1. Let \( \{a_n\}_{n=1}^{\infty} \) be a bounded sequence of real numbers and let \( f : \mathbb{N} \to \mathbb{N} \) be a surjective function.

(a) Prove that 
\[
\limsup a_{f(n)} \geq \limsup a_n.
\]
(b) Give an example in which \( \limsup a_{f(n)} > \limsup a_n \).

2. Prove that the function \( f : \ell^\infty \to \mathbb{R} \) defined by 
\[
f(\{a_n\}_{n=1}^{\infty}) = \limsup a_n
\]
is well defined and continuous.

3. Let \((M, d)\) be a metric space and let \( A \subseteq M \) be a non-empty subset. Define \( f : M \to \mathbb{R} \) by 
\[
f(x) = \inf \{d(x, a) | a \in A\}
\]
(a) Prove that \( |f(x) - f(y)| \leq d(x, y) \) and deduce that \( f \) is continuous.
(b) Let \( A \) be a closed subset. Prove that \( f(x) = 0 \) implies \( x \in A \). (Bonus question: is this true if we don’t assume \( A \) is closed?)
(c) Let \( M = \mathbb{R} \) with the euclidean metric (i.e. \( d(x, y) = |x - y| \)). Let \( A \subseteq \mathbb{R} \) be a closed subset and \( x \in \mathbb{R} \). Prove that there exists a \( y \in A \) such that \( f(x) = d(x, y) \). (Bonus question: is this true for any metric space \( M \)?)

4. Prove that \( \ell^1 \subseteq \ell^\infty \) and that the map \( i : \ell^1 \to \ell^\infty \) given by \( i(\{a_n\}_{n=1}^{\infty}) = \{a_n\}_{n=1}^{\infty} \) is continuous.

5. Let \( M \) be the set of all subsets of \( \mathbb{N} \). For a subset \( A \subseteq \mathbb{N} \) we shall write \( 1_A : \mathbb{N} \to \mathbb{R} \) for the function 
\[
1_A(x) = \begin{cases} 
1 & \text{if } x \in A \\
0 & \text{if } x \notin A 
\end{cases}
\]
For two subsets \( A, B \subseteq M \) we then define 
\[
d(A, B) = \sum |1_A(n) - 1_B(n)|2^{-n}.
\]
(a) Prove that this defines a well defined function $M \times M \to \mathbb{R}$.

(b) Prove that $(M, d)$ is a metric space.

(c) Let $A_n = \{ k \in \mathbb{N} | k \geq n \}$. Prove that the sequence $\{ A_n \}_{n=1}^\infty$ converges and find its limit.