Math 171: Homework 4 Solutions

Spring 2013

**JP 34.4** For \( n \geq 1 \), let

\[
I_n = \begin{cases} 
(a, b - \frac{1}{n}) & \text{if } a < b - \frac{1}{n} \\
\emptyset & \text{otherwise.}
\end{cases}
\]

For any \( x \in (a, b) \), there exists \( n \) such that \( \frac{1}{n} < b - x \), so \( x \in I_n \). Thus \( (a, b) \subseteq \bigcup_{n \geq 1} I_n \). Any finite subcover is contained in some \( \bigcup_{1 \leq n \leq N} I_n = I_N \) and so cannot contain \( (a, b) \).

**JP 34.6** [A continuous function on a compact space is uniformly continuous. (Theorem 44.5)]

For any fixed \( x \in [a, b] \), by continuity there exists \( \delta_x > 0 \) such that \( |x - y| < \delta_x \) implies \( |f(x) - f(y)| < \frac{\epsilon}{2} \). Clearly \( [a, b] \subseteq \bigcup_{x \in [a,b]} B_{\delta_x/2}(x) \). Since \( [a, b] \) is compact, there exist \( x_1, \ldots, x_n \in [a, b] \) such that \( [a, b] \subseteq \bigcup_{i=1}^{n} B_{\delta_{x_i}/2}(x_i) \). We claim that \( \delta := \min_{1 \leq i \leq n} \frac{\delta_{x_i}}{2} > 0 \) works. Indeed, let \( x, y \in [a, b] \) with \( |x - y| < \delta \). We have \( x \in B_{\delta_{x_i}/2}(x_i) \) for some \( i \). Then

\[
|y - x| \leq |y - x_i| + |x - x_i| < \delta + \frac{\delta_{x_i}}{2} \leq \frac{\delta_{x_i}}{2} + \frac{\delta_{x_i}}{2} = \delta_{x_i},
\]

so both \( x, y \in B_{\delta_{x_i}}(x_i) \). Hence

\[
|f(x) - f(y)| \leq |f(x) - f(x_i)| + |f(x_i) - f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\]

as desired.

**JP 40.12** Denote the product metric space by \((M \times M, d_p)\). For any \( x_1, x_2, y_1, y_2 \in M \),

\[
d(y_1, y_2) = d(y_1, x_1) + d(x_1, x_2) + d(x_2, y_2) = d(x_1, x_2) + (d(x_1, y_1) + d(x_2, y_2)) = d(x_1, x_2) + d_p((x_1, x_2), (y_1, y_2)).
\]

Similarly, \( d(x_1, x_2) \leq d(y_1, y_2) + d_p((x_1, x_2), (y_1, y_2)) \). Hence

\[
|d(x_1, x_2) - d(y_1, y_2)| \leq d_p((x_1, x_2), (y_1, y_2)).
\]

This gives the (uniform) continuity of \( d : (M \times M, d_p) \to \mathbb{R} \) with \( \delta = \epsilon \).
(a) We will use the following characterization of convergence.

**Proposition 0.1.** The following are equivalent in any metric space:

(i) \( x_n \to x \) (i.e. for any \( \epsilon > 0 \), \( d(x_n, x) < \epsilon \) for all large \( n \))

(ii) For any open neighborhood of \( U \) of \( x \), \( x_n \in U \) for all large \( n \).

**Proof.** (ii) \( \Rightarrow \) (i): Note that (i) can be rephrased as follows: for any \( \epsilon > 0 \), \( x_n \in B_\epsilon(x) \) for all large \( n \). This is a special case of (ii) when \( U = B_\epsilon(x) \).

(i) \( \Rightarrow \) (ii): Given \( U \), there exists \( \epsilon > 0 \) such that \( B_\epsilon(x) \subset U \). Then \( x_n \in B_\epsilon(x) \subset U \) for all large \( n \).

(i) \( \iff \) (ii): Closed sets are precisely the complement of open sets (and vice versa) (Theorem 39.5).

