1. (i) For any CW complex $X$ the cellular chains, and hence homology, of $X^{(k)}$ vanishes in degrees $> k$. The cellular chains of $X^{(k)}$ agrees with those of $X$ in degrees $\leq k$. Therefore in degrees $< k$, they have the same groups of cycles and boundaries, and hence the same (reduced) homology.

Now, $\Delta^p$ is contractible so $\tilde{H}_i(\Delta^p) = 0$ for all $i$, and hence $\tilde{H}_i((\Delta^p)^{(k)}) = 0$ for $i \neq k$.

(ii) Let $c_k$ denote the rank of the augmented cellular chain group $C_k(\Delta^p)$, and let $r_k$ denote the rank of the linear map $\partial_k : C_k(\Delta^p) \to C_{k-1}(\Delta^p)$. Then the group of boundaries $B_k = \Im(\partial_{k+1})$ has rank $r_{k+1}$, and the group of cycles $Z_k$ has rank $c_k - r_k$. Since $\tilde{H}_i(\Delta^p) = 0$, we get the equation

$$r_{k+1} = c_k - r_k$$

which together with $r_0 = 1$ (the augmentation) uniquely determines the numbers $r_k$ as a function of $c_k$.

If $S$ has cardinality $k + 1$, then $f_S : \Delta^k \to \Delta^p$ is the attaching map of a cell of dimension $k$. Hence the number of $k$-cells is

$$c_k = \binom{p + 1}{k + 1}.$$  

By induction on (0.1) we see that $r_k = \binom{p}{k}$: this is true for $k = 0$ and the induction step is

$$\binom{p}{k + 1} = \binom{p + 1}{k + 1} - \binom{p}{k},$$

which is true by “Pascal’s triangle”.

(iii) $H_n((\Delta^p)^{(n)})$ is the group of cycles $Z_n \subseteq C_n$. Since $Z_n$ is a subgroup of a free abelian group, it is itself a free abelian group. Its rank is

$$c_n - r_n = r_{n+1} = \binom{p}{n + 1}.$$  

2. (i) Let $\textbf{Top}_*$ denote the category of based topological spaces and based maps. Naturality of $\phi$ means that if $f : (X, x_0) \to (Y, y_0)$ is a based map, then

$$\phi \circ f_\# = f_* \circ \phi : \pi_1(X, x_0) \to H_1(Y).$$

This is true because both maps send the homotopy class of a loop $\lambda : (\Delta^1, \partial\Delta^1) \to (X, x_0)$ to the homology class of $f \circ \lambda$. 
(ii) First assume \( T : \pi_1 \to H_1 \) is a natural transformation. Let \( * \in S^1 \) be a basepoint and regard \( \pi_1(X, x_0) = [(S^1, *), (X, x_0)] \). Let \( \iota \in \pi_1(S^1, *) \) be element represented by the identity map. Then \( T(\iota) \in H_1(S^1) = \mathbb{Z} \), so \( T(\iota) = n\phi(\iota) \) for a unique \( n \in \mathbb{Z} \).

Then for any \([f] \in \pi_1(X, x_0)\) we have \([f] = f_#(\iota)\) and naturality of \( \phi \) demands
\[
T([f]) = T \circ f_#(\iota) = f_*(T(\iota)) = f_*(n\phi(\iota)) = nf_*(\phi(\iota)) = n\phi \circ f_#(\iota) = n\phi([f]).
\]

So any natural transformation \( T : \pi_1 \to H_1 \) must be an integer multiple of \( \phi \). On the other hand it is clear that all these work: For any \( n \in \mathbb{Z} \), the definition \( T_n : \pi_1(X, x_0) \to H_1(X) \)
\[
[f] \mapsto nf
\]
gives a natural transformation of functors from \( \text{Top}_\bullet \) to groups.

(iii) For \( n \geq 2 \) we have \( H_n(S^1) = 0 \), so any natural transformation \( T : \pi_1 \to H_n \) must have \( T(\iota) = 0 \). Proceeding as in (ii) we now get \( T([f]) = f_*(T(\iota)) = 0 \) for all \([f] \in \pi_1(X, x_0)\), so the only natural transformation is the one that sends all elements to 0.

For \( n = 0 \) we have \( H_0(S^1) = \mathbb{Z} \). The generator is the class \([c]\) of the constant map \( c \) to the basepoint. Hence a natural transformation \( T : \pi_1 \to H_0 \) must have \( T(\iota) = n[c] \). For any \([f] \in \pi_1(X, x_0)\) we therefore get \( T([f]) = f_*(T(\iota)) = f_*(n[c]) = nf_*(c) \). In particular \( T([f]) \in H_0(X) \) is independent of \([f] \in \pi_1(X, x_0)\).

Considering again \( X = S^1 \), \( T \) gives a constant map \( \pi_1(S^1, *) \to H_0(S^1) \). The only constant homomorphism \( \mathbb{Z} \to \mathbb{Z} \) is the zero map, so \( T(\iota) = 0 \), and hence \( n = 0 \) so also in this case the only natural transformation is the one that gives the zero map for all spaces.

