1. Exact couples

Following Hatcher’s notes, we considered an exact couple, i.e. two abelian groups $A$ and $E$, and maps

$$
\begin{array}{ccc}
A & \xrightarrow{i} & A \\
\downarrow{k} & & \downarrow{j} \\
E & & \\
\end{array}
$$

making the triangle exact at each vertex. Recall that we set $d = jk : E \to E$ and defined $E' = \ker(d)/\text{im}(d)$, $A' = iA$, $i'(a) = ia$, $j'(ia) = [ja]$, $k[e] = ke$ and proved that this gives the derived couple

$$
\begin{array}{ccc}
A' & \xrightarrow{i'} & A' \\
\downarrow{k'} & & \downarrow{j'} \\
E' & & \\
\end{array}
$$

Iterating this construction we get a sequence of exact sequences

$$
\begin{array}{ccc}
A^r & \xrightarrow{i^r} & A^r \\
\downarrow{k^r} & & \downarrow{j^r} \\
E^r & & \\
\end{array}
$$

which for $r = 1$ is the one we started with, and where each is the derived couple of the previous exact couple.

In applications, the groups $A$ and $E$ are usually (bi-)graded, and the maps $i$, $j$, $k$ have some fixed degree. We saw that the differential $d^r : E^r \to E^r$ has degree given by

$$
\deg(d^r) = \deg(j) + \deg(k) - (r - 1) \deg(i).
$$

1.1. The spectral sequence of a filtered space. We can now construct our first spectral sequence. Let $X$ be a space and $X_p \subseteq X$, $p \in \mathbb{Z}$ spaces, such that $X_p \subseteq X_{p+1}$ for all $p$. Now set

$$
\begin{align*}
A_{p,q} &= H_{p+q}(X_p) \\
E_{p,q} &= H_{p+q}(X_p, X_{p-1}),
\end{align*}
$$

let $i : A \to A$ be induced by the inclusions $X_{p-1} \to X_p$, let $j : A \to E$ be induced by the maps of pairs $(X_p, \emptyset) \to (X_p, X_{p-1})$, and let $k : E \to A$ be the connecting homomorphisms $H_{p+q}(X_p, X_{p-1}) \to H_{p+q-1}(X_{p-1})$. 

We see that in this case
\[ \deg(i) = (1, -1) \]
\[ \deg(j) = (0, 0) \]
\[ \deg(k) = (-1, 0). \]

We get a bigraded spectral sequence with
\[ \deg(d^r) = (-r, r - 1). \]

The \( E^1 \) term is given by \( E^1_{p,q} = H_{p+q}(X_p, X_{p-1}) \), and the differential \( d^1 \) is the composition
\[ H_{p+q}(X_p, X_{p-1}) \to H_{p+q-1}(X_{p-1}) \to H_{p+q-1}(X_{p-1}, X_{p-2}) . \]

**Remark 1.1.** This need not be a “first quadrant” spectral sequence. In general, all we can say is that \( E^1_{p,q} = 0 \) when \( p + q < 0. \)

1.2. **The terms of the spectral sequence.** Let us study the meaning of the terms \( E^r \) in the spectral sequence. It easier to do this more generally in the context of exact couples. Recall that a subquotient of an abelian group \( G \) is a quotient of a subgroup.

“A subquotient of a subquotient is a subquotient”: If \( A \subseteq B \subseteq G \), and \( A' \subseteq B' \subseteq B/A \), then \( B'/A' \) is canonically isomorphic to \( B''/A'' \), where \( A'' \subseteq G \) and \( B'' \subseteq G \) are the unique subgroups containing \( A \) such that \( (A''/A) = A' \) and \( B''/A = B' \).

In the derived couple of an exact couple, \( E' = \ker(d)/\im(d) \) is a subquotient of \( E \), so iteratively \( E^r \) is a subquotient of \( E \). Thus, there are canonical subgroups and isomorphisms
\[ B^r \subseteq Z^r \subseteq E \]
\[ E^r \cong Z^r/B^r \]

The meaning of \( Z^r \) can best be explained if we introduce some (slight) abuse of notation: An element \( e \in E \) with \( de = 0 \) represents an element \( [e] \in E' \). If also \( d'[e] = 0 \in E' \), then \( [e] \) represents an element which should properly be written \( [[e]] \in E''. \) In this case we will write just \( [e] \) instead of \( [[e]] \). Confusion between the two meanings of \( [e] \) can be avoided by emphasizing the group: \( [e] \in E' \) or \( [e] \in E'' \). Similarly, starting with an element \( e \in E \), we could potentially get an element of \( E^r \) which should be written with \( (r - 1) \) pairs of brackets. We will again write just \( [e] \).

Then, start with \( e \in E^1 \), we are allowed to write \( [e] \in E^2 \) if \( d^1e = 0 \). If also \( d^2[e] = 0 \) we are allowed to write \( [e] \in E^3 \). Etc. Then \( Z^r \) is the subset consisting of \( e \in E \) for which we are allowed to write \( [e] \in E^r \).
In that case \( e \in B^r \) precisely if \( [e] = 0 \in E^r \). Equivalently, \( [e] = d^{r-1}[x] \) for some \( x \in Z^{r-1} \).

