1.

Let $X$ be a topological space and consider the set of pairs $(M, f)$, where $M \subset \mathbb{R}^n$ is a compact, codimension 0 submanifold with boundary, and $f : \partial M \to X$ is a continuous map. If $(M_0, f_0)$ and $(M_1, f_1)$ are two such pairs, a concordance between them is a pair $(V, f)$ as follows. $V$ is a compact subset of $[0, 1] \times \mathbb{R}^n$ is a compact subset such that $V \cap ([0, 1) \times \mathbb{R}^n)$ is a smooth submanifold such that there exists an $\epsilon > 0$ with

$$V \cap [0, \epsilon) \times \mathbb{R}^n = [0, \epsilon) \times M_0$$

$$V \cap (1 - \epsilon, 1] \times \mathbb{R}^n = (1 - \epsilon, 1] \times M_1.$$

If we write $\partial' V = \{0\} \times \partial M_0 \cup \{1\} \times \partial M_1 \cup \partial(V \cap (0, 1) \times \mathbb{R}^n)$, then $f : \partial' V \to X$ is a continuous map with $f(0, -) = f_0$ and $f(1, -) = f_1$. Let us say that $(M_0, f_0)$ and $(M_1, f_1)$ are concordant if there exists a concordance between them.

(a) Prove that concordance is an equivalence relation.

(b) Let $\Sigma X$ be the unreduced suspension of $X$, i.e. the space obtained from $[-1, 1] \times X$ by collapsing $\{-1\} \times X$ and $\{1\} \times X$ to two points, and let $\pi : \Sigma X \to [-1, 1]$ be the natural map. We shall regard the point $s = \{-1\} \times X \in \Sigma X$ as a basepoint. Prove that any element

$$[g] \in \pi_n(\Sigma X, s)$$

admits a representative $g : (S^n, *) \to (\Sigma X, s)$ for which $\pi \circ g$ is smooth and has 0 as a regular value. Then prove that $M = (\pi \circ f)^{-1}(0, 1)$ is a compact codimension 0 submanifold of $\mathbb{R}^n$ and $g$ restricts to a continuous map $f : \partial M \to \{0\} \times X = X$.

(c) Prove that if $g, g'$ are two such representatives (for the same homotopy class), then the resulting $(M, f)$ and $(M', f')$ are concordant.

(d) Prove that this construction gives a bijection between $\pi_n(\Sigma X, s)$ and the set of concordance classes of $(M, f)$’s.

(e) In the case $n = 1$ and $X = S^0$, we’re describing $\pi_1(S^1) \cong \mathbb{Z}$. Draw a picture of a generator and a picture of its inverse. Then draw a concordance indicating why they’re inverse to each other.

(f) If $h : X \to Y$ is a continuous function and $C_h$ denotes its mapping cone, give an interpretation of the homotopy groups $\pi_n(C_h)$.

[Guidelines on detail levels: Doing all of these in every detail would likely be unreasonably long. In (d) you should explain properly why the correspondence is surjective, but you can be sketchy about the injectivity. In (e) and (f), you need just give your answer.]