Math 173, Final Exam, March 14/17, 2005

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Acknowledgement and acceptance of honor code:

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This problem set consists of three pages. All problems, labelled with letters a, b, c, . . . below, have equal weight.

1a Determine the signs of the following permutations

(i) \((\sigma(1), \sigma(2), \sigma(3), \sigma(4), \sigma(5)) = (1, 2, 3, 5, 4)\)
(ii) \((\sigma(1), \sigma(2), \sigma(3), \sigma(4), \sigma(5)) = (5, 2, 1, 4, 3)\)
(iii) \((\sigma(1), \sigma(2), \sigma(3), \sigma(4), \sigma(5)) = (5, 2, 1, 3, 4)\)

1b Which of the following equations will define a 2-tensor \(f\) on \(V = \mathbb{R}^4\)?

(i) \(f(x, y) = x_1x_2 + y_1y_2\)
(ii) \(f(x, y) = x_1y_1 + x_2y_4 - 27x_3y_3\)
(iii) \(f(x, y) = x_1 + y_3x_3 + (x_3)^2\)
(iv) \(f(x, y) = \det(\varphi(x), \varphi(y))\) for any linear map \(\varphi : \mathbb{R}^4 \rightarrow \mathbb{R}^2\),

and which will define an alternating tensor?

1c Define \(f \in \mathcal{L}^2(\mathbb{R}^5)\) by

\[f(x, y) = (x_1 + x_2)y_3\]

(i) Write \(f\) as a sum of elementary 2-tensors.
(ii) Write \(f \otimes f\) as a sum of elementary tensors.

1d Let \(f\) be as in exercise 1c. Let

\[g(x, y) = f(x, y) - f(y, x)\]

(i) Write \(g\) as a sum of elementary alternating tensors.
(ii) Write \(g \wedge g\) as a sum of elementary alternating tensors.
1e Let $\omega_1$ and $\omega_2$ be the differential forms on $\mathbb{R}^4$ defined by

\[
\omega_1 = x_1 \, dx_2 + x_2 \, dx_3 - x_4^2 \, dx_4 \\
\omega_2 = x_1 \, dx_2 \wedge dx_4 + e^{x_2} \, dx_1 \wedge dx_3
\]

(i) Calculate $\omega_1 \wedge \omega_2$, $\omega_2 \wedge \omega_2$ and $d(\omega_1 \wedge \omega_2)$.

(ii) For $p = (0, 1, 2, 3) \in \mathbb{R}^4$ and vectors

\[
v_1 = (p; e_1 + e_2) \\
v_2 = (p; e_2 - e_3) \\
v_3 = (p; e_4)
\]

in $T_p\mathbb{R}^4$, calculate $(\omega_1 \wedge \omega_2)_p(v_1, v_2, v_3)$ (in the notation of the book, this would be $(\omega_1 \wedge \omega_2)(p)(v_1, v_2, v_3)$).

1f Define $\alpha : \mathbb{R} \to \mathbb{R}^2$ by $\alpha(t) = (\cos t, \sin t)$

(i) Calculate $\alpha^*(y \, dx)$

(ii) Calculate $\int_{S^1} y \, dx$
Let $M \subseteq \mathbb{R}^4$ be defined by

$$M = \{ x \in \mathbb{R}^4 \mid x_1^2 + x_2^2 = 1 = x_3^2 + x_4^2 \}$$

and let $\beta : \mathbb{R}^2 \rightarrow M$ be defined by

$$\beta(s, t) = (\cos s, \sin s, \cos t, \sin t)$$

For $p = (s_0, t_0) \in \mathbb{R}^2$, let $U_p = (s_0 - \pi, s_0 + \pi) \times (t_0 - \pi, t_0 + \pi) \subseteq \mathbb{R}^2$, let $V_p = \beta(U_p) \subseteq M$, and let $\alpha_p : U_p \rightarrow V_p$ be given by $\alpha_p(s, t) = \beta(s, t)$. It can be shown that $\alpha_p$ is a coordinate patch on $M$ for all $p$, and thus that $M$ is a 2-manifold without boundary. You may use that without proof in the following exercises.

(a) Prove that the $\alpha_p$’s overlap positively and thus define an orientation on $M$.

Now let $p = (0, 0)$ and let $U = U_p = (-\pi, \pi) \times (-\pi, \pi)$ and $\alpha = \alpha_p : U \rightarrow M$. Let $A = (\mathbb{R}^2 - \{0\}) \times (\mathbb{R}^2 - \{0\}) \subseteq \mathbb{R}^4$ and define $\omega, \tau \in \Omega^1(A)$ by

$$\omega = \frac{x_1 dx_2 - x_2 dx_1}{x_1^2 + x_2^2}, \quad \tau = \frac{x_3 dx_4 - x_4 dx_3}{x_3^2 + x_4^2}$$

(b) (i) Calculate $\alpha^* \omega$ and $\alpha^* \tau$.

(ii) Prove that $\int_M \omega \wedge \tau = 4\pi^2$. [Hint: You may use that $K = M - \alpha(U)$ has measure 0 in $M$. Apply Theorem 35.2]

(c) Does there exist an oriented compact 3-manifold $W \subseteq A$ with $\partial W = M$?

(d) (i) Define $f : \mathbb{R}^2 - \{0\} \rightarrow A$ by $f(x, y) = (x, y, 1, 0)$. Calculate $f^* \omega$ and $f^* \tau$.

(ii) Define $F : A \rightarrow A$ by $F(x, y) = (y, x)$, where $x, y \in \mathbb{R}^2 - \{0\}$. Prove that $F$ is not $C^\infty$ homotopic to the identity map.