Math 171: Homework 5 Solutions

Spring 2013

**JP 37.11** Suppose \( a_n^{(k)} \to a_n \) for all \( n \geq 1 \). Let \( \epsilon > 0 \) be given. Since \( \sum_{n>M} \frac{2^n}{2^n} = \frac{2}{2^n} \to 0 \) as \( M \to \infty \), we can choose \( M \) such that \( \sum_{n>M} \frac{2^n}{2^n} < \frac{\epsilon}{2} \). Now using \( a_n^{(k)} \to a_n \), for each \( n \) there exists \( K_n \) such that \( k \geq K_n \) implies \( |a_n^{(k)} - a_n| < \frac{\epsilon}{2M} \). Then \( k \geq K := \max(K_1, \ldots, K_M) \) implies

\[
\sum_{n=1}^{M} \frac{|a_n^{(k)} - a_n|}{2^n} < \sum_{n=1}^{M} \frac{\epsilon}{2M} = \frac{\epsilon}{2}.
\]

Note that \( |a_n^{(k)} - a_n| \leq |a_n^{(k)}| + |a_n| \leq 2 \) for all \( k \) and \( n \). So \( k \geq K \) implies

\[
d(a^{(k)}, a) = \sum_{n=1}^{\infty} \frac{|a_n^{(k)} - a_n|}{2^n} \leq \sum_{n=1}^{M} \frac{|a_n^{(k)} - a_n|}{2^n} + \sum_{n>M} \frac{2^n}{2^n} < \frac{\epsilon}{2} + \frac{\epsilon}{2}.
\]

Thus \( a^{(k)} \to a \).

For the converse, note that

\[
|a_n^{(k)} - a_n| = 2^n \cdot \frac{|a_n^{(k)} - a_n|}{2^n} \leq 2^n \sum_{m=1}^{\infty} \frac{|a_m^{(k)} - a_m|}{2^m} = 2^n \cdot d(a^{(k)}, a).
\]

So \( d(a^{(k)}, a) \to 0 \) implies \( |a_n^{(k)} - a_n| \to 0 \) for all \( n \geq 1 \).

**JP 43.4** By Exercise 40.12 (HW4), \( d : M \times M \to \mathbb{R} \) is continuous, where \( M \times M \) is the product metric space. Since \( M \) is compact, \( M \times M \) is compact. (Fact: The product of two compact spaces is compact.) So \( d \) achieves its supremum, i.e. there exist \( x, y \in M \) such that \( d(x, y) = \text{diam } M \).

Alternative solution not using Fact: Define \( f : M \to \mathbb{R} \) by \( f(x) = \sup_{y \in M} d(x, y) \). Taking supremum over \( y \) in \( d(x, y) \leq d(x, x') + d(x', y) \) yields \( f(x) \leq d(x, x') + f(x') \). By symmetry, \( f(x') \leq d(x, x') + f(x) \) as well, so \( |f(x) - f(x')| \leq d(x, x') \). Thus \( f \) is continuous. Since \( M \) is compact, \( \sup_{x \in M} f(x) = f(a) \) for some \( a \). Distance to \( a \) defines a continuous function (Theorem 40.3), so again by compactness, \( f(a) = d(a, b) \) for some \( b \). Now, \( \text{diam } M = \sup_{x \in M} f(x) = f(a) = d(a, b) \).
Theorem 43.5 asserts that, for metric spaces, compactness is equivalent to sequential compactness (every sequence has a convergent subsequence). We will show that $H^\infty$ is sequentially compact. So fix an arbitrary sequence $\{a(k)\}_{k \geq 1}$ in $H^\infty$. We will inductively choose subsequences, each time ensuring that one more coordinate converges, and at the end take the “diagonal” subsequence. First, since $a_1^{(k)} \in [-1,1]$ for all $k \geq 1$, by the (sequential) compactness of $[-1,1]$, there exists a subsequence $\{a((k_1,i))\}_{i \geq 1}$ of $\{a(k)\}_{k \geq 1}$ such that $a_1^{(k_1,i)}$ converges to some $a_1 \in [-1,1]$. Next, since $a_2^{(k_1,i)} \in [-1,1]$ for all $l \geq 1$, there exists a further subsequence $\{a((k_2,i))\}_{i \geq 1}$ of $\{a((k_1,i))\}_{i \geq 1}$ such that $a_2^{(k_2,i)}$ converges to some $a_2 \in [-1,1]$. Continuing inductively, for every $n \geq 1$ choose $\{a((k_n,i))\}_{i \geq 1}$ such that $a^{(k_n,i)} \to a_n \in [-1,1]$. Finally, let $a^{(k_i)} = a^{(k_i,i)}$. Then for any $n \geq 1$, $\{a^{(k_i)}\}_{i \geq 1}$ is eventually (namely, for $l \geq n$) a subsequence of $\{a((k_n,i))\}_{i \geq 1}$, so $\lim_{i \to \infty} a^{(k_i)} = \lim_{i \to \infty} a^{(k_n,i)} = a_n$. By Exercise 37.11, this implies $a^{(k_i)} \to (a_1,a_2,\ldots)$. 

1. (a) $\text{im}(f)$ consists exactly of sequences with every entry either 0 or 1. We show that $H^\infty \setminus \text{im}(f)$ is open. Let $a \in H^\infty \setminus \text{im}(f)$, so $a_n \notin \{0,1\}$ for some $n \geq 1$. Let $\epsilon = \min(|a_n - 0|,|a_n - 1|) > 0$. Then $B_{2^{-n}}(a) \subset H^\infty \setminus \text{im}(f)$; indeed, if $d(a,b) < 2^{-n}\epsilon$, then 

$$|a_n - b_n| = 2^n \cdot \frac{|a_n - b_n|}{2^n} \leq 2^n \cdot d(a,b) < \epsilon,$$

so $b_n \notin \{0,1\}$.

(b) This follows from Exercise 43.6 and (a) since a closed subset of a compact space is compact (Theorem 43.8).

[Alternatively, first show (b): Since $\{0,1\}$ is compact, the proof of Exercise 43.6 goes through word-for-word with $[-1,1]$ replaced by $\{0,1\}$. Then (a) follows since any compact subset of a metric space is closed (Exercise 42.2, HW4).]

(c) 

$$d(f(A), f(B)) = d(\{1_A(n)\}_{n=1}^\infty, \{1_B(n)\}_{n=1}^\infty) = \sum 2^{-n}|1_A(n) - 1_B(n)| = d(A,B).$$

Since $f$ is moreover injective, it defines a metric-preserving bijection between $M$ and $\text{im}(f)$. (That is, $M$ and $\text{im}(f)$ are “the same” as metric spaces, up to renaming points.) The compactness of $\text{im}(f)$ therefore implies that of $M$. 

2