Math 42 — Midterm 2 — Solutions

1. A cargo plane flies at height h above the ground with speed v. At time $t = 0$ the cargo doors open and an item is pushed out of the cargo bay, whence it plummets to the ground. Let x denote a coordinate in the direction of motion of the plane and let z denote the height coordinate, with the ground located at $z = 0$. Assume that the trajectory of the item satisfies the differential equation

$$z'' = -g$$
$$x'' = 0$$

with initial conditions $x(0) = 0$, $x'(0) = v$, $z(0) = h$ and $z'(0) = 0$.

To answer these questions, we need the solution of the differential equation, which is

$$x(t) = vt \qquad \text{and} \qquad z(t) = -\frac{g}{2}t^2 + h\,.$$

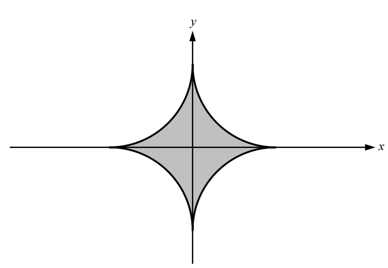
The item hits the ground at time $t = T$ satisfying $z(T) = 0$ or else $T = \sqrt{2h/g}$. The location where the item hits the ground is the point $(x(T), 0)$ where $x(T) = v\sqrt{2h/g}$. The velocity of impact is $(x'(T), z'(T)) = (v, -\sqrt{2hg})$. (The speed is $\sqrt{v^2 + 2gh}$ which is also an acceptable answer.)

2. Find the length of the curve $y = \frac{1}{6}(x^2 + 4)^{3/2}$ for $x \in [0, 3]$. (Hint: $4 + 4y + y^2 = (y + 2)^2$.)

The first thing to do is to write this in parametric form: $x(t) = t$ and $y(t) = \frac{1}{6}(x^2 + 4)^{3/2}$. Now we can compute the derivatives $x'(t) = 1$ and $y'(t) = \frac{1}{2}x(x^2 + 4)^{1/2}$. Now we insert this into the arc length formula

$$\mathit{Length} = \int_0^3 \sqrt{1 + \tfrac{1}{4}x^2(x^2 + 4)}\,dx = \frac{1}{2}\int_0^3 \sqrt{4 + 4x^2 + x^4}\,dx = \frac{1}{2}\int_0^3 (x^2 + 2)dx = \frac{7}{2}$$

3. The height of a monument is h. The shape of its horizontal cross-section at height z is given in the following picture, where each of the four curves is one quarter of a circle of radius $h - z$. Find the volume of the monument.



We need the surface area of the cross section at height z. This is equal to four times the surface are under the inverted circular arc in the first quadrant, by symmetry. The equation for the circle to which this circular arc belongs is $(x-(h-z))^2+(y-(h-z))^2=(h-z)^2$ since the center of this circle is at the point $(h-z,h-z)$. Thus we have

$$y=(h-z)-\sqrt{(h-z)^2-(x-(h-z))^2}$$

with $x\in[0,h-z]$ describing the circular arc in question. The area under this arc is

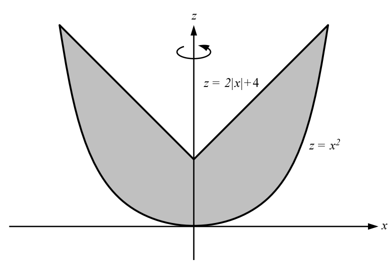
$$\begin{aligned} Area(z) &= \int_0^{h-z} \left((h-z)-\sqrt{(h-z)^2-(x-(h-z))^2}\right)dx \\ &= (h-z)^2\int_{-1}^{0}\left(1-\sqrt{1-w^2}\right)dw \\ &= (h-z)^2\left(1-\frac{\pi}{4}\right) \end{aligned}$$

using the change of variables $w=(x-(h-z))/(h-z)$ and standard integrals. Another way of arriving at this formula is to note the that the shaded region in question is obtained by removing a circle of radius $h-z$ from square of side lengths $2(h-z)$ and then re-arranging the pieces.

Now we can find the volume of the monument by integrating

$$Vol=\int_0^h Area(z)dz=\left(1-\frac{\pi}{4}\right)\int_0^h(z-h)^2dz=\left(1-\frac{\pi}{4}\right)\frac{h^3}{3}.$$

4. Let us generate a solid of revolution by rotating the shaded region pictured below about the z axis.



(a) Write down (but do not evaluate) an integral for the volume of the solid using the method of washers/disks.

(b) Write down (but do not evaluate) an integral for the volume of the solid using the method of shells.

The coordinates of the point of intersection of the two boundary curves is $(1+\sqrt{5},6+2\sqrt{5})$.

$$Vol\ by\ washers=\int_0^4\pi z dz+\int_4^{6+2\sqrt{5}}\pi\left(z-\frac{(z-4)^2}{4}\right)dz$$

$$Vol\ by\ shells=\int_0^{1+\sqrt{5}}2\pi x(2x+4-x^2)dx$$

5. Find all solutions of the ordinary differential equation

$$y'''(t) = \cos(t)\,.$$

Let $u = y''$. Then $u' = \cos(t)$ so by integration and the FTC we have

$$u(T) - u(0) = \sin(T) \qquad \Leftrightarrow \qquad y''(T) = y''(0) + \sin(T)\,.$$

Now let $v = y'$. Then $v' = y''(0) + \sin(t)$ so that by integration and the FTC we have

$$v(T) - v(0) = -\cos(T) + y''(0)T \qquad \Leftrightarrow \qquad y'(T) = y'(0) + y''(0)T - \cos(T)\,.$$

Now we integrate one more time and apply the FTC to get

$$y(T) = y(0) + y'(0)T + \frac{1}{2}y''(0)T^2 - \sin(T)\,.$$

Thus all solutions of this ODE are obtained by choosing the quantities $y(0), y'(0), y''(0)$ arbitrarily.

6. Find a solution of the ordinary differential equation

$$y'(t) = \frac{te^{[y(t)]^2}}{y(t)}$$

satisfying the initial condition $y(0) = 1$.

This is a separable equation. Thus we must integrate

$$\int_0^T y(t)e^{-[y(t)]^2}y'(t)dt = \int_0^T tdt \qquad \text{or} \qquad \int_{y(0)}^{y(T)} ue^{-u^2}du = \frac{T^2}{2}$$

after making the change of variables $u = y(t)$ and $du = y'(t)dt$. By standard integrals, we get

$$\frac{T^2}{2} = -\frac{1}{2}e^{-u^2}\Big|_{y(0)}^{y(T)} = \frac{1}{2}e^{-[y(0)]^2} - \frac{1}{2}e^{-[y(T)]^2}\,.$$

To finish, we isolate $y(T)$, which is

$$y(T) = \sqrt{-\ln\left(e^{-[y(0)]^2} - T^2\right)}\,.$$