(i) \( \Rightarrow \) (iii): The Proposition characterizes convergence purely in terms of open sets.

(iii) \( \Rightarrow \) (ii): The definition of a closed set (contains all its limit points) depends only on the notion of convergence.

[Make sure you understand why these one-line proofs are completely rigorous. Here’s another way to say the argument in e.g. (i) \( \Rightarrow \) (iii). The notion of convergence was defined relative to a given metric \( d \). However, by the Proposition, it in fact depends only on the open sets defined by \( d \), not on \( d \) itself. So if two metrics define the same open sets, then they also define the same notion of convergence.]

(b) See the midterm solutions for the proof that \( d \) and \( d'' \) are equivalent. Alternatively, \( d(x_n, x) \to x \) iff \( \min\{d(x_n, x), 1\} \to 0 \) (essentially by the same argument as on the midterm), so \( x_n \to x \) w.r.t. \( d \) iff \( x_n \to x \) w.r.t. \( d'' \). Now use (a).

Similarly, to show the equivalence of \( d \) and \( d' \), it suffices to show that \( d(x_n, x) \to 0 \) iff \( \frac{d(x_n, x)}{1 + d(x_n, x)} \to 0 \). The forward direction is clear since \( \frac{d(x_n, x)}{1 + d(x_n, x)} \leq d(x_n, x) \). So suppose \( \frac{d(x_n, x)}{1 + d(x_n, x)} \to 0 \). Let \( \epsilon > 0 \) be given. Since \( \frac{\delta}{1 - \delta} \to 0 \) as \( \delta \to 0^+ \), there exists \( \delta > 0 \) such that \( \frac{\delta}{1 - \delta} < \epsilon \). For all large \( n \),

\[
\frac{d(x_n, x)}{1 + d(x_n, x)} < \delta \\
\Rightarrow d(x_n, x) < \delta + \delta d(x_n, x) \\
\Rightarrow d(x_n, x) < \frac{\delta}{1 - \delta} < \epsilon.
\]

Thus \( d(x_n, x) \to 0 \).

(c) For any \( x, y \in \mathbb{R}^n \),

\[
d(x, y) = \left( \sum (x_i - y_i)^2 \right)^{1/2} \leq \left( \sum |x_i - y_i|^2 \right)^{1/2} = d'(x, y).
\]
Also,
\[
  d'(x, y) = n \cdot \sum_{i=1}^{n} \frac{|x_i - y_i|}{n} \leq n \cdot \left( \frac{(x_i - y_i)^2}{n} \right)^{1/2} = \sqrt{n} \cdot d(x, y)
\]
by the RMS-AM inequality. Hence \(d(x^{(k)}, x) \to 0\) iff \(d'(x^{(k)}, x) \to 0\).

[Alternatively, show that \(x^{(k)} \to x\) w.r.t. \(d'\) iff \(x_i^{(k)} \to x_i\) for all \(i = 1, \ldots, n\). This shows that \(d'\) defines the same notion of convergence as \(d\) (Theorem 37.2).]

**JP 41.3** (a) \((\frac{1}{2}, 2)\) is open in \(\mathbb{R}\), so \(B = (\frac{1}{2}, 2) \cap A\) is open in \(A\). It is not closed in \(A\) because its complement in \(A\), \(A \setminus B = [0, \frac{1}{2}]\), is not open in \(A\): no open ball (of \(A\)) centered at \(\frac{1}{2}\) lies entirely in \(A \setminus B\).

(b) \((\frac{1}{4}, \frac{3}{4})\) is open in \(\mathbb{R}\), so \(C = (\frac{1}{4}, \frac{3}{4}) \cap A\) is open in \(A\). As in (a), \(C\) is not closed in \(A\) because \(A \setminus C = [0, \frac{1}{4}] \cup [\frac{3}{4}, 1]\) is not open in \(A\).

(c) \(A\) is closed but not open in \(\mathbb{R}\).

(d) \(C\) is open but not closed in \(\mathbb{R}\).