3.

(i) We can use
\[
\cdots \to \Lambda \xrightarrow{2} \Lambda \xrightarrow{2} \Lambda \to M \to 0
\]

(ii) We toss out \( M \) from the projective resolution and tensor with \( M \) and get a chain complex
\[
\cdots \to \Lambda \otimes_\Lambda M \to \Lambda \otimes_\Lambda M \to 0.
\]

We have the isomorphism \( \Lambda \otimes_\Lambda M = M \), and multiplication by 2 vanishes. Hence we get
\[
\text{Tor}_n^\Lambda(M, M) = \mathbb{Z}/2
\]
for all \( n \geq 0 \).
(iii) Use the same projective resolution as in (i) and apply $\text{Hom}_\Lambda(-, M)$. We have $\text{Hom}_\Lambda(\Lambda, M) = M$, and multiplication by 2 vanishes, so we get

$$\text{Ext}^n_\Lambda(M, M) = \mathbb{Z}/2$$

for all $M$.

Applying $\text{Hom}_\Lambda(-, \Lambda)$ and using $\text{Hom}_\Lambda(\Lambda, \Lambda) = \Lambda$ we get the cochain complex

$$0 \rightarrow \Lambda \xrightarrow{2} \Lambda \xrightarrow{2} \Lambda \xrightarrow{2} \ldots$$

which is exact in positive degrees. Hence $\text{Ext}^0_\Lambda(M, \Lambda) = \text{Hom}_\Lambda(M, \Lambda) = \mathbb{Z}/2$ and all the higher Ext groups vanish.

4.

(i) To see that $(\Delta_\ast(X), \partial)$ is a chain complex of $\Lambda$-modules we only need to check that $\partial$ is a homomorphism of $\Lambda$-modules. But this just the fact that $T_\ast$ is a chain map.

To see that $\Delta_n(X)$ is a free $\Lambda$-module, notice that each $\sigma : \Delta^n \rightarrow Y$ has exactly two lifts to $Y$ (because $\Delta^n$ is contractible). If $\tilde{\sigma} : \Delta^n \rightarrow X$ is one lift, then the other is $t\tilde{\sigma} = T \circ \tilde{\sigma}$. Pick a specific lift $\tilde{\sigma}$ for each $\sigma$. Then any element of $\Delta_n(X)$ can be written uniquely as a finite $\Lambda$-linear combination of the simplices $\tilde{\sigma}$ and $T\tilde{\sigma}$, as $\sigma$ ranges through simplices of $Y$. But this just says that any element of $\Delta_n(X)$ can be written uniquely as a $\Lambda$-linear combination of the elements $\tilde{\sigma}$, so these form a basis.

(ii) The expression $n p_\ast(c)$ is obviously $\mathbb{Z}$-bilinear as a function of $(n, c)$. Since $t \in \Lambda$ acts trivially on both $\mathbb{Z}$ and $\Delta_\ast(Y)$, it is $\Lambda$-linear as a function of $n$. That it is $\Lambda$-linear in $c$ amounts to $n p_\ast(T_\ast(c)) = n p_\ast(c)$ which follows from $p \circ T = p$.

To construct an inverse, pick as in (i) a particular lift $\tilde{\sigma}$ of each $\sigma : \Delta^n \rightarrow Y$. Then $\sigma \mapsto \tilde{\sigma} \otimes 1$ extends to a $\mathbb{Z}$-linear map $\Delta_\ast(Y) \rightarrow \Delta_\ast(X) \otimes_\Lambda \mathbb{Z}$. It is also $\Lambda$-linear because $t \in \Lambda$ acts trivially on both modules. It follows from (i) that $\Delta_n(X) \otimes_\Lambda \mathbb{Z}$ is generated as a $\mathbb{Z}$-module by the elements $\tilde{\sigma} \otimes 1$. It is clear that the two homomorphisms are each others inverse when applied to these generators.

(iii) We already proved that $\Delta_n(X)$ is a free $\Lambda$-module. When $X$ is contractible $H_\ast(X) = 0$ which says precisely that the complex is exact. It remains to see that the augmentation is $\Lambda$-linear: on generators we have $\epsilon(t\sigma) = 1 = \epsilon(\sigma)$.

(iv) We can calculate $\text{Tor}^1_\Lambda(\mathbb{Z}, \mathbb{Z})$ by using the resolution in (iii). The isomorphism in (ii) commutes with the boundary map: $(\partial c) \otimes n = \partial(n p_\ast(c))$ since $p_\ast$ is a chain map. Hence we get an isomorphism of chain complexes

$$\Delta_\ast(X) \otimes_\Lambda \mathbb{Z} \cong \Delta_\ast(Y).$$

One chain complex calculates Tor and the other calculates $H_\ast(Y)$, hence these are isomorphic.