Homework due 1/12: 1.3, 1.4, 1.11, 1.12, 1.14, and the following exercises.

1. Let $X$ be a set and write $\mathbb{F}^X$ for the set of functions $X \to \mathbb{F}$. Define addition by $(f + g)(x) = f(x) + g(x)$ and define scalar multiplication by $(af)(x) = a(f(x))$ for $f, g \in \mathbb{F}^X$ and $a \in \mathbb{F}$. Prove that $\mathbb{F}^X$ is a vector space with these operations. (This was asserted in class.)

2. Prove that the only subspaces of $\mathbb{F}^1$ are $\{0\}$ and $\mathbb{F}^1$.

3. Let $U$ denote the set of functions $f : \mathbb{Z} \to \mathbb{F}$ for which there exists an $N \in \mathbb{Z}$ such that $f(t) = 0$ for $t \geq N$. Prove that $U$ is a subspace of the space of functions.

4. For each of the following statements, either prove it or give a counterexample.

(a) Let $U_1$, $U_2$ and $U_3$ be subspaces of $V$. Then their sum is direct if $U_1 \cap U_2 \cap U_3 = \{0\}$.

(b) Let $U_1$, $U_2$ and $U_4$ be subspaces of $V$. Then their sum is direct if $U_1 \cap U_2 = U_1 \cap U_3 = U_2 \cap U_3 = \{0\}$.

5. Let $\mathcal{U}$ be a collection of subspaces of $V$. (I.e. $\mathcal{U}$ is a set, all of whose elements are subspaces of $V$.) Do not assume that $\mathcal{U}$ is finite. Recall that the intersection of $\mathcal{U}$ is defined by

$$\bigcap \mathcal{U} = \{v \in V \mid v \in U \text{ for all } U \in \mathcal{U}\}.$$  

(a) Prove that $\bigcap \mathcal{U}$ is a subspace of $V$.

(b) Prove that if $W_1, \ldots, W_m$ are subspaces of $V$, and $\mathcal{U}$ is the collection

$$\mathcal{U} = \{U \subseteq V \mid U \text{ is a subspace, and } W_i \subseteq U \text{ for all } i = 1, 2, \ldots, m\},$$

then $\bigcap \mathcal{U} = W_1 + \cdots + W_m$.

(Hint for (b): First prove $\bigcap \mathcal{U} \subseteq W_1 + \cdots + W_m$, then prove $\bigcap \mathcal{U} \supseteq W_1 + \cdots + W_m$.)