Define $g : M \to \mathbb{R}$ by $g(x) = d(f(x), x)$. For any distinct $x, y \in M$,

\[
d(f(x), x) \leq d(f(x), f(y)) + d(f(y), y) + d(y, x)
\]

$\Rightarrow g(x) - g(y) \leq d(f(x), f(y)) + d(x, y) < 2d(x, y),$

and by symmetry, indeed $|g(x) - g(y)| < 2d(x, y)$. Thus $g$ is (uniformly) continuous, hence achieves its infimum on the compact space $M$, at some $x_0 \in M$. Then $x_0$ is a fixed point; otherwise, $f(x_0) \neq x_0$, so $g(f(x_0)) = d(f(f(x_0)), f(x_0)) < d(f(x_0), x_0) = g(x_0)$, contradicting the choice of $x_0$. This is the unique fixed point; if $x'_0$ were a different fixed point, then $d(x_0, x'_0) > d(f(x_0), f(x'_0)) = d(x_0, x'_0)$, a contradiction.

**JP 44.7** 1. Given $\epsilon > 0$, take $\delta = \epsilon/2$. Then $d(x, y) < \delta$ implies $d(f(x), f(y)) \leq cd(x, y) < \epsilon$.

2. Since $|\frac{1}{2}x - \frac{1}{2}y| \leq \frac{1}{2}|x - y|$, $x \mapsto \frac{1}{2}x : \mathbb{R} \to \mathbb{R}$ is a contraction mapping with $c = \frac{1}{2}$. It is surjective.

3. By exercise 43.4, there exist $x, y \in M$ achieving $d(x, y) = \text{diam}(M)$. Since $M$ contains more than one point, $\text{diam}(M) > 0$, so $x \neq y$. If such $f$ existed, then by surjectivity, $f(x') = x$, $f(y') = y$ for some $x', y' \in M$, necessarily distinct, and $d(x', y') \leq \text{diam}(M) = d(x, y) = d(f(x'), f(y')) < d(x', y')$, a contradiction.

**JP 46.1** There are only finitely many pairs of distinct points, all positive distance apart, so we can take $\epsilon > 0$ less than all such distances. Then applying the definition of a Cauchy sequence with $\epsilon$ shows that every Cauchy sequence is eventually constant, hence convergent.

**JP 46.5** 1. Let $x_n \in C$, $x_n \to x \in M$. Then $\{x_n\}$ is Cauchy in $M$, hence in $C$, so by completeness, it converges to some $y \in C$. Then also $x_n \to y$ in $M$. Since the limit of a convergent sequence is unique in a metric space, $x = y \in C$. Thus $C$ is closed.

2. Let $\{x_n\}$ be Cauchy in $C$. Then it is Cauchy in $M$, so by completeness, it converges to some $x \in M$. Since $C$ is closed, $x \in C$. Thus $C$ is complete.

**JP 46.10** ($X_n$ are of course assumed nonempty.) For each $n \geq 1$, choose any $x_n \in X_n$. For any $n \geq 1$, $\{x_m\}_{m \geq n}$ is a sequence in $X_n$. We claim that it is Cauchy. Given $\epsilon > 0$, since
diam($X_n$) $\to 0$, diam($X_N$) $< \epsilon$ for some $N$. Then $m, m' > N$ implies $x_m, x_{m'} \in X_N$, so $d(x_m, x_{m'}) \leq$ diam($X_N$) $< \epsilon$. This proves the claim. Each $X_n$ is closed in a complete space, hence complete, so $\{x_m\}_{m \geq n}$ converges in $X_n$ to some $x^{(n)}$. For $n \geq 2$, this convergence also holds in $X_{n-1}$. But $\{x_m\}_{m \geq n-1}$ also converges in $X_{n-1}$ to $x^{(n-1)}$. Since the limit of convergence is unique in a metric space, $x^{(n)} = x^{(n-1)}$. Hence by induction, all $x^{(n)}$ are equal; call this $x$. Then $x = x^{(n)} \in X_n$ for all $n$, so $x \in \bigcap_{n=1}^{\infty} X_n$. This is the unique point in $\bigcap_{n=1}^{\infty} X_n$; if it contained distinct points $x, y$, then diam($X_n$) $\geq d(x, y) > 0$ for all $n$, contradicting diam($X_n$) $\to 0$.

**JP 46.11** 1. We first show by induction that $d(x_n, x_{n+1}) \leq c^{n-1}d(x_1, x_2)$ for all $n \geq 1$. The case $n = 1$ is clear. Assuming this for $n$,

$$d(x_{n+1}, x_{n+2}) = d(f(x_n), f(x_{n+1})) \leq cd(x_n, x_{n+1}) \leq c^n d(x_1, x_2),$$

completing the induction. Now, since $c < 1$, given $\epsilon > 0$ there exists $N$ such that $\frac{c^n}{1-c} d(x_1, x_2) < \epsilon$. Then for $n, m \geq N$, say with $n \leq m$,

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m)$$

$$\leq c^{n-1} d(x_1, x_2) + c^n d(x_1, x_2) + \cdots + c^{m-2} d(x_1, x_2)$$

$$\leq (c^{n-1} + c^n + \cdots) d(x_1, x_2)$$

$$= \frac{c^{n-1}}{1-c} d(x_1, x_2) \leq \frac{c^{N-1}}{1-c} d(x_1, x_2) < \epsilon.$$

2. Since $f$ is continuous, $f(x) = \lim_{n \to \infty} f(x_n)$. But $\{f(x_n)\}$ is just $\{x_n\}$ reindexed, so also converges to $x$.

