Notation: If $C$ is a category and $x, y$ are two objects of $C$, we shall write $C(x, y)$ for the set of morphisms from $x$ to $y$. (Another common notations is $\text{Mor}_C(x, y)$ or just $\text{Mor}(x, y)$.)

Notation: If $C$ is an object and $x$ is an object, we shall write either $1_x$ or $\text{id}_x$ for the identity morphism of $x$.

Definition: Let $C$ be a category. A morphism $f \in C(x, y)$ is called an isomorphism if there exists a morphism $g \in C(y, x)$ such that $f \circ g = 1_y$ and $g \circ f = 1_x$.

Two objects $x$ and $y$ are called isomorphic if $C(x, y)$ contains an isomorphism.

1. Prove that if $F : C \to D$ is a functor and $f \in C(x, y)$ is an isomorphism, then $F(x) \in D(F(x), F(y))$ is also an isomorphism. (“Functors send isomorphisms to isomorphisms”.)

2. Let $C$ denote the category of topological spaces and continuous maps. Prove that there exists a category $D$ with the following properties: (i) $\text{Ob}(C)$ is the collection of all topological spaces; (ii) $C(X, Y) = [X, Y]$ (i.e. the set of homotopy classes of maps) for all spaces $X, Y$; (ii) there is a functor $F : C \to D$ given on objects by $F(X) = X$ and on morphisms by $F(f) = [f]$ (i.e. it sends a map to its equivalence class modulo homotopy).

3. Let $C$ and $D$ be the categories from the previous exercise ($D$ is sometimes called “the homotopy category”, although that name is also sometimes used about a slightly different category). Prove that two spaces $X$ and $Y$ are homotopy equivalent if and only if $F(X)$ and $F(Y)$ are isomorphic objects in $D$.

Remark: The “if and only if” part of this exercise shows that the functor $F$ above is in some sense the “strongest possible” homotopy invariant of a space. This is of course a rather useless invariant, because it is no easier to determine whether $F(X)$ and $F(Y)$ are isomorphic than to determine whether $X \simeq Y$ in the first place. (This example shows that we shouldn’t be looking for the strongest possible invariant: neither $\pi_1$ nor the invariants defined later in this class are as strong as the functor $F$ from this exercise, but they are much more useful.)