Math 42 — Midterm 1 — Solutions

1. Compute the integral $I = \int \frac{dx}{x^{8/15}\left(x^{1/3} + x^{1/5}\right)}$.

   Let $x = u^{15}$ so that $dx = 15u^{14}du$. The integral becomes

   $$I = \int \frac{15u^3 du}{1 + u^2}\,.$$

   Now let $v = 1 + u^2$ so that $dv = 2udu$ and $u^2 = v - 1$. The integral becomes

   $$I = \frac{15}{2}\int \frac{(v-1)dv}{v} = \frac{15}{2}\big(v - \ln(v)\big) + C\,.$$

   After back-substitution we get

   $$I = \frac{15}{2}\left(1 + x^{2/15} - \ln(x^{2/15} + 1)\right) + C\,.$$

2. Compute the integral $I = \int \sin^2(2x)\sin(x)dx$.

   Use the double angle formula for sine:

   $$I = \int (2\sin(x)\cos(x))^2 \sin(x)dx = 4\int \sin^3(x)\cos(x)dx\,.$$

   Now substitute $v = \sin(x)$ so that $dv = \cos(x)dx$. The integral becomes

   $$I = 4\int v^3 dv = v^4 + C = \sin^4(x) + C\,.$$

3. Compute the integral $I = \int \frac{x^3}{(x-1)(x+1)^2} dx$.

First observe that $(x-1)(x+1)^2 = x^3 + x^2 - x - 1$ so that $x^3 = (x-1)(x+1)^2 - x^2 + x + 1$. Hence the integral becomes

$$I = \int dx - \int \frac{x^2 - x - 1}{(x-1)(x+1)^2} dx\,.$$

This requires a "Type 2" partial fraction expansion:

$$\begin{aligned} \frac{x^2-x-1}{(x-1)(x+1)^2} &= \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \\ &= \frac{(A+B)x^2 + (2A+C)x + (A-B-C)}{(x-1)(x+1)^2} \end{aligned}$$

which implies that $A = \frac{-1}{4}$, $B = \frac{5}{4}$, $C = -\frac{1}{2}$. Consequently,

$$I = -\frac{1}{4}\ln(x-1) + \frac{5}{4}\ln(x+1) + \frac{1}{2}\frac{1}{(x+1)^2} + C\,.$$

4. Write down the **form** of the partial fraction expansion (i.e. that which is given to you by the relevant "guiding principle" — don't actually solve for the coefficients in the expansion) for each of the following rational functions.

(a) $\frac{1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$

(b) $\frac{1}{(x^2+1)^2(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} + \frac{E}{x-1} + \frac{F}{(x-1)^2}$

(c) For this fraction, the nature of the quadratics in the denominator is in doubt. For the first one, the discriminant is -4 so it is irreducible. For the second one, the discriminant is 8 so there must be two distinct roots. Indeed, $x^2 + 2x - 3 = (x-1)(x+3)$. Hence

$$\frac{1}{(x^2+2x+2)(x^2+2x-3)} = \frac{Ax+B}{x^2+2x+2} + \frac{C}{x-1} + \frac{D}{x+3}\,.$$

(d) Here we must do some work to factor the denominator (since it must have a root, being of degree three). In fact, we can check that -1 is a root and so $(x+1)$ is a factor. By long division, we get $x^3 + 1 = (x+1)(x^2 - x + 1)$. The quadratic factor is irreducible because its discriminant is -3. Hence we have a "type 2" partial fraction:

$$\frac{1}{x^3+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}\,.$$

5. Define the function $P(x) \equiv \int_0^x e^{-t^2}\,dt$. Express the following quantities in terms of P.

(a) After performing the substitution $s = 2t$ we get

$$\int_{1/2}^{3/2} e^{-4t^2}\,dt = \frac{1}{2}\int_1^3 e^{-s^2}\,ds = \frac{1}{2}\big(P(3) - P(1)\big)\,.$$

(b) The antiderivative of the function $f(x) = e^{-x^2}$ that has the value 1 at $x = 1$ can be found as follows. First, P is an antiderivative of f and it satisfies $P(0) = 0$. All other antiderivatives can be gotten by adding a constant. The antiderivative we want is

$$F(x) \equiv P(x) - P(1) + 1\,.$$

(c) The function $g(x) = 2xe^{-x^4}$ is equal to the derivative of $P(x^2)$ by the chain rule and the Fundamental Theorem of Calculus.

(d) Complete the square:

$$\int_{-1}^{0} e^{-t^2-2t-2}dt = \int_{-1}^{0} e^{-(t+1)^2-1}dt = \frac{1}{e}\int_{-1}^{0} e^{-(t+1)^2}\,.$$

Now make the substitution $s = t+1$ and so $ds = dt$. The new bounds of integration are thus 0 and 1. Thus

$$\int_{-1}^{0} e^{-t^2-2t-2}dt = \frac{1}{e}\int_0^1 e^{-s^2}\,ds = \frac{1}{e}P(1)\,.$$

6. Find the area of the region between the curves $y = -x^2 + a^2$ and $y = x^2 - b^2$.

The points of intersection of the two curves occur when $-x^2 + a^2 = x^2 - b^2$ or at the points $x = \pm\sqrt{\frac{1}{2}(a^2 + b^2)}$. Call these points $\pm A$. Hence

$$\text{Area} = \int_{-A}^{A} (-2x^2 + a^2 + b^2)dx = -\frac{2x^3}{3} + (a^2 + b^2)x\Big|_{-A}^{A} = \frac{8A^3}{3}\,.$$