Math 111, Take-home mid-term, due 2/12, 4PM
Open book, open notes. All problems have equal weight.

Name: ________________________________

Acknowledgement and acceptance of honor code:

Signature: ________________________________

1. Let \( k = \mathbb{F}_3 = \{0, 1, 2\} \). Equip \( k \) with addition and multiplication given by the following tables:

\[
\begin{array}{c|ccc}
+ & 0 & 1 & 2 \\
\hline
0 & 0 & 1 & 2 \\
1 & 1 & 2 & 0 \\
2 & 2 & 0 & 1 \\
\end{array}
\quad
\begin{array}{c|ccc}
\cdot & 0 & 1 & 2 \\
\hline
0 & 0 & 0 & 0 \\
1 & 0 & 1 & 2 \\
2 & 0 & 2 & 1 \\
\end{array}
\]

(a) Explain why \( k \) is a field. (You need not check the associative and distributive properties, but you should verify the existence of identities and inverses, both additive and multiplicative).

(b) Let \( f = x^6 - 1 \in k[x] \) and \( g = x^5 + x^3 + x^2 + x + 2 \in k[x] \). Compute \( \gcd(f, g) \).

Do this by hand.

(c) Check your answer to (b) with a computer. How does your answer change if we instead work with \( k = \mathbb{Q} \)?

2. 2.3.2(a) from the book.

3. 2.6.2 from the book.

4. Exercise 1.5.11 from the book.

5. (a) Let \( k = \mathbb{R} \) and consider the subset \( V = k^2 - \{(0, 0)\} \) (the plane except the origin). Prove that \( V \) is not an affine variety. (Hint: Indirect proof. Assume \( V = V(f_1, \ldots, f_s) \) and use continuity of the \( f_i \) to prove \( (0, 0) \in V \)).

(b) Now let \( k = \mathbb{F}_2 \). Is \( V = k^2 - \{(0, 0)\} \) an affine variety?

6. Let \( k \) be a field and \( V \subseteq k^n \) a variety. Prove that \( V = V(I(V)) \) in the following two steps.

(a) Assume \( a = (a_1, \ldots, a_n) \in V \). Prove that \( a \in V(I(V)) \).

(b) Assume \( a \in V(I(V)) \). Prove that \( a \in V \). (Hint: this step uses that \( V \) is a variety, i.e. \( V = V(f_1, \ldots, f_s) \) for some polynomials \( f_i \). Proving \( a \in V \) now amounts to proving \( f_i(a) = 0 \).)

7. Let \( A \) be a \( k \times n \) matrix with entries in \( \mathbb{N} \), of rank \( n \). Let \( \prec \) be a monomial
order on $\mathbb{N}^k$. Define a relation $\prec_A$ on $\mathbb{N}^n$ by the requirement

$$\alpha \prec_A \beta \iff A\alpha \prec A\beta.$$ 

(a) Prove that $\prec_A$ is a monomial ordering. (Does your proof use that $A$ has rank $n$?)

(b) Find an $(n + 1) \times n$ matrix $A$ with entries in $\mathbb{N}$ such that for $\alpha, \beta \in \mathbb{N}^n$ we have

$$\alpha \prec_{\text{grlex}} \beta \iff A\alpha \prec_{\text{lex}} A\beta.$$ 

Deduce that $\prec_{\text{grlex}}$ is a monomial order.

(c) Repeat (b) for $\prec_{\text{grevlex}}$.

8. 2.4.1 from the book.

9. 2.5.13 from the book.

10. 2.5.15 from the book.