1. (12p) In this problem (only!) you need not justify answers. True or false:

(a) If $U$ and $W$ are subspaces of a finite dimensional vector space $V$ and $\dim(U) + \dim(W) > \dim(V)$, then $U \cap W \neq \{0\}$.

(b) If $U, W \subseteq V$ are subspaces of a finite dimensional vector space $V$ and $\dim(U) + \dim(W) \leq \dim(V)$, then $U \cap W = \{0\}$.

(c) If $X$ is a finite set with $n$ elements and $V$ a finite dimensional vector space with $\dim(V) = m$, then $\dim(V^X) = m + n$. [$V^X$ is the vector space of all functions from $X$ to $V$, as in HW1.]

(d) If $V$ is a vector space, $T \in \mathcal{L}(V)$ is invertible, and $S_1, S_2 \in \mathcal{L}(V)$ are such that $S_1TS_2 = 0$, then $S_1S_2 = 0$.

2. (10p) Let $V$ be any vector space over a field $\mathbb{F}$, and let $T \in \mathcal{L}(V)$. Prove that if there exist vectors $v_1, \ldots, v_n \in V$ such that $V = \text{span}(Tv_1, \ldots, Tv_n)$, then $T$ is invertible.

3. (20p) Let $V$ and $W$ be finite dimensional vector spaces over a field $\mathbb{F}$. Let $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(W, V)$ and assume $TS = I$.

   (i) Prove that $\dim(W) \geq \dim(V)$.

   (ii) Prove that $ST = I$ if and only if $\dim(W) = \dim(V)$.

4. (20p) Let $V$ be a finite dimensional complex vector space and let $n = \dim(V)$. Let $T \in \mathcal{L}(V)$.

   (i) Prove that the list $I, T, T^2, \ldots, T^n \in \mathcal{L}(V)$ is linearly dependent.

   (ii) Deduce that there exists a monic polynomial $p \in \mathcal{P}(\mathbb{C})$ such that $p(T) = 0$ and $\deg(p) \leq n^2$.

5. (40p + 10p) Let $T \in \mathcal{L}(\mathcal{P}_3(\mathbb{C}))$ be the operator given by $(Tp)(t) = p(2t + 1)$.

   (i) Prove that 1 is an eigenvalue of $T$ and find a basis for $\text{Null}(T - I)$.

   (ii) Prove that 2, 4, and 8 are also eigenvalues of $T$.

   (iii) Prove that 1, 2, 4, and 8 are the only eigenvalues of $T$.

   (iv) Does there exist a basis of $\mathcal{P}_3(\mathbb{C})$ such that $M(T)$ is diagonal?

   (v) (Bonus problem) For each $a \in \mathbb{C}$, define an operator $T_a \in \mathcal{L}(\mathcal{P}_3(\mathbb{C}))$ by the formula $(T_ap)(t) = p(at + 1)$. For which $a \in \mathbb{C}$ does $\mathcal{P}_3(\mathbb{C})$ have a basis consisting of eigenvectors of $T_a$?