of ${}^1E^1_{p-r,q+r-1}$ which in the subquotient represents $d^r[e]$.

Starting with an element $f \in {}^2E^1_{p',q'}$ which is in ${}^2Z^r$, we use the same recipe to find $d^r[f]$: Write $f = [y]$ for a representative $y \in {}^2C^{p'}_{p'+q'}/{}^2C^{p'-1}_{p'+q'}$, pick a lift $\overline{y} \in {}^2C^{p'}_{p'+q'}$ and write

$$\partial\overline{y} = \overline{\beta} + \partial\tau,$$

for elements $\overline{\beta} \in {}^2C^{p'-r}_{p'+q'-1}$ and $\tau \in {}^2C^{p'-1}_{p'+q'}$.

Finally we use the recipe to calculate $d^r[e \times f]$. Write $p'' = p + p'$ and $q'' = q + q'$. First, $x \times y \in {}^3C^{p''}_{p''+q''}/{}^3C^{p''-1}_{p''+q''}$ is a chain representing $e \times f$. Here are the steps in the recipe.

(1) $x \times y$ lifts to $\overline{x} \otimes \overline{y} \in {}^3C^{p''}_{p''+q''}$.

(2) We have

$$\begin{aligned}\partial(\overline{x}\otimes\overline{y}) &= (\partial\overline{x})\otimes y + (-1)^{|x|}\overline{x}\otimes(\partial\overline{y})\\ &= (\overline{\alpha}+\partial\sigma)\otimes\overline{y} + (-1)^{|x|}\overline{x}\otimes(\overline{\beta}+\partial\tau)\\ &= (\overline{\alpha}\otimes\overline{y}+(-1)^{|x|}\overline{x}\otimes\overline{\beta}) + ((\partial\sigma)\otimes\overline{y}+(-1)^{|x|}\overline{x}\otimes(\partial\tau))\\ &= \overline{\gamma}+\partial\upsilon,\end{aligned}$$

where

$$\upsilon = \sigma\otimes\overline{y} + (-1)^{|x|}\overline{x}\otimes\tau \in {}^3C^{p''-1}_{p''+q''}$$

and

$$\overline{\gamma} = \overline{\alpha}\otimes\overline{y} + (-1)^{|x|}\overline{x}\otimes\overline{\beta} \in {}^3C^{p''-r}_{p''+q''-1}.$$

(3) Let γ be the class of $\overline{\gamma}$ in the quotient modulo ${}^3C^{p''-r-1}_{p''+q''-1}$. Then $\gamma = \alpha\times y + (-1)^{|x|}x\times\beta$ which in the homology groups ${}^3E^1$ gives

$$[\gamma] = [\alpha]f + (-1)^{|e|}e[\beta].$$

In the subquotient ${}^3E^r$, the elements $[\alpha]$, $[\beta]$ and $[\gamma]$ become $d^r[e]$, $d^r[f]$ and $d^r([e]\times[f])$ which is the desired Leibnitz rule.

□

Proof of Proposition 5.2. That the product is defined on all E^r for $r < \infty$ means that

$$\begin{aligned}{}^1Z^r\times{}^2Z^r &\subseteq {}^3Z^r\\ {}^1B^r\times{}^2Z^r &\subseteq {}^3B^r\\ {}^1Z^r\times{}^2B^r &\subseteq {}^3B^r.\end{aligned}$$

If these hold for all r, it is easy to see they hold for $r=\infty$, so the product on E^1 induces a product on E^∞.

In the filtration on $H_*(C_*)$, F^p is the image of $H_*(C^p_*)$. It is then clear that $({}^1F^p)\times({}^2F^{p'})\subseteq {}^3F^{p+p'}$.

The definition of ϕ: Starting with $e =\in H_*(C^p_*/C^{p-1}_*)$, lift to $H_*(C^p_*)$ and map it to $H_*(C_*)$. This obviously preserves products. □