1. Let $M$ be a smooth manifold and $\pi : E \to M$ a topological vector bundle. (This means that $E$ is a topological space, $\pi$ is continuous, for each $m \in M$ there is specified a vector space structure on $\pi^{-1}(m) \subset E$, subject to the condition that each $m \in M$ has an open neighborhood $U \subset M$ and a homeomorphism $\pi^{-1}(U) \approx U \times \mathbb{R}^n$ which commutes with the projection to $U$ and is fiberwise linear.) Prove that when $M$ is smooth, there exists a smooth structure on $E$ making $\pi : E \to M$ into a smooth vector bundle.

2. Let $M$ be a smooth $n$-manifold and $j : M \to \mathbb{R}^N$ an immersion. Prove that if $N > 2n$ then there exists a projection $\mathbb{R}^N \to \mathbb{R}^{N-1}$ such that the composite map $M \to \mathbb{R}^N \to \mathbb{R}^{N-1}$ is an immersion.

It follows by induction that there exists a projection $\mathbb{R}^N \to \mathbb{R}^{2n}$ such that $M \to \mathbb{R}^{2n}$ is an immersion. Prove that it’s possible to avoid “triple intersections,” i.e. three different points in $M$ mapping to the same point in $\mathbb{R}^{2n}$. (For example, this is used every time a “knot” is represented by a “knot diagram”.)

3. Let $M$ be a smooth $n$-dimensional manifold and let $E \to M$ be a smooth vector bundle with $k$-dimensional fibers. In class we proved that there exists a smooth fiberwise linear embedding $E \to M \times \mathbb{R}^N$ with $N = 2(n + k) + 1$. Use Sard’s theorem to prove that there exists such an embedding with $N = n + k$.