Math 171: Midterm Solutions

Spring 2013

1. (a) Check the axioms:
   (i) Since $C > 0$, $d'(x, y) = \min(d(x, y), C) = 0$ iff $d(x, y) = 0$ iff $x = y$.
   (ii) $d'(x, y) = \min(d(x, y), C) = \min(d(y, x), C) = d'(y, x)$.
   (iii) We need to show that $\min(d(x, z), C) \leq \min(d(x, y), C) + \min(d(y, z), C)$. If either $d(x, y) \geq C$ or $d(y, z) \geq C$, then LHS $\leq C \leq$ RHS. If both $d(x, y) < C$ and $d(y, z) < C$, then LHS $\leq d(x, z) \leq d(x, y) + d(y, z) = \text{RHS}$.

(b) For $\epsilon \leq C$, $\min(d(x, y), C) < \epsilon$ iff $d(x, y) < \epsilon$, so $B_\epsilon(x) = B'_\epsilon(x)$ (open $\epsilon$-balls for $d$ and $d'$, respectively). If $U$ is open w.r.t. $d$, then for all $x \in U$, there exists $\epsilon > 0$ such that $B_\epsilon(x) \subset U$. Replacing $\epsilon$ by $\min(\epsilon, C)$, we may assume that $\epsilon \leq C$, so $B'_\epsilon(x) \subset U$. Thus $U$ is open w.r.t. $d$. The same argument gives the converse.

2. For $\{a_n\} \in \ell^1$, $\sum a_n$ converges absolutely, hence converges. Thus $f$ is well defined. For $\{a_n\}, \{b_n\} \in \ell^1$,

$$|f(\{a_n\}) - f(\{b_n\})| = \left| \sum_{n=1}^N a_n - \sum_{n=1}^N b_n \right| = \left| \sum_{n=1}^N (a_n - b_n) \right| \leq \sum_{n=1}^N |a_n - b_n| = d(\{a_n\}, \{b_n\}),$$

where $d$ is the $\ell^1$-metric. This gives the (uniform) continuity of $f$ with $\delta = \epsilon$.

[More carefully, using triangle inequality first for finite sums and then passing to the limit: For all $N \geq 1$,

$$\left| \sum_{n=1}^N a_n - \sum_{n=1}^N b_n \right| \leq \sum_{n=1}^N |a_n - b_n| \leq \sum_{n=1}^\infty |a_n - b_n| = d(\{a_n\}, \{b_n\}).$$

Since $x \mapsto |x|$ is continuous and the limit of a difference is the difference of the limits, taking $N \to \infty$ yields $|f(\{a_n\}) - f(\{b_n\})| \leq d(\{a_n\}, \{b_n\})].$]

3. Let $A = \limsup a_n$, $B = \limsup b_n$. Let $\epsilon > 0$. By the definition of lim inf and lim sup,

$$0 < a_n < A + \epsilon, \quad 0 < b_n < B + \epsilon$$

(1)
for all large $n$. Then $a_nb_n < (A + \epsilon)(B + \epsilon)$ for all large $n$, so $\limsup a_nb_n \leq (A + \epsilon)(B + \epsilon)$. Taking $\epsilon \to 0$, $\limsup a_nb_n \leq AB = (\limsup a_n)(\limsup b_n)$.

[More carefully: By the definition of lim inf, there exists $N_1$ such that

$$n \geq N_1 \text{ implies } a_n > 0.$$ Similarly, there exist $N_2, N_3, N_4$ such that

$$n \geq N_2 \text{ implies } b_n > 0,$$$$
n \geq N_3 \text{ implies } a_n < A + \epsilon,$$$$
n \geq N_4 \text{ implies } b_n < B + \epsilon.$$

Thus $n \geq \max(N_1, N_2, N_3, N_4)$ implies (1).]

4. Let $d$ be the $\ell^2$-metric. For $n \in \mathbb{N}$, let $a_n = \left(\frac{1}{n}, 0, 0, \ldots\right) \in \ell^2$. Let $A = \{a_n : n \in \mathbb{N}\}$. As $n \to \infty$, $d(a_n, (0, 0, \ldots)) = \frac{1}{\sqrt{n}} \to 0$, so $a_n \to (0, 0, \ldots) \in \ell^2 \setminus A$. Thus $A$ is not closed.

5. (a) $(0, \infty)$ is open in $\mathbb{R}$, so $f^{-1}((0, \infty)) = \{x \in M : f(x) > 0\}$ is open is $M$.

Alternatively: Let $x_0 \in U := \{x \in M : f(x) > 0\}$. By the continuity $f$ at $x_0$ with $\epsilon = f(x_0)$, there exists $\delta > 0$ such that $d(x, x_0) < \delta$ implies $|f(x) - f(x_0)| < f(x_0)$, so in particular $f(x) > 0$. Thus $B_\delta(x_0) \subset U$. Since $x_0$ was arbitrary, this shows $U$ is open.

(b) Define $f(x) = \inf_{a \in U^c} d(x, a)$. Since $d(x, a) \geq 0$ for all $x, a \in M$, $f(x) \geq 0$ for all $x \in M$. If $x \in U^c$, then takig $a = x$, $f(x) \leq d(x, x) = 0$, so $f(x) = 0$. Now let $x \in U$. Since $U$ is open, $B_\epsilon(x) \subset U$ for some $\epsilon > 0$. If $a \in U^c$, then $a \notin B_\epsilon(x)$, i.e. $d(x, a) \geq \epsilon$. Taking infimum, $f(x) \geq \epsilon > 0$. Thus $U = \{x \in M : f(x) > 0\}$.

Fix $x, y \in M$. For any $a \in U^c$, $d(x, a) \leq d(x, y) + d(y, a)$. Taking infimum over $a \in U^c$, $f(x) \leq d(x, y) + f(y)$. By symmetry, $f(y) \leq d(x, y) + f(x)$ as well, so

$$|f(x) - f(y)| \leq d(x, y) \quad \text{for all } x, y \in M,$$

which gives the (uniform) continuity of $f$ with $\delta = \epsilon$. 

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