Let us find more explicit formulas for \( Z^r \) and \( B^r \), and let us also find formulas for the maps \( j_r : A^r \to E^r \) and \( k_r : E^r \to A^r \) in terms of the identification \( E^r = Z^r / B^r \). We can set \( Z^1 = E \) and \( B^1 = 0 \). Also we have

\[
Z^2 = \text{Ker}(d) = \text{Ker}(jk) = k^{-1}(\text{Ker}(j)) = k^{-1}(\text{Im}(i)) \\
B^2 = \text{Im}(d) = \text{Im}(jk) = j(\text{Im}(k)) = j(\text{Ker}(i)).
\]

Fiddling around with the formula for \( d^2 \), you will find \( Z^3 = k^{-1}(\text{Im}(r^2)) \) and \( B^3 = j(\text{Ker}(r^2)) \) where we write \( r^2 = i \circ i \). Fiddling some more leads us to guess a closed (non-recursive) formula for the \((r-1)\)st derived couple of an exact couple.

**Lemma 1.2.** We have

\[
A^r = \text{Im}(i^{r-1}) \\
Z^r = k^{-1}(\text{Im}(i^{r-1})) \\
B^r = j(\text{Ker}(i^{r-1})).
\]

With respect to the canonical isomorphism \( E^r = Z^r / B^r \), we have

\[
i_r(a) = ia \\
j_r(i^{r-1}a) = [ja] \\
k_r[e] = ke.
\]

**Proof.** It is easy to see that \( i_r, j_r \) and \( k_r \) are well-defined. The statement is tautological for \( r = 1 \), and we proceed by induction. Assume the lemma is proved for \( r \) and let \( d^r = j_k k_r \). Given an element \( e \in E \) with \( ke = i^{r-1}a \) (i.e. \( e \in Z^r \)), we have

\[
d^r[e] = j_r k_r[e] = j_r i^{r-1}a = [ja] \in Z^r / B^r
\]

If \( ke = i^r b \), then \( d^r[e] = [jib] = 0 \). Conversely, assume \( ke = i^{r-1}a \) and that \( d^r[e] = [ja] = 0 \). Using the induction hypothesis we get \( ja = jb \) for some \( b \in \text{Ker}(i^{r-1}) \). Then \( a - b \in \text{Ker}(j) = \text{Im}(i) \), so \( a = b + ix \), and we have

\[
ke = i^{r-1}a = i^{r-1}(b + ix) = i^r x.
\]

and we have proved the formula for \( Z^{r+1} \).

\( B^{r+1} \) is the set of \( b \in E \) such that \( [b] = d^r[e] \) for some \( [e] \in E^r = Z^r / B^r \). Writing again \( ke = i^{r-1}a \) we have

\[
b - ja \in B^r = j(\text{Ker}(i^{r-1})),
\]
so \( b = j(a + x) \) with \( i^r a x = 0 \). Then
\[
  i^r (a + x) = i^r a = ike = 0,
\]
so \( b \in j(\text{Ker}(i^r)) \). Conversely, if \( b = ja \) with \( i^r a = 0 \), then \( i^{r-1} a \in \text{Ker}(i) = \text{Im}(k) \), so we can write \( i^{r-1} a = ke \) and then \( d^r[e] = [ja] = [b] \).

This proves the induction formula for \( B^r \).

The formula for \( j_{r+1} \) is obvious, and \( j_{r+1} \) is easy: \( j_{r+1} (i(i^{r-1} a)) = [j, i^{r-1} a] = [ja] \), where the first equality is the definition of derived couple and the second is the induction hypothesis. The formula for \( k_r \) is also easy: \( k_{r+1}[e] = k_r[e] = ke \) where the first equality is the definition of derived couple and the second is the induction hypothesis. □

For example in the spectral sequence of a filtered space, the element \( e \in E^1_{p,q} = H_{p+q}(X_p, X_{p-1}) \)
is in \( Z^r \) if its image in \( H_{p+q-1}(X_{p-1}) \) comes from \( H_{p+q-1}(X_{p-r}) \). The resulting class \([e] \in E^r_{p,q}\) is zero if \( e \) comes from an element in \( H_{p+q}(X_p) \) which vanishes in \( H_{p+q}(X_{p+r-1}) \).

Our next goal will be to prove that the spectral sequence “converges” to \( H^*(X) \) under suitable assumptions, but first we need to study convergence in more generality.

2. Convergence

Last lecture I discussed how the Serre spectral sequence eventually stabilizes: For fixed \((p, q)\) we have \( E^r_{p,q} = E^{r+1}_{p,q} = \ldots \) for large \( r \), and we called this value \( E^\infty_{p,q} \). In fact it makes sense to talk about \( E^\infty \) even without the stabilization phenomenon, and even without gradings, by taking an appropriate “limit” of the groups \( E^r \).

The abelian group \( E \) has subgroups \( Z^r, B^r \) arranged as follows
\[
  0 = B^1 \subseteq B^2 \subseteq \cdots \subseteq B^r \subseteq \cdots \subseteq Z^r \subseteq \cdots \subseteq Z^2 \subseteq Z^1 = E,
\]
and \( E^r = Z^r / B^r \) is the quotient of a group that becomes smaller and smaller, by a groups that becomes bigger and bigger. Thus the natural definition to make is
\[
  Z^\infty = \bigcap_r Z^r, \quad B^\infty = \bigcup_r B^r, \quad E^\infty = Z^\infty / B^\infty.
\]
This agrees with the previous definition in the graded case where the sequences \( Z^r_{p,q} \) and \( B^r_{p,q} \) stabilize for fixed \( p, q \).

