Math 177: Homework 1 Solutions

Spring 2013

JP 20.10 Let $a_n = (-1)^{n+1}$. Then $\{a_n\}$ is not Cauchy (take say $\epsilon = \frac{1}{2}$), so does not converge (Theorem 19.1). [Alternatively, note that $\liminf a_n = -1 \neq 1 = \limsup a_n$ and use Theorem 20.4.] Let $s_n = a_1 + \cdots + a_n$. Then $\frac{s_n}{n} = \frac{1}{n}$ if $n$ is odd and 0 if $n$ is even, so $0 \leq \frac{s_n}{n} \leq \frac{1}{n}$ for all $n$. Taking $n \to \infty$ yields $\lim s_n = 0$ (Theorem 14.3 (Squeeze Theorem)).

JP 23.1 Let $s_n = a_1 + \cdots + a_n$ and $t_n = b_1 + \cdots + b_n$. Then $s_n - t_n$ and $cs_n$ are the partial sums of the series $\sum (a_n - b_n)$ and $\sum ca_n$, respectively. The conclusion follows since $\lim (s_n - t_n) = L + M$ (Theorem 12.2) and $\lim cs_n = cL$ (Theorem 12.3).

JP 23.4 Suppose otherwise, so $\sum (a_n + b_n)$ converges. Then applying Theorem 23.1 to $\sum (a_n + b_n)$ and $\sum a_n$, $\sum ((a_n + b_n) - a_n) = \sum b_n$ converges, a contradiction.

JP 24.5 Fix $K \geq 1$. Since all terms are nonnegative, $\sum_{k=1}^{K} a_{nk} \leq \sum_{n=1}^{N} a_n$ for all $N \geq n_k$. Taking $N \to \infty$ yields $\sum_{k=1}^{K} a_{nk} \leq \sum_{n=1}^{\infty} a_n$. This holds for all $K$, so the series $\sum_{k=1}^{\infty} a_{nk}$ has nonnegative terms and bounded partial sums, hence converges by Theorem 24.1. Taking $K \to \infty$ in the last inequality yields $\sum_{n=1}^{\infty} a_{nk} \leq \sum_{n=1}^{\infty} a_n$.

JP 26.9 Since $0 \leq a_n \leq 9$,

$$0 \leq \sum_{n=1}^{N} \frac{a_n}{10^n} \leq \sum_{n=1}^{N} \frac{9}{10^n} = 0. \underbrace{9\cdots 9}_{N \text{ digits}} < 1$$

for any $N \geq 1$. In particular, the partial sums are bounded above. Since all terms are nonnegative, the series converges by Theorem 24.1. Taking $N \to \infty$ in the inequality above yields $0 \leq \sum_{n=1}^{\infty} a_n \frac{1}{10^n} \leq 1$.

If $a_n = 9$ for all $n$, then $\sum_{n=1}^{\infty} \frac{9}{10^n} = 1$. Indeed, $1 - \sum_{n=1}^{N} \frac{9}{10^n} = 10^{-N} \to 0$ as $N \to \infty$.

JP 27.1 (a) Apply the Ratio Test:

$$\lim_{n \to \infty} \left| \frac{x^{2(n+1)-1}/(2(n+1)-1)!}{x^{2n-1}/(2n-1)!} \right| = \lim_{n \to \infty} \frac{x^2}{(2n+1)(2n)} = 0 < 1$$

for all $x \in \mathbb{R}$, so the series converges everywhere absolutely, i.e. $R = \infty$. 

1
1. Let $a_n^+ = \max(a_n, 0)$ and $a_n^- = -\min(a_n, 0)$, so $a_n = a_n^+ - a_n^-$ with $a_n^+, a_n^- \geq 0$. Since $a_n^+, a_n^- \leq |a_n|$ and $\sum |a_n|$ converges, $\sum a_n^+$ and $\sum a_n^-$ converge by Theorem 26.3 (Comparison Test). Then by Theorem 23.1,

$$\sum a_n = \sum (a_n^+ - a_n^-) = \sum a_n^+ - \sum a_n^-.$$ 

By the case for non-negative terms, $\sum |a_{f(n)}| = \sum |a_n| < \infty$, so the same argument repeated for $\sum a_{f(n)}$ shows that

$$\sum a_{f(n)} = \sum (a_{f(n)}^+ - a_{f(n)}^-) = \sum a_{f(n)}^+ - \sum a_{f(n)}^-.$$ 

Now,

$$\sum a_n = \sum a_n^+ - \sum a_n^- = \sum a_{f(n)}^+ - \sum a_{f(n)}^- = \sum a_{f(n)}^-,$$

where the middle equality holds by the case for non-negative terms applied to each of $\sum a_n^+$ and $\sum a_n^-$. 

