**Theorem 0.1** (Poincare). There exists a closed 3-manifold \( M \) which is not simply connected, but which has \( H_k(M) \cong H_k(S^3) \) for all \( k \).

**Proof.** Let \( A_5 \subseteq \Sigma_5 \) be the alternating group. We will use without proof that \( A_5 \) is simple. We will also use without proof that \( SO(3) \) is homeomorphic to \( RP^3 \). Include \( A_5 \subseteq SO(3) \) as the isometries of an icosahedron, and let \( M = (SO(3))/A_5 \). Since the projection \( SO(3) \to M \) is a covering map and hence a local homeomorphism, \( M \) is again a closed 3-manifold.

The universal cover of \( M \) is \( S^3 \). Let \( G = \pi_1(M) \). Then \( G \) acts on \( S^3 \) through deck transformations. Deck transformations of \( S^3 \to RP^3 \) gives a subgroup \( \{ \pm 1 \} < G \), and the quotient is \( G/\{ \pm 1 \} = A_5 \).

**Lemma 0.2.** \( M \) is orientable.

**Proof.** \( SO(3) \cong RP^3 \) is orientable. We will show that the action of \( A_5 \) on \( SO(3) \) preserves orientation; then it follows that \( M \) inherits an orientation from \( SO(3) \). We get a group homomorphism \( \epsilon : A_5 \to \mathbb{Z}/2 \) where \( \epsilon(g) = 1 \) iff \( g : SO(3) \to SO(3) \) reverses orientation. If \( M \) were not orientable, \( \epsilon \) would be surjective. This contradicts the fact that \( A_5 \) is simple. \( \square \)

This proves that \( H_3(M) = \mathbb{Z} = H_3(S^3) \). It remains to see that \( H_1(M) = H_2(M) = 0 \). Let \( G' = [G,G] < G \) be the commutator subgroup. It maps onto the commutator subgroup of \( A_5 \) which is \( A_5 \), so \( |G : G'| \) is either 1 or 2.

**Lemma 0.3.** \( |G : G'| = 1 \).

**Proof.** Suppose it were 2. Then the composition \( \{ \pm 1 \} \to G \to G/G' \) would be an isomorphism. By Hurewicz' theorem, the covering map \( RP^3 \to M \) would then induce an isomorphism \( \mathbb{Z}/2 = H_1(RP^3) \to H_1(RP^3) = \mathbb{Z}/2 \) and hence by the universal coefficient theorem we get an isomorphism \( \mathbb{Z}/2 = H^1(RP^3) \to H^1(RP^3) = \mathbb{Z}/2 \) and by Poincare duality an isomorphism \( H_2(RP^3) \to H_2(M) \).

This implies that the induced map

\[
H^*(M; \mathbb{F}_2) \to H^*(RP^3; \mathbb{F}_2) = \mathbb{F}_2
\]

is an isomorphism in degrees \( \leq 2 \). Since the cup product \( H^1(M; \mathbb{F}_2) \times H^2(M; \mathbb{F}_2) \to H^3(M; \mathbb{F}_2) = \mathbb{F}_2 \) is non-degenerate, (0.1) is also an isomorphism in degree 3. By the universal coefficient theorem, this implies that the induced map in integral cohomology

\[
\mathbb{Z} = H^3(M) \to H^3(RP^3) = \mathbb{Z}
\]

is multiplication by an odd number. But it follows from Exercise 3.3.9 in Hatcher that (0.2) is multiplication by 60 = \( |A_5| \).

Therefore we get \( H_1(M) = G/G' = 0 \) by Hurewicz, and then \( H_2(M) = 0 \) by universal coefficient theorem and Poincare duality. \( \square \)