We can immediately write a completely general formula for \( E^\infty \):
\[
  E^\infty = \frac{k^{-1}(\cap_r \text{Im}(i^r))}{j(\cup_r \text{Ker}(i^r))}. \tag{2.1}
\]
In this generality this formula is of little use, but under additional assumptions we can rewrite it in a more useful way. Viz. let us study the additional assumption on the map $i : A \to A$ that
\[ \bigcap_r \text{Im}(i^r) = 0 \tag{2.2} \]
i.e. if $a \in A$ can be written as $i^r b$ for arbitrarily large $r$, then $a = 0$. For the exact couple associated to a filtered spaces, this holds if $X_{-1} = \emptyset$. In that case we have
\[ E^\infty = \frac{\text{Ker}(k)}{j(\cup_r \text{Ker}(i^r))} = \frac{j(A)}{j(\cup_r \text{Ker}(i^r))}. \]
j : $A \to E$ induces an isomorphism $jA \cong A/iA$, and we can write
\[ E^\infty = \frac{A}{iA + \cup_r \text{Ker}(i^r)}. \]
Writing $F = A/(\cup_r \text{Ker}(i^r))$, there is an induced map $i : F \to F$, and we have $E^\infty = F/iF$.

Finally, notice that $F$ is the same as the image of $A$ in the direct limit $A^\infty = \lim_{\to} A$ of the direct system
\[ \cdots \to A \xrightarrow{i} A \xrightarrow{i} A \to \ldots, \tag{2.3} \]
so the $E^\infty$ term is given very concisely by two pieces of data:

- The limit group $A^\infty = \lim_{\to} A$, together with the isomorphism $i : A^\infty \to A^\infty$ induced by $i : A \to A$.
- The subgroup $F \subseteq A^\infty$ which is the image of $A$ in the direct limit.

Then $E^\infty \cong F/iF$. The isomorphism is also very explicit: It is the composition
\[ E \xrightarrow{j} A \to A^\infty. \]
of $j^{-1}$ (restricted to $Z^\infty$) with the canonical map $A \to \lim_{\to} A = A^\infty$.

2.1. The spectral sequence of a filtered space. Convergence.
Let us calculate the $E^\infty$ term of the spectral sequence of a filtered space. Recall that we have
\[ A_{p,q} = H_{p+q}(X_p) \]
and that $i : A_{p,q} \to A_{p+1,q-1}$ is the map induced from the inclusion $X_p \to X_{p+1}$. Let us assume that assumption (2.2) holds, e.g. that $X_{-1} = \emptyset$. Let us also assume that the map
\[ \lim_{p \to -\infty} H_n(X_p) \to H_n(X) \]
is an isomorphism. This holds e.g. if \( X = \bigcup X_p \) and \( X \) has the weak topology (e.g. \( X \) is a CW-complex and the \( X_p \) are subcomplexes).

Then \( A^\infty_{p,q} \) is the direct limit of the system

\[
H_{p+q}(X_p) \to H_{p+q}(X_{p+1}) \to \ldots,
\]

i.e. \( A^\infty_{p,q} = H_{p+q}(X) \) by our assumptions. The subspace \( F_{p,q} \) is the image of \( H_{p+q}(X_p) \to H_{p+q}(X) \), and \( iF_{p-1,q+1} \) is the image of \( H_{p+q}(X_{p-1}) \) in \( H_{p+q}(X) \). Changing notation to \( F^{p,q} = F_{p,q} \), we see that

\[
E^\infty_{p,q} = F^p_{p+q} / F^{p-1}_{p+q}.
\]

In symbols we often write

\[
E^1_{p,q} = H_{p+q}(X_p, X_{p-1}) \Rightarrow H_{p+q}(X),
\]

meaning that we are considering a spectral sequence with the specified \( E^1 \) term, and where the \( E^\infty \) term is the filtration quotients in a filtration of \( H_*(X) \). In words, we say that the spectral sequence “converges” to \( H_*(X) \). Note that this spectral sequence need still not be first quadrant. \( E^1_{p,q} \) vanishes unless \( p \geq 0 \) and \( q \geq -p \), so it can occupy \( \frac{3}{8} \) of the plane in general.

Let us briefly discuss more explicitly what the spectral sequence does. That an element \( x \in E^1_{p,q} \) is in \( Z^r \) means that \( d^1 x = 0 \), \( d^2 x = 0 \), \ldots, \( d^{r-1} x = 0 \), so \( x \) defines an element in \( E^r \). The formula \( Z^r = k^{-1}(\text{Im}(i^{r-1})) \) can be interpreted in the diagram

\[
\begin{array}{ccc}
H_{p+q}(X_p, X_{p-r}) & \longrightarrow & H_{p+q}(X_p, X_{p-1}) \\
\downarrow & & \downarrow k \\
H_{p+q-1}(X_{p-r}) & \longrightarrow & H_{p+q-1}(X_{p-1})
\end{array}
\]

with exact rows. It follows from the diagram that \( Z^r_{p,q} \subseteq E^1_{p,q} \) consists of the elements \( x \in H_{p+q}(X_p, X_{p-1}) \) that come from \( H_{p+q}(X_p, X_{p-r}) \). By a similar argument, \( B^r_{p,q} \) is the group of elements \( x \in H_{p+q}(X_p, X_{p-1}) \) that vanish when mapped to \( H_{p+q}(X_{p+r-1}, X_{p-1}) \).

\( Z^\infty \) is the elements that come from \( H_{p+q}(X_p) \), and the isomorphism \( E^\infty_{p,q} \to F^p_{p+q} / F^{p-1}_{p+q} \) is given by lifting to \( H_{p+q}(X_p) \) and mapping to \( H_{p+q}(X) \).

3. The Serre spectral sequence

The Serre spectral sequence is associated to a (Serre) fibration \( p : E \to B \). There is both a homology and a cohomology version. The homology version has

\[
E^2_{p,q} = H_p(B; H_q(F))
\]
and converges to $H_\ast(E)$. First we need a few preliminaries about the fundamental group of $B$.