3. If $y \in M$ were a different fixed point, then $d(x, y) = d(f(x), f(y)) \leq cd(x, y)$ with $d(x, y) > 0$, contradicting $0 \leq c < 1$.

1. Let $\{a^{(n)}\}_{n \geq 1}$ be Cauchy in $\ell^\infty$, where $a^{(n)} = (a_1^{(n)}, a_2^{(n)}, \ldots)$. For any $i \geq 1$, $|a_i^{(n)} - a_i^{(m)}| \leq \sup_j |a_j^{(n)} - a_j^{(m)}| = d(a^{(n)}, a^{(m)})$. Hence Cauchyness implies that for any $\epsilon > 0$, there exists $N(\epsilon)$ such that $|a_i^{(n)} - a_i^{(m)}| < \epsilon$ for all $n, m > N(\epsilon)$ and all $i \geq 1$. In particular, each $\{a_i^{(n)}\}_{n \geq 1}$ is Cauchy in $\mathbb{R}$, hence converges to some $a_i \in \mathbb{R}$. Taking $m \to \infty$,

$$|a_i^{(n)} - a_i| \leq \epsilon \text{ for all } n > N(\epsilon) \text{ and all } i \geq 1. \quad (1)$$

Then $|a_i| \leq |a_i^{(n)}| + 1$ for all $n > N(1)$ and all $i \geq 1$. Since $a^{(n)} \in \ell^\infty$, this implies $a := (a_1, a_2, \ldots) \in \ell^\infty$. Moreover, taking supremum over $i$ in (1) shows that $a^{(n)} \to a$ in $\ell^\infty$.

2. a
(a) $s_0 = 1$, $s_1 = 1 + 4$, $s_2 = 1 + 4 + 4^2$, ..., so

$$s_n = 1 + 4 + \cdots + 4^n = \frac{4^{n+1} - 1}{4 - 1}.$$  

Then $s_n - s_m = \frac{1}{3}(4^{n+1} - 4^{m+1})$ is divisible by $2(\min(n, m)+1)$ factors of 2, so $d(s_n, s_m) = 2^{-2(\min(n, m)+1)}$. Hence $\{s_n\}$ is Cauchy.

(b) $3s_n - (-1) = 4^{n+1}$, so $d(3s_n, -1) = 2^{-(2n+2)} \to 0$ as $n \to \infty$.

(c) $f(x) - f(y) = 3(x - y)$ has as many factors of 2 as $x - y$, so $d(f(x), f(y)) = d(x, y)$. Thus $f$ is (uniformly) continuous.

(d) If $s_n \to s \in \mathbb{Z}$, then since $f$ is continuous, $3s = \lim_{n \to \infty} 3s_n = -1$, but no such $s \in \mathbb{Z}$ exists. Thus $\{s_n\}$ is a Cauchy sequence in $\mathbb{Z}$ with the 2-adic metric that does not converge.

2. b

(a) For $n \in \mathbb{Z}$, let $v(n)$ be the number of factors of 5. Thus $d(x, y) = 5^{-v(x-y)}$. Note that $v(nm) = v(n) + v(m)$. For any $x, y \in \mathbb{Z},$

$$f(x) - f(y) = x^5 - y^5 = (x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$$

$$\Rightarrow v(f(x) - f(y)) = v(x - y) + v(x^4 + x^3y + x^2y^2 + xy^3 + y^4).$$

This implies $v(f(x) - f(y)) \geq v(x - y)$, so $d(f(x), f(y)) \leq d(x, y)$. Thus $f$ is (uniformly) continuous.

Suppose $d(x, y) < 1$, i.e. $x \equiv y \mod 5$. Then $x^4 + x^3y + x^2y^2 + xy^3 + y^4 \equiv 5x^4 \equiv 0 \mod 5$, so (2) implies $v(f(x) - f(y)) \geq v(x - y) + 1$, hence $d(f(x), f(y)) \leq \frac{d(x, y)}{5}$.

(b) For any $x \in \mathbb{Z},$

$$x^5 - 2 = (x^5 - 2^5) + (2^5 - 2) = (x - 2)(x^4 + 2x^3 + 4x^2 + 8x + 16) + 5 \cdot 6.$$  

So $5|x - 2$ implies $5|x^5 - 2$, i.e. $d(x, 2) < 1$ implies $d(x^5, 2) < 1$. Thus $f(X) \subset X$. For $x, y \in X$, $5|x - 2, y - 2$, so $5|(x - 2) - (y - 2) = x - y$, i.e. $d(x, y) < 1$. Hence by (a), $f|_X : X \to X$ is a contraction.

(c) For $x \in \mathbb{Z}$, $x^5 = x$ implies $x = \pm 1$, and $\pm 1 \notin X$ since $d(\pm 1, 2) = 1$. Thus $f|_X : X \to X$ has no fixed point. By (b) and the Banach contraction principle, $X$ cannot be complete.

(d) Since $d$ has values in $5^{\mathbb{Z}_{\leq 0}}$, $X = \{x \in \mathbb{Z} : d(x, 2) < 1\} = \{x \in \mathbb{Z} : d(x, 2) \leq \frac{1}{5}\}$.  

This is closed, being the preimage of the closed set $(-\infty, \frac{1}{5}]$ under the continuous map $x \mapsto d(x, 2) : (\mathbb{Z}, d) \to \mathbb{R}$. Since $X$ is not complete and any closed subset of complete space is complete, $(\mathbb{Z}, d)$ is not complete.