1. Let $M_1$ and $M_2$ be connected oriented (topological) $n$-manifolds. Let $h_1 : \mathbb{R}^n \rightarrow M_1$ and $h_2 : \mathbb{R}^n \rightarrow M_2$ be homeomorphisms onto open subsets, such that $h_1$ preserves local orientation and $h_2$ reverses local orientation. Let $M = M_1 \# M_2$ be the “connected sum”, i.e. the quotient

$$M = \left( (M_1 - h_1(\text{Int}D^n)) \sqcup (M_2 - h_2(\text{Int}D^n)) \right) / \sim$$

where $\sim$ identifies $h_1(x) \sim h_2(x)$ for all $x \in \partial D^n$. (This definition, involving orientations, is a bit more precise than the one given in Hatcher's exercise 3.3.6).

(i) Prove that $M$ has a unique orientation in which the inclusion maps $M_\nu - h_\nu(\text{Int}D^n) \rightarrow M$ are orientation preserving.

(ii) As you know, there is a ring isomorphism $H^4(\mathbb{C}P^2) = \mathbb{Z}[\alpha]/\alpha^3$. Let $M = \mathbb{C}P^2 \# \mathbb{C}P^2$. Prove that the map $M \rightarrow M/\partial D^4 \cong \mathbb{C}P^2 \vee \mathbb{C}P^2$ induces an isomorphism in $H^2$, where $\partial D^4 \subset M$ is the sphere along we glued. In particular, we get an explicit isomorphism $H^2(M) \cong \mathbb{Z}^2$.

(iii) Equip $\mathbb{C}P^2$ with the orientation in which $[\mathbb{C}P^2] \cap \alpha^2 = 1$. Find the matrix of the cup product pairing

$$H^2(M; \mathbb{Z}) \otimes H^2(M; \mathbb{Z}) \rightarrow H^4(M; \mathbb{Z}) \xrightarrow{|M| \cap} \mathbb{Z}$$

and prove that it is positive definite. Deduce that the signature of $M$ is 2.

(iv) Let $\overline{\mathbb{C}P^2}$ denote $\mathbb{C}P^2$ equipped with the opposite orientation, and let $N = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. Prove that the cup product pairing

$$H^2(M; \mathbb{Z}) \otimes H^2(M; \mathbb{Z}) \rightarrow H^4(M; \mathbb{Z}) \xrightarrow{|N| \cap} \mathbb{Z}$$

is indefinite. Deduce that the signature of $N$ is 0.

[The manifold $V = [0,1] \times (\mathbb{C}P^2 - \text{Int}(D^4))$ is compact and has $\partial V = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. That gives another proof that $N$ has signature 0.]

NOTE: Choose one of the two problems 2a and 2b below.

2a Let $A$ be a free abelian group of finite rank, and let $b : A \otimes A \rightarrow \mathbb{Z}$ be a non-degenerate pairing, i.e. the adjoint $A \rightarrow \text{Hom}(A, \mathbb{Z})$ is an isomorphism. Assume that $b$ is symmetric, i.e. $b(x \otimes y) = b(y \otimes x)$. (For example, $A$ could be $H^{2n}(M)/\text{torsion}$ for $M$ an oriented connected compact manifold of dimension $4n$ and $b$ could be induced from the cup product.)

(i) Prove that $A \otimes \mathbb{R}$ has a basis $\{v_1, \ldots, v_n\}$ such that $b(v_i \otimes v_j) = 0$ for $i \neq j$ and $b(v_i \otimes v_i) \neq 0$. Deduce that there exists a basis $\{x_1, \ldots, x_n\}$ of $A \otimes \mathbb{R}$ such that the matrix of $b$ is of the form $\text{diag}(1, \ldots, 1, -1, \ldots, -1)$. (The number of +1’s minus the number of −1’s is then the signature of $b$.)

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(ii) Deduce that if there exists a subspace \( L \subset A \otimes \mathbb{R} \) such that \( \dim(A \otimes \mathbb{R}) = 2 \dim(L) \) and \( b(x \otimes y) = 0 \) for \( x, y \in L \), then the signature of \( b \) is zero.