(g) \((e)–(g)\) assume that \(\mathbb{R}\) is identified as a subset of \(\mathbb{R}^2\) with the induced (i.e. relative) metric. For example, identify \(\mathbb{R}\) with the \(x\)-axis, \(\{(x, 0) : x \in \mathbb{R}\} \subset \mathbb{R}^2\). Note that the Euclidean metric on \(\mathbb{R}^2\) then restricts to that on \(\mathbb{R}\).

Since \(\mathbb{R}\) contains no open ball of \(\mathbb{R}^2\), no nonempty subset of \(\mathbb{R}\) can be open in \(\mathbb{R}^2\).

\(\mathbb{R}\) is closed in \(\mathbb{R}^2\) because \(\mathbb{R}^2 \setminus \mathbb{R}\) is open in \(\mathbb{R}^2\): if \((x, y) \in \mathbb{R}^2 \setminus \mathbb{R}\), then \(y \neq 0\), so \(B_{\mathbb{R}^2}(x, y) \subset \mathbb{R}^2 \setminus \mathbb{R}\). It follows that a subset of \(\mathbb{R}\) is closed in \(\mathbb{R}^2\) iff it is closed in \(\mathbb{R}\).

(e) \(A\) is closed in \(\mathbb{R}^2\).

(f) \(C\) is not closed in \(\mathbb{R}^2\).

**JP 41.5** Suppose \(x \in Y^{-(X)}\), i.e. \(y_n \to x\) in \(X\) for some \(y_n \in Y\). The metric on \(X\) is induced from that on \(M\), so \(y_n \to x\) also in \(M\), hence \(x \in Y^{-}\). Thus \(Y^{-(X)} \subset Y^{-}\). We have \(Y^{-(X)} \subset X\) by definition, so \(Y^{-(X)} \subset Y^{-} \cap X\). Conversely, suppose \(x \in Y^{-} \cap X\). Then \(x \in Y^{-}\), so \(x_n \to x\) in \(M\) for some \(x_n \in Y\). But \(x \in X\), so as before, \(x_n \to x\) in \(X\) as well. Thus \(x \in Y^{-}(X)\), which shows \(Y^{-} \cap X \subset Y^{-}(X)\).

In the same set-up, let \(Y^{o}(X)\) be the interior of \(Y\) in the metric space \(X\). We claim that \(Y^{o} = Y^{o}(X) \cap X^{o}\).

Write \(B\) and \(B^{X}\) for open balls in \(M\) and \(X\), respectively. Suppose \(y \in Y^{o}\), i.e. \(B_{\epsilon}(y) \subset Y\) for some \(\epsilon > 0\). Then \(B^{X}_{\epsilon}(y) = B_{\epsilon}(y) \cap X \subset B_{\epsilon}(y) \subset Y\), so \(y \in Y^{o}(X)\), and \(B_{\epsilon}(y) \subset X\), so \(y \in X^{o}\). Thus \(Y^{o} \subset Y^{o}(X) \cap X^{o}\). Conversely, suppose \(y \in Y^{o}(X) \cap X^{o}\), i.e. \(B_{\epsilon}(y) \subset Y\) and \(B_{\epsilon'}(y) \subset X\) for some \(\epsilon, \epsilon' > 0\). Then
\[
  B_{\min(\epsilon, \epsilon')}(y) \subset B_{\epsilon}(y) \cap B_{\epsilon'}(y) \subset B_{\epsilon}(y) \cap X = B_{\epsilon}(y) \subset Y,
\]
so \(y \in Y^{o}\). Thus \(Y^{o}(X) \cap X^{o} \subset Y^{o}\).
JP 42.1 More generally, let \((M,d)\) be a metric space, and suppose that \(M\) contains points arbitrarily far away from some fixed point \(x\). (That is, given any \(r > 0\), there exists \(y \in M\) such that \(d(x,y) > r\).) Then \(B_1(x) \subsetneq B_2(x) \subsetneq \cdots \) and \(M = \bigcup_{n \geq 1} B_n(x)\), so this open cover has no finite subcover. (Recall that open balls are open (Theorem 39.4).) Thus \(M\) is not compact.

