We now know that $A$ must be of the form

$$A = \mathbb{Z}/p^k \oplus \mathbb{Z}/p^l, \quad 0 \leq k \leq l, \quad k + l = n + m.$$

**Claim:** $l \geq \max(n, m)$.

**Proof:** The maximal order of an element in $A$ is $p^k$. On the other hand, the generator of $A$ has order $p^m$, and $n \cdot A$ has order at least $p^m$ (since $j(\alpha^m) = 0$ for $\alpha < p^m$). Therefore $l \geq n$, $l \geq m$, so $l \geq \max(n, m)$.

It remains to see how all these can be realized. The map

$$\mathbb{Z}/p^m \xrightarrow{i} \mathbb{Z}/p^k \oplus \mathbb{Z}/p^l$$

is well defined and injective.

The map

$$\mathbb{Z}/p^k \oplus \mathbb{Z}/p^l \xrightarrow{j} \mathbb{Z}/p^m$$

is well defined and injective. It is easy to see $j(i) = 0$, so $\text{Im}(i) \subseteq \text{Ker}(j)$.

On the other hand, $|\text{Im}(i)| = p^n$, $|\text{Ker}(j)| = \frac{p^k}{p^m} = p^{k-m} = p^n$, so we must have $\text{Im}(i) = \text{Ker}(j)$, so $0 \to \mathbb{Z}/p^n \to \mathbb{Z}/p^k \oplus \mathbb{Z}/p^l \to \mathbb{Z}/p^m$ is exact.
We know that any finite group $A$ is a direct sum of groups of the form $\mathbb{Z}/p^n$, for $p$ a prime and $n \geq 1$. If

$$0 \rightarrow \mathbb{Z}/p^n \rightarrow A \rightarrow \mathbb{Z}/p^n \rightarrow 0$$

is exact, then $A$ has $p^n$ elements (by Lagrange's theorem), so only the prime $p$ will appear.

Pick $a \in A$ with $j(a)$ = generator of $\mathbb{Z}/p^n$. Let $b \in A$ be $b = i(\text{gen of } \mathbb{Z}/p^n)$.

Then for any $x \in A$, there is $\lambda \in \mathbb{Z}$ such that $x = \lambda + \langle a \rangle$, $j(x - xa) = j(x) - \lambda j(a) = 0$ for suitable $\lambda \in \mathbb{Z}$. Hence $x - xa = (\beta \cdot b$ for some $\beta \in \mathbb{Z}$.

So $A$ is generated by $a$ and $b$.

**Claim:** $A$ is the direct sum of at most 2 groups of the form $\mathbb{Z}/p^n$.

**Proof:** Since $A$ is generated by $a, b$, so is $A/pA$. But $A/pA$ is a $\mathbb{Z}/p$ vector space whose dimension is the number of $\mathbb{Z}/p^n$ summands in $A$. Therefore, $A$ is the direct sum of at most 2 groups of the form $\mathbb{Z}/p^n$. 