(iii) Now assume that \( A \) admits a Lagrangian, i.e. a subgroup \( L \subset A \) such that \( \text{rank}(A) = 2 \text{rank}(L) \) and that \( b(x \otimes y) = 0 \) for \( x, y \in L \). Prove that there also exists a Lagrangian \( L' \subset A \) such that \( A/L' \) is a free abelian group. [Hint: let \( L' = (A \otimes \mathbb{Z}) \cap (L \otimes \mathbb{R}) \subset (A \otimes \mathbb{R}) \), i.e. the elements of \( A \) such that some multiple is in \( L \).]

(iv) Prove that if \((A, b)\) has a Lagrangian \( L \subset A \), then there is a basis \( \{e_1, f_1, \ldots, e_n, f_n\} \) of \( A \) such that
\[
\begin{align*}
    b(e_i \otimes e_j) &= 0 \quad \text{for all } i, j \\
    b(e_i \otimes f_j) &= \delta_{i,j} \quad \text{for all } i, j \\
    b(f_i \otimes f_j) &= 0 \quad \text{for } i \neq j \\
    b(f_i \otimes f_i) &\in \{0, 1\} \quad \text{for all } i
\end{align*}
\]

(v) We say that \( b \) is even if \( b(x \otimes x) \in 2\mathbb{Z} \) for all \( x \in A \). Prove that it is possible to achieve \( b(f_i \otimes f_i) = 0 \) for all \( i \) if and only if \( b \) is even. [I erroneously omitted the condition of evenness when I talked about this in class. The example \( A = H^2(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}) \) gives an example which has a Lagrangian (because the manifold is the boundary of an oriented compact manifold) but the cup product is not even.]

\[2b\] Let \( V \subset S^1 \) be an open subset homeomorphic to \( \mathbb{R} \) which is not dense, for example the (open) upper hemisphere. Let \( M = (\{0, 1\} \times S^1)/\sim \), where \( \sim \) is the equivalence relation generated by \((0, t) \sim (1, t)\) for \( t \in V \). Give \( M \) the quotient topology.

(i) Prove that every \( p \in M \) has an open neighborhood \( U \subset M \) homeomorphic to \( \mathbb{R} \).

(ii) Prove that \( H_0(M; \mathbb{F}_2) \not\cong H^1(M; \mathbb{F}_2) \). Is this a counterexample to the Poincare duality theorem?

(iii) What part of the proof of Poincare duality fails for the space \( M' \)? [You should point to at least one specific step in the proof which is invalid for the “manifold” \( M \).]

3. Let \( C^\infty(\mathbb{R}^n, 0) \) denote the ring of germs of functions \( \mathbb{R}^n \to \mathbb{R} \) at 0, i.e.
\[
\lim_{\substack{U \to \{0\}}} C^\infty(U, \mathbb{R}),
\]
where \( U \) runs over smaller and smaller neighborhoods of \( 0 \in \mathbb{R}^n \). There are ring maps
\[
c : \mathbb{R} \to C^\infty(\mathbb{R}^n, 0) \quad \quad \epsilon : C^\infty(\mathbb{R}^n, 0) \to \mathbb{R}
\]
given by \( c(t) \) being the germ of the constant function with value \( t \) and \( \epsilon(f) = f(0) \). An \( \mathbb{R} \)-linear function \( X : C^\infty(U, p) \to \mathbb{R} \) is a derivation if \( X(fg) = X(f)g(0) + f(0)X(g) \). The goal of this exercise is to prove that the set of derivations is an \( n \)-dimensional \( \mathbb{R} \)-vector space with basis \( \partial/\partial x_i \).

(i) Prove that \( X = \partial/\partial x_i \) is a derivation. Prove that these are linearly independent over \( \mathbb{R} \).

(ii) Prove that any derivation sends the germ of a constant function to 0.

(iii) The projection \( x_i : \mathbb{R}^n \to \mathbb{R} \) onto the \( i \)th coordinate defines a germ which we also denote \( x_i \in C^\infty(\mathbb{R}^n, 0) \). Prove that if \( X : C^\infty(\mathbb{R}^n, 0) \to \mathbb{R} \) is a derivation with \( X(x_i) = 0 \) for all \( i \), then \( X(f) = 0 \) for all germs \( f \). [Hint: First consider the case where \( f(x) = x_i g(x) \) for some smooth germ \( g \) with \( g(0) = 0 \). Then use Taylor expansion.]

(iv) Let \( X : C^\infty(\mathbb{R}^n, 0) \to \mathbb{R} \) be an arbitrary derivation. Prove that there exist \( a_i \in \mathbb{R} \) such that \( X = \sum a_i(\partial/\partial x_i) \).