Let $\lambda : [0,1] \to B$ be a path, and let $E_\lambda \to [0,1]$ be the pullback of the fibration $p$. Let $F_\nu = p^{-1}(\lambda(\nu))$ for $\nu = 0, 1$, and let $i_\nu : F_\nu \to E_\sigma$ be the inclusions. The long exact sequence in homotopy groups shows that $i_\nu$ are both weak equivalences, so they induce isomorphisms in homology. We get an induced isomorphism

$$\lambda_\ast = (i_1)_\ast \circ (i_0)_\ast^{-1} : H_\ast(F_0) \to H_\ast(F_1).$$

Thus, if $B$ is path connected, all fibers have isomorphic homology, although in general the isomorphism will depend on a choice of $\lambda$. We have $(\lambda \ast \sigma)_\ast = \lambda_\ast \circ \sigma_\ast$, and hence for each $b \in B$ we get an action of $\pi_1(B,b)$ on $H_\ast(p^{-1}(b))$. If this action is trivial, we have a canonical isomorphism between the homologies of two fibers. Therefore we can write $H_\ast(F)$ for the homology of a fiber: up to canonical isomorphism it doesn’t depend on which fiber we pick.

**Theorem 3.1.** Let $p : E \to B$ be a Serre fibration with $B$ path connected and assume that $\pi_1(B)$ acts trivially on the homology of the fiber. Then there is a spectral sequence, functorial with respect to all maps of fibrations

$$E^2_{p,q} = H_p(B; H_q(F)) \Rightarrow H_{p+q}(E).$$

**Proof.** We first consider the case where $B$ is a path connected CW complex. Let $B^p$ denote the $p$-skeleton of $B$, and let us make the following assumptions on $B$, to be used later.

(i) $B$ has only one 0-cell $b_0$, which we consider the basepoint. Let $F = p^{-1}(b_0)$.

(ii) The attaching maps $\partial \sigma : \partial D^p \to B^{p-1}$ are based maps, they map finitely many disjoint open $(p-1)$-disks in $\partial D^p$ homeomorphically onto open $p-1$-cells of $B^{p-1}$, and maps the complement of these disks to $B^{p-2}$.

At the end, we will reduce the general case to this special case (easy).

Define a filtration on $E$ by $E_p = p^{-1}(B^p)$. We get a spectral sequence with

$$E^1_{p,q} = H_{p+q}(E_p, E_{p-1}).$$

We obviously have $E_{-1} = \emptyset$ and $\lim H_\ast(E_p) = H_\ast(E)$, so the spectral sequence converges to $H_\ast(E)$. We need to prove that the $E^2$ term is what we claim. When $B$ is a CW complex, the groups $H_p(B; H_q(F))$ can be calculated by the cellular chain complex

$$C^\text{CW}_p(B; H_q(F)) = H_p(B^p; B^{p-1}) \otimes H_q(F).$$

$$C^\text{CW}_p(B; H_q(F)) = H_p(B^p; B^{p-1}) \otimes H_q(F).$$
The proof in the case $B$ is a CW complex will consist of identifying the groups (3.2) with the $E^1$ term (3.1), and identifying the cellular boundary map with the $d^1$ differential in the spectral sequence.

Let $\sigma : D^p \to B^p$ be the characteristic map of a cell in $B$. Let $p_\sigma : E\sigma \to D^p$ be the pullback fibration, and let $E_{\partial \sigma} \to \partial D^p$ be the restriction. Let $V_p \subseteq B^p$ denote the complement of the center points of the $p$-cells and let $U_p = p^{-1}(V_p)$. Then $B_{p-1} \to V_p$ is a homotopy equivalence, and the five-lemma gives that $E_{p-1} \to U_p$ is a weak equivalence. Similarly if we let $U_{\sigma} = p_{\sigma}^{-1}(D^p - \{0\})$, then $E_{\partial \sigma} \to U_{\sigma}$ is a weak equivalence. Using these open sets, an excision argument gives that the characteristic maps induce isomorphisms

$$\bigoplus_{\sigma} H_{p+q}(E\sigma, E_{\partial \sigma}) \to H_{p+q}(E_p, E_{p-1}) = E^1_{p,q}.$$  

Let us calculate the groups $H_{p+q}(E\sigma, E_{\partial \sigma})$. Pick once and for all a CW approximation $K \to F$. Then, the constant map $K \times D^p \to D^p$ to the basepoint $* \in \partial D^p$ is homotopic to the projection map. Lift this homotopy to a homotopy of maps $K \times D^p \to E\sigma$, which starts at the projection $K \times D^p \to K \to F \subseteq E\sigma$. We can also assume that the lifted homotopy is relative to $K \times \{*\}$. That gives a diagram of fibrations

$$
\begin{align*}
K \times D^p & \xrightarrow{g_\sigma} E\sigma \\
\pi \downarrow & \downarrow p_\sigma \\
D^p & \xrightarrow{\text{proj}} D^p
\end{align*}
$$

where $\pi$ denotes the projection to the second coordinate. The vertical maps are Serre fibrations, and over $* \in D^p$, the map of fibers is the fixed CW approximation $K \to F$. Then the LES of homotopy groups and the five-lemma shows that the map $g_\sigma : K \times D^p \to E\sigma$ is a weak equivalence. Restricting to $\partial D^p$ does not change the fibers, so the five-lemma gives that $g_{\partial \sigma} : K \times \partial D^p \to E_{\partial \sigma}$ is also a weak equivalence. Thus $g_\sigma$ gives an equivalence of pairs $K \times (D^p, \partial D^p) \to (E\sigma, E_{\partial \sigma})$ which induces an isomorphism

$$H_{p+q}(E\sigma, E_{\partial \sigma}) \cong H_q(K) \otimes H_p(D^p, \partial D^p) = H_q(K) \tag{3.3}$$