2. For any finite subset $F \subset Y$,

$$\sum_{y \in F} b(y) = \sum_{y \in F} a(f(y)) = \sum_{x \in f(F)} a(x) \leq \sum_{x \in X} a(x),$$

where the second equality uses $f$ injective, and the inequality holds by definition since $f(F) \subset X$ is finite. Thus $\sum_{y \in F} b(y) \leq \sum_{x \in X} a(x)$. Taking sup over $F$ yields $\sum_{y \in Y} b(y) \leq \sum_{x \in X} a(x)$. The inequality may not hold if $f$ is not injective. E.g. Let $Y = \{p, q\}$, $X = \{r\}$, $f : Y \to X$ the unique map, and $a(r) = 1$. Then $b(p) = b(q) = 1$, so $\sum_{y \in Y} b(y) = 1 + 1$ while $\sum_{x \in X} a(x) = 1$.

3. We first show that $\sum_{x \in X} a(x) \leq \sum_{y \in Y} b(y)$. Fix $F \subset X$ finite. Partitioning $x \in F$ according to the image $p(x) = y \in Y$,

$$\sum_{x \in F} a(x) = \sum_{y \in p(F)} \sum_{x \in F_y} a(x),$$

where $F_y := F \cap p^{-1}(y) \subset p^{-1}(y)$. Since $F_y$ and $p(F)$ are finite, $\sum_{x \in F_y} a(x) \leq \sum_{x \in p^{-1}(y)} a(x)$ and $\sum_{y \in p(F)} b(y) \leq \sum_{y \in Y} b(y)$ by definition, so

$$\sum_{y \in p(F)} \sum_{x \in F_y} a(x) \leq \sum_{y \in p(F)} \sum_{x \in p^{-1}(y)} a(x) = \sum_{y \in p(F)} b(y) \leq \sum_{y \in Y} b(y).$$
Thus $\sum_{x \in F} a(x) \leq \sum_{y \in Y} b(y)$. Taking sup over $F$ yields $\sum_{x \in X} a(x) \leq \sum_{y \in Y} b(y)$. 

For the reverse inequality, fix $G \subset Y$ finite and $\epsilon > 0$. We will show that $\sum_{y \in G} b(y) < \sum_{x \in X} a(x) + \epsilon$; then, taking sup over $G$ yields $\sum_{y \in Y} b(y) \leq \sum_{x \in X} a(x) + \epsilon$, and since $\epsilon$ is arbitrary, it follows that $\sum_{y \in Y} b(y) \leq \sum_{x \in X} a(x)$. For each $y \in G$, by the definition of $b(y)$ there exists $F_y \subset p^{-1}(y)$ finite with

$$b(y) < \sum_{x \in F_y} a(x) + \frac{\epsilon}{|G|}.$$ 

Then

$$\sum_{y \in G} b(y) < \sum_{y \in G} \left( \sum_{x \in F_y} a(x) + \frac{\epsilon}{|G|} \right) = \sum_{y \in G} \sum_{x \in F_y} a(x) + \epsilon \leq \sum_{y \in G} \left( \sum_{x \in X} a(x) + \epsilon \right),$$

as desired; (1) holds since $F_y \subset p^{-1}(y)$ are disjoint, and (2) holds by definition since $\bigcup_{y \in G} F_y$ is finite.

4. Let $M = \sum_{x \in X} a(x)$ and, for $n \geq 1$, $A_n = \{ x \in X : a(x) > \frac{1}{n} \}$. For any finite subset $F \subset A_n$,

$$M \geq \sum_{x \in F} a(x) > \frac{1}{n} |F|,$$

so $|F| < nM$. Hence $A_n$ is finite for any $n \geq 1$. Now, $\{ x \in X : a(x) > 0 \}$ is $\bigcup_{n=1}^\infty A_n$, a countable union of finite sets, hence countable.