Math 113 midterm, 2/7/11, 7pm-9pm.

Open books, open notes. No calculators. No computers, cell phones, or other internet capable devices.

1. (10p) Let \( V \) and \( W \) be finite dimensional vector spaces. Prove that there exists an injective linear map \( T : V \to W \) if and only if \( \dim(V) \leq \dim(W) \).

2. (10p) Let \( T \in \mathcal{L}(V) \).
   (i) Prove that if \( T^2 = I \), then \( V = \text{Null}(T + I) \oplus \text{Null}(T - I) \).
   (ii) Prove that if \( V = \text{Null}(T + I) + \text{Null}(T - I) \), then \( T^2 = I \).

3. (10p) Let \( T \in \mathcal{L}({\mathbb{C}}^2) \) be the map given by \( T(x,y) = (2x-y,x) \).
   (i) Prove that 1 is an eigenvalue of \( T \) and find a basis for \( \text{Null}(T - I) \).
   (ii) Does \( T \) have other eigenvalues? (as usual, you must prove your answer).

4. (10p) Let \( T \in \mathcal{L}(V) \) and \( p \in \mathcal{P}(\mathbb{F}) \).
   (i) Prove that if \( \lambda \) is an eigenvalue of \( T \), then \( p(\lambda) \) is an eigenvalue of \( p(T) \).
   (ii) Now assume \( p(T) = 0 \). Prove that all eigenvalues of \( T \) are roots of \( p \).

5. (10p) Let \( T \in \mathcal{L}(\mathbb{P}_2(\mathbb{R})) \) be the linear map given by \( (Tp)(t) = p(t+1) \).
   (i) Find the matrix of \( T \) with respect to the basis \( \{1, t, t^2\} \) of \( \mathbb{P}_2(\mathbb{R}) \).
   (ii) Let \( A \) denote the matrix of \( T \) from (i). Prove that there exists a matrix \( B \in \text{Mat}(3,3,\mathbb{R}) \) such that \( AB = BA = M(I) \).

6. (20p) Let \( V \) be finite dimensional and let \( T \in \mathcal{L}(V) \). For \( k \geq 0 \), set \( U_k = \text{Null}(T^k) \).
   (i) Prove that \( U_k \subseteq U_{k+1} \) for all \( k \geq 0 \).
   (ii) Prove that \( U_k \) is invariant under \( T \) for all \( k \geq 0 \).
   (iii) Let \( n = \dim(V) \). Prove that if \( U_n \neq V \), then \( U_{k-1} = U_k \) for some \( k \leq n \).
   (iv) Prove that if \( U_{k-1} = U_k \), then \( U_k = U_{k+1} \).
   (v) Prove that if \( U_n \neq V \), then \( U_k \neq V \) for all \( k \geq n \).
   (vi) Deduce that if \( T^k = 0 \) for some \( k \), then \( T^n = 0 \) for \( n = \dim(V) \).