We have constructed the desired isomorphism

$$E^1_{p,q} = \bigoplus_{\sigma} H_q(K) \cong C_p^{CW}(B; H_q(K)) \cong C_p^{CW}(B; H_q(F))$$

and it remains to see that the $d^1$ differential and the cellular differential correspond under this isomorphism.
Let us first recall the definition of the cellular boundary map in a convenient form. It is the composition of the connecting homomorphism $H_p(B^p, B^{p-1}) \rightarrow H_{p-1}(B_{p-1})$ with the map $H_{p-1}(B^{p-1}) \rightarrow H_{p-1}(B^{p-2})$. The characteristic maps $\sigma : D^p_o \rightarrow B^p$ gives an isomorphism $\bigoplus \sigma H_p(D^p_o, \partial D^p_o) \rightarrow H_p(B^p, B^{p-1})$, and we have a commutative diagram

$$
\begin{array}{ccc}
\bigoplus \sigma H_p(D^p_o, \partial D^p_o) & \cong & \bigoplus \sigma H_{p-1}(\partial D^p_o) \\
\downarrow & & \downarrow \\
H_p(B^p, B^{p-1}) & \rightarrow & H_{p-1}(B^{p-1})
\end{array}
$$

(3.4)

Let us use our special assumption on the attaching map $\phi : \partial D^p_o \rightarrow B^{p-1}$. It implies that

$$
\phi^{-1}(B^{p-1} - B^{p-2}) = \bigcup_i A_i \subseteq \partial D^p_o
$$

where the $A_i$ are (finitely many) disjoint open disks, and $\phi$ restricts to a homeomorphism from each $A_i$ to an open $(p - 1)$-cell in $B$. Let $a_i \in A_i$ be the point that maps to the center the $(p - 1)$-cell; then $\phi^{-1}(V_{p-1} - B^{p-2}) = \bigsqcup_i A_i - \{a_i\}$. To understand the cellular boundary map on the summand $H_p(D^p_o, \partial D^p_o)$, we must study the diagram $\partial D^p_o \rightarrow (B^{p-1}, B^{p-2}) \leftarrow \bigsqcup \sigma (D^p_{\sigma}, \partial D^p_{\sigma})$, where the first map is the attaching map of $\sigma$, and the second induces an isomorphism in homology. This is the left column in the following diagram.

(3.5)

$$
\begin{array}{ccc}
\partial D^p_o & \rightarrow & (\partial D^p_o, \partial D^p_o - \bigsqcup \{a_i\}) \\
\downarrow & & \downarrow \\
(B^{p-1}, B^{p-2}) & \rightarrow & (B^{p-1}, V_{p-1}) \\
\downarrow & & \downarrow \\
\bigsqcup \sigma (D^p_{\sigma}, \partial D^p_{\sigma}) & \leftarrow & \bigsqcup \sigma (D^p_{\sigma}, V_{\sigma}).
\end{array}
$$

The maps labeled $\sim$ become isomorphisms in homology, either because they are homotopy equivalences, or because they are excision maps. The diagram is commutative, so instead of following the left column from the top left corner to the bottom left corner, we can go along the other three edges of the diagram (6 maps forming a △ shape). In any case, the cellular boundary map is given by the resulting maps $H_{p-1}(\partial D^p_o) \rightarrow H_{p-1}(\partial D^p_{\sigma})$, one for each $(\sigma, \tau)$. The cellular boundary map in $C_*(B; H_q(K))$ is obtained by tensoring these maps.
with (the identity map of) $H_q(K)$ or equivalently, by multiplying all spaces in the above diagrams with (the identity map of) $K$ and then applying homology.

We now compare this to the differential $d^1 = jk$ in the spectral sequence. This is the composition connecting homomorphism $H_{p+q}(E_p, E_{p-1}) \to H_{p+q-1}(E_{p-1})$ with the map $H_{p+q-1}(E_{p-1}) \to H_{p+q-1}(E_{p-1}, E_{p-2})$. This is the exact same construction as the cellular boundary, except that all spaces are replaced by their inverse image under $p : E \to B$, and thus it can be analyzed by diagrams similar to (3.4) and (3.5). For the diagram (3.4), we get

$$\bigoplus_\sigma H_{p+q}(E_\sigma, E_{\partial \sigma}) \xrightarrow{\partial} \bigoplus_\sigma H_{p+q-1}(E_{\partial \sigma})$$

We can use the weak equivalence $g_\sigma : K \times D^p_\sigma \to E_\sigma$ to replace the top row by the top row of (3.4) tensored with (the identity map of) $H_q(K)$:

$$\bigoplus_\sigma H_q(K) \otimes H_p(D^p_\sigma, \partial D^p_\sigma) \xrightarrow{(g_\sigma)_*} \bigoplus_\sigma H_q(K) \otimes H_{q-1}(\partial D^p_\sigma)$$

A similar procedure works to analyze the map $E_{p-1} \to (E_{p-1}, E_{p-2}) \leftarrow \coprod_r(E_r, E_{\partial r})$: All the spaces in diagram (3.5) have a preferred map to $B$, so we pull back the fibration $p : E \to B$ to each space in the diagram and write the resulting total space instead. For example $A_i$ becomes $(p_{\partial \sigma})^{-1}(A_i)$ and the map $\phi$ becomes

$$\bigcup_i (p_{\partial \sigma}^{-1}(A_i), p_{\partial \sigma}^{-1}(A_i - \{a_i\}) \xrightarrow{\hat{\phi}} (E_{p-1} - E_{p-2}, U_{p-1} - E_{p-2}).$$

