Stanford University Department of Mathematics

Math 42 — Final Exam

Examiner: Adrian Butscher

Date: 17 March 2008

Duration: 180 minutes

Family Name: _____

Given Name(s): _____

Student Number: _____

The time of your discussion section: _____

The name of your TA: _____

Jarod Alper Dung-Huang Nguyen Yu-Jong Tzeng Ian Weiner

Your signature: _____

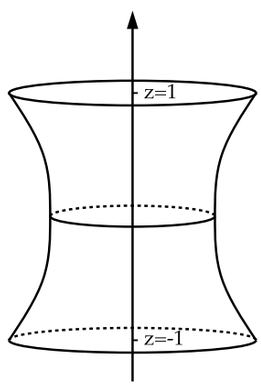
DO NOT OPEN THIS TEST UNTIL INSTRUCTED TO DO SO.

Instructions:

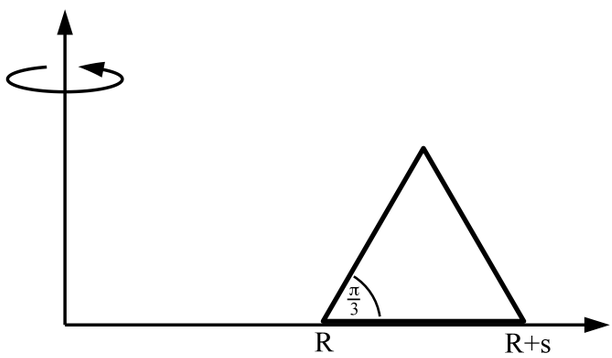
- Your signature above indicates that you have abided by the Stanford Honour Code while writing this test.
- All questions have equal value (20 points). There are seven questions.
- You may quote theorems from your textbook if you make an appropriate reference.
- Show all your work.
- No electronic devices of any kind (e.g. calculators, cell-phones) are allowed.

Question	Marks
1	
2	
3	
4	
5	
6	
7	
Total (140 points)	

1. The *hyperboloid of one sheet* is the surface in $\mathbb{R}^3$ given by the equation $x^2+y^2-z^2=1$. Find the volume of the region contained within this surface and bounded above and below by the planes $z=\pm 1$.



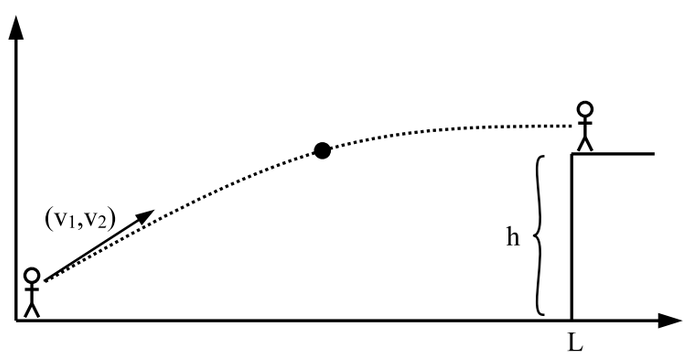
2. Consider an equilateral triangle of side length s whose base is located on the x-axis between $x = R$ and $x = R + s$ as indicated in the figure below. Suppose this triangle is rotated about the y-axis to generate a solid of revolution. Write down the integrals which give the volume of this solid.



3. (a) A person stands at the origin and wants to throw a ball to a friend who is standing a distance L away and at a height h. Suppose the person imparts the initial velocity (v_1, v_2) to the ball and the ball travels subject only to the force of gravity. The differential equation satisfied by the position $(x(t), z(t))$ of the ball is thus

$$x''(t) = 0$$
$$z''(t) = -g\,.$$

Find v_1 and v_2 so that the friend can catch the ball exactly when it is at the point of maximal height on its trajectory, as indicated in the figure below.



(a) Continued.

(b) Write down the integral for the arc length of the ball's trajectory. (For partial credit if you have not found the trajectory of part (a) of this problem: write down the formula for the arc length of the curve $(x(t), z(t))$, for $t \in [0, T]$.)

4. The rate of proliferation of a good idea amongst students in a 100-person calculus class is proportional to the product of the number of people who have heard the idea and the number of people who have not. In other words, if $x(t)$ is the number of people who have heard the idea, then

$$x'(t) = Kx(t)\big(100 - x(t)\big)$$

for some $K > 0$. Suppose that at time $t = 0$, a study group of eight people have a good idea and begin discussing it with their classmates. If by time $t = 4$ exactly half the class has heard of this idea, then when will 90% of the class have heard of it?

5. Consider the following differential equation.

$$x'(t) = -x(t)\big(1 + [x(t)]^2\big)$$

   (a) Draw a slope field for this differential equation in the range $t \in [0, 2]$ and $x \in [0, 2]$. Draw a few sample solution curves, and indicate if there are any equilibrium solutions.

(b) Find a solution of this differential equation having $x(0) = 1$.

6. Do the following series converge or diverge? Prove your assertions.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1+2n^3}{1+n^2+4n^3}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n+1}}{5^{n-1}+2^n}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{1/2}}{n^{2/3}+1}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(3n)!}$

7. A sequence is defined recursively by $a_1 = 1$ and $a_{n+1} = \dfrac{3}{(2/a_n) + 1}$.

   (a) Use induction to prove that $a_n \geq 1$ for all n.

   (b) Show that $a_{n+1} - a_n \leq 0$.

   (c) Why can you conclude that the limit exists? What is the limit of the sequence?

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