Math 173, Mid-term, October 31, 2005

Due Monday Nov. 7, 11:00AM

Name: ________________________________

Acknowledgement and acceptance of honor code:

Signature: ________________________________

All problems (labelled with letters below) have equal weight.

   (b) The book, exercise 2, section 9, page 78.
   (c) Let $A$ be a $2 \times 2$ matrix. Let $B$ be the $4 \times 2$ matrix $B = \begin{pmatrix} I_2 \\ A \end{pmatrix}$. Prove that $V(B) = (1 + \text{Tr}(A^tA) + \det^2(A))^\frac{1}{2}$.
   (Here, $\text{tr}$ denotes the transpose of a matrix, and $\text{Tr}$ denotes the trace of a square matrix, i.e. the sum of the diagonal entries: $\text{Tr}(M) = \sum_{i=1}^{n} M_{ii}$ if $M$ is an $n \times n$ matrix).

2. Let $A \subseteq \mathbb{R}^{n-1}$ be open. Let $p \in \mathbb{R}^n$ be a point with positive last coordinate $p_n$. Let $i : \mathbb{R}^{n-1} \to \mathbb{R}^n$ be the map given by $i(x) = (x, 0)$, and let $f : A \times (0, 1) \to \mathbb{R}^n$ be the map given by
   
   $f(x, t) = (1-t)i(x) + tp$

   (a) Let $B \subseteq \mathbb{R}^n$ be the image of $f$. Prove that $\det(Df(x, t)) = (1-t)^{n-1}p_n$ and that $B \subseteq \mathbb{R}^n$ is open. (Hint: Use the result of exercise 5 in section 8)
   (b) Prove that $f : A \times (0, 1) \to B$ is a diffeomorphism.
   (c) Now assume that $v(A) < \infty$. Prove that
   
   $v(B) = \frac{p_n}{n} v(A)$
   
   (d) Recall that the centroid of $A$ is the point $c(A) = (c_1(A), \ldots, c_{n-1}(A)) \in \mathbb{R}^{n-1}$ with coordinates
   
   $c_i(A) = \frac{1}{v(A)} \int_A \pi_i,$
   
   where $\pi_i : A \to \mathbb{R}$ is the projection to the $i$th coordinate. Define $c(B) \in \mathbb{R}^n$ similarly. Show that $c(B)$ lies on the line segment joining $i(c(A))$ and $p$. Express it in terms of $c(A)$ and $p$. 
3. Let $A \subseteq \mathbb{R}^n$ be a subset. Recall that a map $f : A \rightarrow \mathbb{R}^m$ is $C^r$ if there exists a $C^r$ function $\tilde{f} : U \rightarrow \mathbb{R}^m$ with $U \supseteq A$ open and $\tilde{f}|_A = f$.

(a) Prove that if $A \subseteq \mathbb{R}^n$ is closed and $f : A \rightarrow \mathbb{R}^m$ is $C^r$, then there exists a $C^r$ function $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that $g|_A = f$. (Hint: Partition of unity).

(b) Prove that the conclusion in (a) fails if we remove the assumption that $A$ is closed (Hint: Try $n = m = 1$, $A = \mathbb{R} - \{0\}$ and a suitable $f : A \rightarrow \mathbb{R}$).

Let us say that a map $f : A \rightarrow \mathbb{R}^m$ is a diffeomorphism if it is bijective and both $f$ and $f^{-1}$ are $C^r$.

(c) Prove that two linear maps $A, B \subseteq \mathbb{R}^n$ be closed. Is it true that any diffeomorphism $f : A \rightarrow B$ can be extended to a diffeomorphism $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$? (Hint: Consider the case $n = 1, A = B = \{0, 1, 2\}$).

4. Let us say that two linear maps $T_0 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $T_1 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are transverse if any $v \in \mathbb{R}^n$ can be written as $v = T_0 v_0 + T_1 v_1$ for some $v_0 \in \mathbb{R}^n$ and $v_1 \in \mathbb{R}^n$. Let us say that two $C^r$ maps $f_0 : U_0 \rightarrow \mathbb{R}^n$ and $f_1 : U_1 \rightarrow \mathbb{R}^n$, with $U_0 \subseteq \mathbb{R}^n$ and $U_1 \subseteq \mathbb{R}^n$ open, are transverse if the linear maps $Df_0(x)$ and $Df_1(y)$ are transverse for all $x \in U_0$ and $y \in U_1$ with $f_0(x) = f_1(y)$.

(a) Let $\alpha : \mathbb{R} \rightarrow \mathbb{R}^2$ be given by $\alpha(t) = (\cos(t), \sin(t))$, and for each $r \in \mathbb{R}$, let $f_r : \mathbb{R} \rightarrow \mathbb{R}^2$ be given by $f_r(t) = (t, r)$. Prove that $\alpha$ and $f_r$ are transverse if and only if $r \neq \pm 1$.

(b) Let $f_0 : U_0 \rightarrow \mathbb{R}^n$ and $f_1 : U_1 \rightarrow \mathbb{R}^n$ be transverse. Prove that the set

$$M = \{(x, y) \in U_0 \times U_1 \mid f_0(x) = f_1(y)\}$$

is a manifold without boundary. (Hint: Let $g(x, y) = f_0(x) - f_1(y)$ and prove that $0 \in \mathbb{R}^n$ is a regular value of $g : U_0 \times U_1 \rightarrow \mathbb{R}^n$)

Now let $M \subseteq \mathbb{R}^m$ be a manifold without boundary (not the same $M$ as in (b)!). Let $W \subseteq \mathbb{R}^n$ be open, and let $f : W \rightarrow \mathbb{R}^m$ be $C^r$, $r \geq 1$. Say that $f$ is transverse to $M$ if $f$ is transverse to all coordinate patches on $M$.

(c) Prove that $f$ is transverse to $M$ if for any point $p \in M$ there exists a coordinate patch $\alpha : U \rightarrow V$ around $p$ such that $f$ and $\alpha$ are transverse.

(d) Prove that if $M \subseteq \mathbb{R}^m$ is a manifold without boundary and $f : W \rightarrow \mathbb{R}^m$ is transverse to $M$, then $f^{-1}(M)$ is a manifold without boundary.