We will not write out the resulting diagram, but let us call it the “lifted diagram”. We can use the weak equivalences $g_{\partial \sigma} : K \times \partial D^p_\sigma \to E_\sigma$ and $g_\sigma : K \times D^{p-1} \to E_\sigma$ to compare the diagram (3.5) and the lifted diagram. It suffices to compare the spaces in the outer 6-map $\supset$-shaped part of the diagram. If we multiply all these spaces in diagram (3.5) by $K$, then either a restriction of $g_{\partial \sigma}$ or of some $g_\sigma$ will define a map from $K \times (3.5)$ to the lifted diagram, and the resulting ladder will be
commutative, except at one point, namely the diagram

\[ K \times ( \bigcup_i (A_i, A_i - \{a_i\})) \xrightarrow{\sim} \bigcup_i (p_{\partial \sigma}^{-1}(A_i), p_{\partial \sigma}^{-1}(A_i - \{a_i\})) \]

\[ K \times (B_{p-1} - B_{p-2}, V_{p-1} - B_{p-2}) \xrightarrow{\sim} (E_{p-1} - E_{p-2}, U_{p-1} - E_{p-2}). \]

In this diagram, the horizontal maps are weak equivalences, but the top map is induced by \( g_{\sigma} \), and the bottom map by various \( g_{\tau}'s \), and they need not agree: if \( b \in B \) is in the image of the characteristic maps of both \( \sigma \) and \( \tau \), there results two weak equivalences of \( K \) with \( p^{-1}(b) \).

Thus, to make the diagram commutative, we should use a different map \( K \to K \) for each disk \( A_i \). However, our assumptions on \( \pi_1(B) \) imply that these maps \( K \to K \) all induce the identity map on homology, and hence the diagram is commutative after applying homology.

We have proved that \( d^1 = H_q(K) \otimes \partial^C_{*} \) by splitting both \( d^1 \) as a (fairly long) composition of maps, and proving that each map in the composition is \( H_q(K) \) tensor the corresponding map in a decomposition of \( \partial^C_{*} \).

\[ \square \]

4. Chain complexes and cohomology

The spectral sequence of a filtered space is a special case of the spectral sequence of a filtered complex, which we briefly describe. Let \((C_n, \partial)\) be a chain complex of abelian groups. (We assume \( \partial : C_n \to C_{n-1} \).) Let \( C^p_n \subseteq C_n \) be subgroups with \( C^p_{n-1} \subseteq C^p_n \) and \( d(C^p_n) \subseteq C^p_n \). Thus \( \partial \) restricts to a differential on each \( C^p_* \), and we have short exact sequences of chain complexes

\[ 0 \to C^{p-1}_n \xrightarrow{i} C^p_n \xrightarrow{i} C^p_n/C^{p-1}_n \to 0, \]

inducing long exact sequences in homology. We get an exact couple by setting

\[ A_{p,q} = H_{p+q}(C^p_*) \]
\[ E_{p,q} = H_{p+q}(C^p/C^{p-1}_*). \]

The exact couple of a filtered space is the special case \( C_* = C_*(X) \) and \( C^p_* = C_*(X^p) \). The discussion of convergence in that case applies in the exact same way here. More precisely, let us make the following two assumptions.

1. The map \( \lim \to H_*(C^p_*) \to H_*(C_*) \) is an isomorphism. This holds if the filtration is exhausting, i.e. that \( C_* = \bigcup p C^p_*. \)

2. The map \( i^r : H_*(C^{p-r}) \to H_*(C^p) \) has \( \cap \text{Im}(i^r) = 0. \)
Then the spectral sequence converges to $H_*(C_*)$: There is a filtration on $H_n(C_*)$ and an isomorphism

$$E^\infty_{p,q} \cong F^p_{p+q}/F^p_{p+q-1}.$$  

Again, we write

$$E^r_{p,q} = H_{p+q}(C_*/C^{p-*}_*) \Rightarrow H_{p+q}(C_*)$$

4.1. **Cohomology spectral sequence of a filtered space.** Suppose we want to use a spectral sequence to “calculate” the cohomology $H^*(X)$, starting with $H^*(X, X_{p-1})$. This can be done, using a special case of the spectral sequence of a filtered complex. To make the degrees work out, we need to change the signs of many indices. Let

$$C_{-n} = C^n(X)$$

$$C^{-p}_{-n} = C^n(X, X_{p-1}).$$

Then the quotient $C^{-p}_{-n}/C^{-p-1}_{-n}$ is canonically isomorphic to $C^n(X, X_{p-1})$ and we get a spectral sequence with $E^1_{-p,-q} = H^{p+q}(X, X_{p-1})$. It converges to $H^*(X)$ provided

(i) The map $\lim H^*(X, X_p) \to H^*(X)$ is an isomorphism. This holds if $X_{-1} = \emptyset$.

(ii) The map $i^r : H^n(X, X_{p+r}) \to H^n(X, X_p)$ has $\cap \text{Im}(i^r) = 0$. This holds if $X = \cup X_p$ with the direct limit topology, and $H^n(X, X_{p+r})$ is independent of $r$ for large $r$.

