1.

Let \( n \geq 1 \) and let \( M \) be a smooth compact oriented connected \( 2n \)-dimensional manifold, and assume that \( \tilde{H}_k(M) = 0 \) for \( k \neq n \).

(a) Prove that \( H_n(M) \) is a free abelian group and that \( H_n(M) = H_n(M, \partial M) \).

(b) Let \( b_1, \ldots, b_k \in H_n(M) \) be a basis and let \( A_{i,j} = b_i \cdot b_j \), where \( \cdot \) denotes the intersection product. Let \( A \in M_n(\mathbb{Z}) \) be the matrix with entries \( A_{i,j} \).

Prove that the map \( H_n(M) \otimes H_n(M) \to \mathbb{Z} \) is non-degenerate if and only if \( \det(A) = \pm 1 \).

(c) Prove that the intersection pairing is non-degenerate if and only if \( H_*(\partial M) \cong H_*(S^{2n-1}) \).

2.

(a) Let \( \iota \in C_n(\Delta^n) \) be the chain represented by the identity map of \( \Delta^n \). Prove that \([\iota] \in H_n(\Delta^n, \partial \Delta^n) \cong \mathbb{Z} \) is a generator (hint: use the isomorphism between simplicial homology and singular homology) and deduce that the quotient map \( q : \Delta^n \to \Delta^n/\partial \Delta^n \) represents a generator \([q] \in H_n(\Delta^n/\partial \Delta^n, *) \), where \(* \in \Delta^n/\partial \Delta^n \) denotes the basepoint (i.e. the class of \( \partial \Delta^n \)).

(b) Let \( n > 0 \). Prove that if \( c \in S^n \) is a point and \( \sigma : \Delta^n \to S^n \) is a map inducing a homeomorphism \( \Delta^n/\partial \Delta^n \to (S^n, c) \), then \( \sigma - c \in C_n(S^n) \) is a cycle representing a generator of \( H_n(S^n) \).

(c) Prove that the diagram

\[
\begin{array}{c}
H^n_{dR}(S^n) \quad f_{\mathbb{R}}^n \\
\downarrow \quad \downarrow
\end{array}
\]

\[
\begin{array}{c}
H^n(S^n; \mathbb{R}) \quad f_{\mathbb{R}[S^n]}^n
\end{array}
\]

commutes (for a suitable orientation of \( S^n \)).

3.

For a smooth manifold \( M \) and a smooth submanifold \( N \subset M \), write \( \Omega^p(M, N) \) for the kernel of the restriction map \( \Omega^p(M) \to \Omega^p(N) \).

(a) Prove that \((\Omega^*(M, N), d)\) is a cochain complex. Write \( H^*_{dR}(M, N) \) for its cohomology, and construct a long exact sequence with \( H^*_{dR}(M) \), \( H^*_{dR}(N) \) and \( H^*_{dR}(M, N) \).

(b) Construct a natural isomorphism \( H^*_{dR}(M, N) \cong H^*(M, N; \mathbb{R}) \).
4. Let $\mathbb{R}P^3 = S^3/\{\pm 1\}$ have its usual smooth structure, i.e. the one in which an atlas is given as $\mathcal{A} = \{i_1^{-1}, \ldots, i_4^{-1}\}$ where $i_\nu : \mathbb{R}^3 \to \mathbb{R}P^3$ is given by

\[
\begin{align*}
i_1(x, y, z) &= [1 : x : y : z] \\
i_2(x, y, z) &= [x : -1 : y : z] \\
i_3(x, y, z) &= [x : y : 1 : z] \\
i_4(x, y, z) &= [x : y : z : -1].
\end{align*}
\]

It can be shown that $\mathbb{R}P^3$ admits a unique orientation in which this atlas is oriented (this may be used without proof in the following).

(a) Use de Rham’s theorem and universal coefficient theorem to prove that $H^k_{dR}(\mathbb{R}P^3) \cong \mathbb{R}$ for $k = 0, 3$ and 0 otherwise. (The integral homology of $\mathbb{R}P^n$ is calculated in Hatcher; the result of this may be used without proof in this exercise.)

(b) Prove that there is a unique form $\omega \in \Omega^3(\mathbb{R}P^3)$ such that

\[
i_1^*(\omega) = e^{-\|x\|^2} dx_1 \wedge dx_2 \wedge dx_3.
\]

[Hint: For existence, let $\omega(p) = 0$ when $p$ is not in the image of $i_1$. Calculate $i_\nu^*(\omega)$, $\nu = 2, 3, 4$ to prove that it is smooth.]

(c) Prove that $[\omega] \in H^3_{dR}(\mathbb{R}P^3)$ forms a basis.

(d) Find a form $\eta \in \Omega^3(\mathbb{R}P^3)$ such that $[\eta] \in H^3_{dR}(\mathbb{R}P^3) \cong H^3(\mathbb{R}P^3; \mathbb{R})$ is in the image of a generator of $H^3(\mathbb{R}P^3; \mathbb{Z})$. 

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