1. (20p) Let $V$ be a complex vector space of dimension 3, and let $T \in \mathcal{L}(V)$ be an operator whose matrix with respect to some basis $B$ is

$$M(T, B) = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

(i) Find all eigenvalues of $T$ and their multiplicities.

(ii) Prove that there exists another basis $B'$ such $M(T, B')$ is diagonal.

Solution.

(i) By Proposition 5.18 the eigenvalues are the diagonal entries 1, 4, 6. The multiplicity of each is the number of times it appears on the diagonal, which is 1 for each.

(ii) Since $T$ has $3 = \dim(V)$ distinct eigenvalues, it is diagonalizable by Proposition 5.20.

2. (30p) Let $V$ be a finite dimensional complex inner-product space.

(i) Prove that if $T, S_1, S_2 \in \mathcal{L}(V)$ satisfy $T = S_1 + iS_2$ and $S_1, S_2$ are self-adjoint, then $2S_1 = T + T^*$ and $2iS_2 = T - T^*$.

(ii) Prove that for any $T \in \mathcal{L}(V)$ there are unique self-adjoint operators $S_1, S_2 \in \mathcal{L}(V)$ such that $T = S_1 + iS_2$.

(iii) Prove that $T \in \mathcal{L}(V)$ is normal if and only if the operators $S_1, S_2$ from the previous question commute.

Solution.

(i) By self-adjointness, $S_1^* = S_1$ and $(iS_2)^* = -iS_2$. Then $T + T^* = (S_1 + iS_2) + (S_1 + iS_2)^* = S_1 + iS_2 + S_1^* + (iS_2)^* = 2S_1 + iS_2 - iS_2 = 2S_1$. Similarly, $T - T^* = (S_1 + iS_2) - (S_1 + iS_2)^* = S_1 + iS_2 - S_1^* - (iS_2)^* = iS_2 + iS_2 = 2iS_2$. These calculations (and the ones in the following questions) use the properties on page 119 in Axler.
(ii) Uniqueness follows from (i). For existence, if we let \( S_1 = (T + T^*)/2 \) and \( S_2 = (T - T^*)/(2i) \) then \( S_1 + iS_2 = T \). To see self-adjointness, \( S_1^* = (T + T^*)^*/2 = (T^* + T)/2 = S_1 \) and \( S_2^* = (T - T^*)^*/(-2i) = -(T^* - T)/(2i) = S_2 \).

(iii) Assume \( T = S_1 + iS_2 \) with \( S_1 \) and \( S_2 \) commuting self-adjoint operators. Then \( T^*T = (S_1 + iS_2)^*(S_1 + iS_2) = (S_1 - iS_2)(S_1 + iS_2) = S_1^2 + S_2^2 \), using commutativity in the last step. Applying the same calculation to \( T = S_1 - iS_2 \) gives \( T^*T = S_1^2 + (-S_2)^2 = TT^* \) so \( T \) is normal. For the converse, suppose \( S_1 \) and \( S_2 \) are self-adjoint operators and \( T = S_1 + iS_2 \) is normal. Then \( T^*T = (S_1 + iS_2)^*(S_1 + iS_2) = (S_1 - iS_2)(S_1 + iS_2) = S_1^2 + S_2^2 + i(S_1S_2 - S_2S_1) \), whereas \( TT^* = S_1^2 + S_2^2 - i(S_1S_2 - S_2S_1) \). Since \( T \) is normal, these are equal, and we get \( 2i(S_1S_2 - S_2S_1) = 0 \), whence \( S_1 \) and \( S_2 \) commute.

3. (20p)

(i) Let \( V \) be a four-dimensional complex vector space, and let \( T \in \mathcal{L}(V) \) be an operator with eigenvalues 1, 2, and 3 (and no other eigenvalues). Prove that
\[
((T - 1)(T - 2)(T - 3))^2 = 0.
\]

(ii) Let \( V \) be a finite dimensional complex vector space and let \( T \in \mathcal{L}(V) \) be an operator. Let \( p \in \mathcal{P}(\mathbb{C}) \) be a polynomial such that \( p(\lambda) = 0 \) for all eigenvalues \( \lambda \) of \( T \). Prove that \( p(T) \) is nilpotent.

Solution.