Written this way, the differentials in the spectral sequence have the same degrees as in the homology case, but the spectral sequence is concentrated in a different $\frac{3}{8}$ of the plane. It is customary to change signs of $p$ and $q$, and indicate this by writing them in the superscript instead of subscript. Thus the cohomology spectral sequence has

$$E^1_{p,q} = H^{p+q}(X, X_{p-1}),$$

the filtration on $H^*(X)$ becomes a decreasing filtration $H^*(X) \supseteq F^0 \supseteq F^1 \supseteq \ldots$, and we have isomorphisms $E^\infty_{p,q} = F^p_{p+q}/F^p_{p+q+1}$.

4.2. **Functoriality.** All the spectral sequences have considered behave well with respect to morphisms. Assume $1 C_*$ and $2 C_*$ are two filtered chain complexes and $f_* : 1 C_* \to 2 C_*$ is a filtration preserving map of chain complexes (i.e. $f_* (1 F^p) \subseteq 2 F^p$ for all $p$). Then $f_*$ induces a map of exact couples, which gives a map of all the derived couples, and hence a morphism of spectral sequences, i.e. maps

$$f_* : 1 E^r_{p,q} \to 2 E^r_{p,q}.$$
which are chain maps \((f_s \circ d_r = d_r' \circ f_s')\) for all \(r\). It also induces a filtration preserving map \(H_*(^1C_*) \to H_*(^2C_*)\) and hence a map of filtration quotients. In the cases where the spectral sequences converge to \(H_*(^\nu C_*)\), the diagram

\[
\begin{array}{ccc}
1^{E_\infty}_{p,q} & \longrightarrow & 1^{F^p}_p / 1^{F^{p-1}} \\
\downarrow & & \downarrow \\
2^{E_\infty}_{p,q} & \longrightarrow & 2^{F^p}_p / 2^{F^{p-1}} \\
\end{array}
\]

is commutative.

If \(X\) and \(Y\) are filtered spaces and \(f : X \to Y\) is a filtration preserving map \((f(X_p) \subseteq Y_p)\), then \(f\) induces filtration preserving maps of chain complexes \(C_*(X) \to C_*(Y)\) and \(C^*(Y) \to C^*(X)\).

5. Multiplicative structure

The cohomology of a space has more structure than the homology: it has a product. It turns out that often this structure can be incorporated into the cohomology spectral sequence. This extra structure is often extremely important for calculations (e.g. figuring out how the differentials go).

What often can be achieved is that all terms \(E_r\) of the spectral sequence have a product \(E^p_r \otimes E^{p'}_r \to E^{p+p'}_{r+q'}\), making it a bigraded ring, all differentials satisfies the Leibnitz rule: \(d_r(xy) = (d_rx)y + (-1)^{|x|}xd_ry\). This induces a ring structure on \(E_\infty\). Moreover the filtration of \(H^*(X)\) will satisfy \((F^p)(F^{p'}) \subseteq F^{p+p'}\), which gives an induced ring structure on \(\oplus_p F^p / F^{p+1}\), and the isomorphism \(E_\infty \cong \oplus_p F^p / F^{p+1}\) is an isomorphism of rings.

It seems difficult to prove this without using chain-level arguments. In fact it seems better to formulate and prove a statement about the spectral sequence of a filtered chain complex.

5.1. Products. Let \(^1C_\ast\) and \(^2C_\ast\) be two chain complexes, filtered by \(^1C^p_n\) and \(^2C^p_n\). Then the tensor product

\[ ^3C_n = (^1C \otimes ^2C)_n = \bigoplus_{s+t=n} ^1C_s \otimes ^2C_t \]

has a differential given by \(\partial(x \otimes y) = (\partial x) \otimes y + (-1)^{|x|}x \otimes (\partial y)\), and there is a compatible filtration in which \(^1C \otimes ^2C\) is the sum of the image of \(^1C^p_s \otimes ^2C^p_t\) over all indices with \(s + t = n, q + r = p\). This situation gives rise to three spectral sequences, and there is an induced
product on filtration quotients

\[(1^{C_p^n}/1^{C_n}) \times (2^{C_p^r}/2^{C_r}) \to (3^{C_p^r}/3^{C_r}),\]
given by \([x] \times [y] = [x \otimes y] \), which in turn induces a product on homology

\[1^{E_{p,q}} \times 2^{E_{p',q'}} \to 3^{E_{p+p',q+q'}}.\]

If we can show that \(d^1\) satisfies a Leibniz rule, then there is an induced product on the \(E^2\) terms; if \(d^2\) satisfies a Leibnitz rule, there is an induced product on \(E^3\), etc.

**Proposition 5.1.** The product \(1^{E_{p,q}} \times 2^{E_{p',q'}} \to 3^{E_{p+p',q+q'}}\) defined above induces well defined products on subquotients

\[1^{E_{p,q}} \times 2^{E_{p',q'}} \to 3^{E_{p+p',q+q'}}\]

for all \(r\), and \(d^r\) satisfies the Leibnitz rule \(d^r(e \times f) = (d^r e) \times f + (-1)^r e \times (d^r f)\) with respect to these. (Here \(|e| = p + q\) if \(e \in E_{p,q}\).)

**Proof.** This is really just a matter of chasing through a lot of definitions while avoiding freaking out about indices. Let’s do it. The proposition has two claims: the products are well defined for all \(r\), and the differential satisfies the Leibnitz rule. If we formulate the proof as an induction proof, we only need to prove the Leibnitz rule on \(E^r\), since then the product on \(E^{r+1}\) is well defined. We will use the explicit formulas for \(d^r\) and for the \(E^r\) terms as subquotients of the \(E^1\) terms.