All spaces in question clearly have this property.

JP 42.2

**Proposition 0.2.** Distinct points of a metric space have disjoint open neighborhoods. More formally, let \((M,d)\) be metric space. Then for any \(x,y \in M\) with \(x \neq y\), there exist open sets \(U, V\) such that \(x \in U\), \(y \in V\), and \(U \cap V = \emptyset\).

**Proof.** Let \(\epsilon = d(x,y) > 0\). Then \(B_{\epsilon/2}(x) \cap B_{\epsilon/2}(y) = \emptyset\); indeed, if \(z\) lies in the intersection, then \(\epsilon = d(x,y) \leq d(x,z) + d(z,y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\), a contradiction. Thus \(U = B_{\epsilon/2}(x)\), \(V = B_{\epsilon/2}(y)\) works. \(\square\)

Let \(X\) be a compact subset of a metric space. We show that \(X^c\) is open. Let \(y \in X^c\). For each \(x \in X\), by the Proposition there exist disjoint open neighborhoods \(U_x\) of \(x\) and \(V_x\) of \(y\). Since \(\bigcup_{x \in X} U_x \supseteq X\), by compactness there exists a finite subcover \(U_{x_1} \cup \cdots \cup U_{x_n} \supseteq X\). The corresponding finite intersection \(V := V_{x_1} \cap \cdots \cap V_{x_n}\) is an open neighborhood of \(y\). We claim that \(V \subseteq X^c\). Indeed, any \(x \in X\) lies in some \(U_{x_i}\), and since \(U_{x_i} \cap V_{x_i} = \emptyset\), this implies \(x \notin V\), so \(x \notin V\).

[Unimportant remark: The Proposition says that metric spaces are Hausdorff. The proof above shows more generally that compact implies closed in a Hausdorff topological space.]

1. (i) Let \(U_t = f^{-1}((\infty, t))\). Every \(x \in [a, b]\) lies in say \(U_{f(x)+1}\), so \([a, b] = \bigcup_{t \in \mathbb{R}} U_t\). By compactness, \([a, b] = U_{t_1} \cup \cdots \cup U_{t_n}\) for some \(t_1, \ldots, t_n \in \mathbb{R}\). Then in fact \([a, b] = U_{\max\{t_1, \ldots, t_n\}}\), i.e. \(f\) is bounded by \(\max\{t_1, \ldots, t_n\}\).

   (ii) By (i), \(T := \sup_{x \in [a,b]} f(x) < \infty\). Suppose towards a contradiction that \(f(x) < T\) for all \(x \in [a,b]\). Then \(g : [a, b] \to \mathbb{R}, g(x) = \frac{1}{T - f(x)}\) is defined and positive everywhere. We claim that \(g((\infty, t))\) is open in \([a,b]\) for all \(t \in \mathbb{R}\). Indeed, if \(t \leq 0\), then \(g^{-1}((\infty, t)) = \emptyset\). If \(t > 0\), then \(g(x) < t\) iff \(\frac{1}{t} < T - f(x)\) iff \(f(x) < T - \frac{1}{t}\), so \(g^{-1}((\infty, t)) = f^{-1}((\infty, T - \frac{1}{t}))\). Hence by (i) applied to \(g\), \(g\) is bounded above; say \(g(x) < S\) for all \(x \in [a,b]\), where \(S > 0\). But then \(f(x) < T - \frac{1}{S}\) for all \(x \in [a,b]\), contradicting \(t = \sup_{x \in [a,b]} f(x)\).

   (iii) \(f : [a, b] \to \mathbb{R}\) defined by

   \[
   f(x) = \begin{cases} 
   0 & \text{if } x \in [a, b) \\
   1 & \text{if } x = b
   \end{cases}
   \]

   is not continuous at \(b\), but \(f^{-1}((\infty, t)) = \emptyset\), \([a,b]\), or \([a,b]\), all open in \([a,b]\).