(i) For \( \lambda = 1, 2, 3 \), let \( U_\lambda = \text{Null}(T - \lambda I)^4 \) be the corresponding generalized eigenspace. We proved in class that \( V = U_1 \oplus U_2 \oplus U_3 \), so \( 4 = \dim(U_1) + \dim(U_2) + \dim(U_3) \). Each \( U_\lambda \) contains an eigenvector, so it has dimension at least 1. If some \( U_\lambda \) had dimension \( \geq 3 \), then we’d have \( \dim(U_1) + \dim(U_2) + \dim(U_3) \geq 5 \); the contradiction proves \( \dim(U_\lambda) \leq 2 \). Since \( (T - \lambda I)|_{U_\lambda}^2 = \mathbf{0} \) and \( \dim(U_\lambda) \leq 2 \) we get \( (T - \lambda I)|_{U_\lambda}^2 = \mathbf{0} \), by Proposition 8.6. This implies that the operator
\[
S = ((T - 1)(T - 2)(T - 3))^2
\]
has \( Sv = 0 \) if \( v \) is in one of the \( U_\lambda \) (because in that case \( Sv = S|_{U_\lambda}v = 0 \)). But the \( U_\lambda \) are the generalized eigenspaces, so \( V = U_1 \oplus U_2 \oplus U_3 \) and hence any vector is a sum of vectors in some \( U_\lambda \). By linearity of \( S \) we get \( Sv = 0 \) for all \( v \in V \).

(ii) I claim that if \( n = \dim(V) \) then \( (p(T))^n = 0 \). Let \( \lambda \) be an eigenvalue. Since \( p(\lambda) = 0 \) we can write \( p(z) = (z - \lambda)q(z) \) for some polynomial \( q \).
Then \( p(T)^n = (q(T))^{n}(T - \lambda I)^n \), so if \( v \in U_\lambda = \text{Null}((T - \lambda I)^n) \), then 
\[
(p(T))^n v = (q(T))^{n}(T - \lambda I)^n v = 0.
\]
Thus \( (p(T))^n v = 0 \) for any generalized eigenvector \( v \), and as before this implies \( p(T)v = 0 \) for any \( v \).

[Both these questions could also be solved using Cayley–Hamilton.]

4. \((40p + 10p)\) Let \( V \) be a finite dimensional complex vector space. Assume \( \dim(V) > 0 \).

(i) Let \( q \in \mathcal{P}(\mathbb{C}) \) be any polynomial with \( q(0) \neq 0 \). Prove that there exist \( p \in \mathcal{P}(\mathbb{C}) \) and \( a \in \mathbb{C} \) such that
\[
1 + aq(z) = zp(z).
\]

(ii) Prove that if \( q_T \) is the characteristic polynomial of an invertible operator \( T \in \mathcal{L}(V) \), then \( q_T(0) \neq 0 \).

(iii) Prove that if \( T \in \mathcal{L}(V) \) is an invertible operator, then there exists a polynomial \( p \in \mathcal{P}(\mathbb{C}) \) such that \( p(T) = T^{-1} \).

(iv) Does there exist a polynomial \( p \in \mathcal{P}(\mathbb{C}) \) such that for all invertible operators \( T \in \mathcal{L}(V) \), \( p(T) = T^{-1} \)?

(v) \((\text{Bonus})\) Now assume \( V \) is an inner-product space and let \( T \in \mathcal{L}(V) \) be a positive operator, with positive square root \( S \). Prove that there exists a polynomial \( p \in \mathcal{P}(\mathbb{C}) \) such that \( p(T) = S \).

Solution.

(i) Let \( q(z) = a_n z^n + \cdots + a_0 \). Then \( q(0) = a_0 \neq 0 \). Then \( a = -1/a_0 \) and 
\[
p(z) = a(a_n z^{n-1} + \cdots + a_1)
\]
works.

(ii) The characteristic polynomial is defined in terms of \( \mathcal{M}(T) \), after picking a basis in which this is an upper triangular matrix. Let the diagonal entries be \( \lambda_1, \ldots, \lambda_n \). We’ve seen that \( T \) is invertible if and only if all \( \lambda_i \) are non-zero (proposition 5.16 in Axler). But since \( q_T(z) = (z - \lambda_1) \cdots (z - \lambda_n) \), we have \( q_T(0) = (-1)^n \lambda_1 \cdots \lambda_n \neq 0 \).

(iii) Let \( q = q_T \) and let \( a \) and \( p \) be as in (i). Then \( I + aq(T) = Tp(T) \).

Cayley–Hamilton implies that \( q(T) = 0 \), so this just says \( Tp(T) = I \), so \( p(T) = T^{-1} \).

(iv) No. Such a \( p \) would have \( Tp(T) = I \) for all \( T \), and in particular \( T = \lambda I \). Since \( (\lambda I)p(\lambda I) = (\lambda p(\lambda))I \), this implies \(zp(z) = 1 \) for all \( z \in \mathbb{C} \setminus \{0\} \), and hence the polynomial \( q(z) = 1 - zq(z) \) has infinitely many roots. Only the zero polynomial has infinitely many roots, since a non-zero polynomial can...
be written \( c(z - \lambda_1) \ldots (z - \lambda_n) \) and has the at most \( n \) roots \( \lambda_1, \ldots, \lambda_n \).

It follows by contradiction that \( z p(z) \) must be the zero polynomial. On the other hand \( q(z) \) is clearly not the zero polynomial, since \( q(0) = 1 \), contradicting the existence of \( p \).