Let \(e \in 1^{E_{p,q}}\) be \(1^{Z^r}\) i.e. \([e]\) represents an element in the subquotient \(1^{E^r}\). In the exact couple, that means we can write \(ke = \sigma^{-1}a\) for some \(a \in H_{p+q-1}^{p+q-1}(\mathbb{C})\) and in that case \(d^r[e] = [ja]\). Let us write \(e = [x]\) for a representative

\[x \in 1^{C^p_{p+q}}/1^{C^p_{p+q}}\]

and spell out how to find a chain level representative for \(d^r[e]\). Here is the recipe.

1. **Lift** \(x\) to \(\bar{x} \in 1^{C^p_{p+q}}\). Then \(\partial \bar{x} \in 1^{C^{p-1}_{p+q-1}}\), and in here, it is a cycle which represents \(k[e]\).

2. **Find elements** \(\bar{\alpha} \in 1^{C^p_{p+q-1}}\) and \(\sigma \in 1^{C^{p-1}_{p+q}}\) such that

\[\partial \bar{x} = \bar{\alpha} + \partial \sigma.\]

Then \(\bar{\alpha}\) is a cycle which represents an element \(a\) with \(i^{-1}(a) = ke\).

3. **Let** \(\alpha \in 1^{C^p_{p+q-1}}/1^{C^p_{p+q-1}}\) be the reduction of \(\bar{\alpha}\). This is a cycle which represents an element of \(1^{E_{p-r,q+r-1}}\) which in the subquotient represents \(d^r[e]\).
Starting with an element \( f \in \mathcal{E}_{p,q} \), which is in \( \mathcal{E}^r \), we use the same recipe to find \( d^r[f] \): Write \( f = [y] \) for a representative \( y \in \mathcal{E}_{p,q} \), pick a lift \( \overline{y} \in \mathcal{E}_{p,q} \) and write
\[
\overline{\partial y} = \overline{\beta} + \partial \tau,
\]
for elements \( \overline{\beta} \in \mathcal{E}_{p,q} \) and \( \tau \in \mathcal{E}_{p,q} \).

Finally we use the recipe to calculate \( d^r[e \times f] \). Write \( p'' = p + p' \) and \( q'' = q + q' \). First, \( x \times y \in \mathcal{E}_{p,q} \) is a chain representing \( e \times f \). Here are the steps in the recipe.

1. \( x \times y \) lifts to \( \exists \otimes \overline{y} \in \mathcal{E}_{p,q} \).
2. We have
\[
\begin{align*}
\partial (\exists \otimes \overline{y}) &= (\exists \otimes y) + (-1)^{|x|} \exists \otimes (\overline{\partial y}) \\
&= (\exists \otimes \sigma \otimes \overline{y} + (-1)^{|x|} \exists \otimes (\overline{\beta} + \partial \tau) \\
&= (\exists \otimes \overline{y} + (-1)^{|x|} \exists \otimes \overline{\beta}) + ((\partial \sigma \otimes \overline{y} + (-1)^{|x|} \exists \otimes (\partial \tau)) \\
&= \overline{y} + \partial \nu,
\end{align*}
\]
where
\[
\nu = \sigma \otimes \overline{y} + (-1)^{|x|} \exists \otimes \tau \in \mathcal{E}_{p,q-1}.
\]
and
\[
\overline{y} = \exists \otimes \overline{y} + (-1)^{|x|} \exists \otimes \overline{\beta} \in \mathcal{E}_{p,q-1}.
\]
3. Let \( \gamma \) be the class of \( \overline{y} \) in the quotient modulo \( \mathcal{E}_{p,q-1} \). Then \( \gamma \) gives
\[
[\gamma] = [\alpha] f + (-1)^{|e|} e[\beta].
\]
In the subquotient \( \mathcal{E}^r \), the elements \( [\alpha] \), \( [\beta] \) and \( [\gamma] \) become \( d^r[e] \), \( d^r[f] \) and \( d^r([e] \times [f]) \) which is the desired Leibnitz rule.

Finally, we would like the relation between \( \mathcal{E}^\infty \) and \( H_*(C_*) \) to reflect something about the products. This is indeed the case.

**Proposition 5.2.** In the above situation, the product on \( \mathcal{E}^1 \) induces a product
\[
1^{\mathcal{E}^\infty}_{p,q} \times 2^{\mathcal{E}^\infty}_{p',q'} \rightarrow 3^{\mathcal{E}^\infty}_{p+p',q+q'}.
\]
The filtrations on $H_s(1C_p)$, $H_s(2C_p)$ and $H_s(3C_p)$ satisfy $(1F^p) \times (2F^{p'}) \subseteq 3F^{p+p'}$, so there is an induced multiplication on filtration quotients. With respect to these products, the isomorphisms $(\nu = 1, 2, 3)$

$$\nu \varphi : \nu E_{p,q}^\infty \to \nu F^p_{p+q}/\nu F^{p-1}_{p+q}$$

satisfy $3\varphi(e \times f) = 1\varphi(e) \times 2\varphi(f)$.

**Proof.** That the product is defined on all $E^r$ for $r < \infty$ means that

$$1Z^r \times 2Z^r \subseteq 3Z^r$$
$$1B^r \times 2Z^r \subseteq 3B^r$$
$$1Z^r \times 2B^r \subseteq 3B^r.$$ 

If these hold for all $r$, it is easy to see they hold for $r = \infty$, so the product on $E^1$ induces a product on $E^\infty$.

In the filtration on $H_s(C_p)$, $F^p$ is the image of $H_s(C^p_p)$. It is then clear that $(1F^p) \times (2F^{p'}) \subseteq 3F^{p+p'}$.

The definition of $\phi$: Starting with $e = \in H_s(C^p_p/C^p_{p-1})$, lift to $H_s(C^p_p)$ and map it to $H_s(C_p)$. This obviously preserves products. $\square$