(v) Pick a polynomial \( p \) with real coefficients, such that \( p(\lambda) = \sqrt{\lambda} \) for all eigenvalues \( \lambda \) of \( T \). Then the polynomial \( q(z) = (p(z))^2 \) has \( q(\lambda) = \lambda \) for all eigenvalues. Therefore \( q(T)v = q(\lambda)v = \lambda v \) whenever \( Tv = \lambda v \), i.e. \( q(T) \) acts as the identity on any eigenvector. By linearity it acts as the identity on any linear combination of eigenvectors. By the spectral theorem (\( T \) is positive, hence self-adjoint), any vector can be written as a linear combination of eigenvectors, so \( q(T) \) is the identity. Therefore \( q(T)v = \lambda v \) whenever \( Tv = \lambda v \), i.e. \( q(T) \) acts as the identity on any eigenvector.

To see that it is possible to choose such a polynomial \( p \) with this property, let \( \lambda_1, \ldots, \lambda_m \) be the distinct eigenvalues. We prove by induction that there exists \( p_i \) with \( p_i(\lambda_j) = \sqrt{\lambda_j} \) for \( j \leq i \). We can set \( p_1(z) = \sqrt{\lambda_1} \), and inductively

\[
p_i(z) = (p_{i-1}(z)) + \frac{(z - \lambda_1) \ldots (z - \lambda_{i-1})}{(\lambda_i - \lambda_1) \ldots (\lambda_i - \lambda_{i-1})}(\sqrt{\lambda_i} - p_{i-1}(\lambda_i)).
\]

This new \( p_i \) is clearly a polynomial and has \( p_i(\lambda_j) = p_{i-1}(\lambda_j) = \sqrt{\lambda_j} \) for \( j < i \) since the second term is 0 for such \( j \), and \( p_i(\lambda_i) = \sqrt{\lambda_i} \). Then \( p = p_m \) has the desired property.

5. (15p) In this problem (only!) you need not justify answers. All vector spaces named \( V \) are assumed finite dimensional. True or false:

(a) If \( V \) is a complex vector space and \( T \in \mathcal{L}(V) \) has characteristic polynomial \( q(z) = z^2 - 3z + 2 \) then there must exist a basis consisting of eigenvectors of \( T \).

(b) If \( V \) is a complex vector space and \( T \in \mathcal{L}(V) \) has characteristic polynomial \( q(z) = z^4 - z^2 \), then \( \dim(\text{Null}(T^2)) \) must be 2.

(c) If \( V \) is a complex vector space and all eigenvalues of \( T \in \mathcal{L}(V) \) are real numbers, then there exists an inner product on \( V \) such that \( T \) is self-adjoint.

(d) If \( V \) is a complex inner-product space and \( T \in \mathcal{L}(V) \) is self-adjoint, then \( T^2 \) is positive.

(e) If \( V \) is a complex inner product space and \( T \in \mathcal{L}(V) \) is positive, then there exists a unique positive \( S \in \mathcal{L}(V) \) with \( S^4 = T \).
Solution.

(a) True. [Since \( q(z) = (z - 1)(z - 2) \) has roots 1 and 2, there is a basis for \( V \), such that \( \mathcal{M}(T) \) is upper triangular and the diagonal entries are 1 and 2 which are therefore the eigenvalues. Since \( \dim(V) = \deg(p) = 2 \) and \( T \) has 2 distinct eigenvalues it must be diagonalizable.]

(b) True. \[ q(z) = z^2(z-1)^2, \] so by the same reasoning as before, the eigenvalues are 0 and 1, both with multiplicity 2. Therefore the generalized eigenspace \( U = \text{Null}(T^4) \) has dimension 2, and hence \( (T|_U)^2 = 0 \), so \( U \subseteq \text{Null}(T^2) \). But clearly also \( \text{Null}(T^2) \subseteq \text{Null}(T^4) = U \), so they are equal.

(c) False. [Example: \( V = \mathbb{C}^2 \), \( T(x, y) = (y, 0) \) has \( T^2 = 0 \) so the only eigenvalue is 0, which is real, but since \( \dim(\text{Null}(T)) = 1 \) the operator is not diagonalizable, and hence not self-adjoint, by the spectral theorem.]

(d) True. [We saw in class that \( T^*T \) is positive for any operator, but if \( T \) is self-adjoint, then \( T^*T = T^2 \).]

(e) True. [We saw there exists positive \( R \) with \( R^2 = T \). Applying that again, there exists positive \( S \) with \( S^2 = R \) but then \( S^4 = T \). For uniqueness, suppose \( S_1^4 = S_2^4 = T \) for positive \( S_1, S_2 \). Then \( S_1^2 \) and \( S_2^2 \) are positive square roots of \( T \), so by uniqueness must be equal: \( S_1^2 = S_2^2 = R \). But then \( S_1 \) and \( S_2 \) are positive square roots of \( R \) so by uniqueness are equal.]