Math 42 — Midterm 1 — Solutions

1. Find the value of the sum $\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N}\left(1+\frac{k}{N}\right)^5$ by expressing it as a definite integral.

This is the Riemann sum for the function $f(x) = x^5$ between 1 and 2. Hence

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N}\left(1+\frac{k}{N}\right)^5 = \int_1^2 x^5 dx = \frac{x^6}{6}\Big|_1^2 = \frac{63}{6}$$

2. Suppose $\int_0^1 f(x)dx = 1$ and $\int_0^3 f(2x)dx = 4$. Find $\int_0^5 f(x+1)dx$.

First,

$$4 = \int_{x=0}^{x=3} f(2x)dx = \frac{1}{2}\int_{u=0}^{u=6} f(u)du$$

so that $\int_0^6 f(x)dx = 8$. Next,

$$\int_{x=0}^{x=5} f(x+1)dx = \int_{u=1}^{u=6} f(u)du = \int_0^6 f(x)dx - \int_0^1 f(x)dx = 8 - 1 = 7\,.$$

3. Find the derivative of $\int_{x^2}^{x^3} \sin(t)dt$.

First

$$\int_{x^2}^{x^3} \sin(t)dt = \int_0^{x^3} \sin(t)dt - \int_0^{x^2} \sin(t)dt\,.$$

Now define $F(y) \equiv \int_0^y \sin(t)dt$ and by FTC we know that $F'(y) = \sin(y)$. Thus what we have to evaluate is

$$\frac{d}{dx}\Big(F(x^3) - F(x^2)\Big) = 3x^2F'(x^3) - 2xF'(x^2) = 3x^2\sin(x^3) - 2x\sin(x^2)\,.$$

Note that this is the way you would do the question if you had $\sin(t)$ replaced by something impossible to integrate explicitly like $\sin(t^2)$ or e^{-t^2}. This is what I had intended to do! However, given that I used $\sin(t)$, one can solve this problem by first integrating

$$\int_{x^2}^{x^3} \sin(t)dt = -\cos(t)|_{x^2}^{x^3} = -\cos(x^3) + \cos(x^2)$$

and then differentiating. This is much more trivial... And good on you if you took this shortcut.

4. Compute the integral $\int x^3 \sin(x^2) dx$.

$$\begin{aligned}
\int x^3 \sin(x^2) dx &= \frac{1}{2} \int x^3 \sin(x^2) 2dx \\
&= \frac{1}{2} \int u \sin(u) du \\
&= \frac{1}{2} \left(-u \cos(u) + \int \cos(u) du \right) \\
&= \frac{1}{2} \left(-x^2 \cos(x^2) + \sin(x^2) + C \right)
\end{aligned}$$

5. Compute the integral $\int \sin(x) \cos(3x) dx$.

$$\begin{aligned}
\int \sin(x) \cos(3x) dx &= \int \sin(x) \Big(\cos^3(x) - 3 \sin^2(x) \cos(x) \Big) dx \\
&= \int \sin(x) \cos^3(x) dx - 3 \int \sin^3(x) \cos(x) dx \\
&= -\int u^3 du - 3 \int v^3 dv \\
&= -\frac{u^4}{4} - 3\frac{v^4}{4} + C \\
&= -\frac{\cos^4(x)}{4} - \frac{3 \sin^4(x)}{4} + C
\end{aligned}$$

6. Compute the integral $\int \frac{1}{(x-2)(x+4)^2} dx$.

Since the integrand is a rational function of Type II, we know that

$$\frac{1}{(x-2)(x+4)^2} = \frac{A}{x-2} + \frac{B}{x+4} + \frac{C}{(x+4)^2}$$

for some $A, B, C \in \mathbf{R}$. Using the procedure given in class, we must compare the numerators

$$1 = (A+B)x^2 + (8A + 2B + C)x + (16A - 8B - 2C)$$

which gives the system of equations

$$\begin{aligned}
A + B &= 0 \\
8A + 2B + C &= 0 \\
16A - 8B - 2C &= 1\,.
\end{aligned}$$

These have solutions $A = 1/36$, $B = -1/36$ and $C = -1/6$. Consequently,

$$\begin{aligned}\int \frac{1}{(x-2)(x+4)^2}dx &= \frac{1}{36}\int \frac{dx}{x-2} - \frac{1}{36}\int \frac{dx}{x+4} - \frac{1}{6}\int \frac{dx}{(x+4)^2} \\ &= \frac{1}{36}\ln|x-2| - \frac{1}{36}\ln|x+4| + \frac{1}{6}\frac{1}{(x+4)}\end{aligned}$$

Bonus. Compute the integral $\int \frac{1}{(x^2+1)^2}dx$.

Make the substitution $x = \tan(\theta)$ so that $1 + x^2 = \sec^2(\theta)$ and $dx = \sec^2(\theta)d\theta$. We get

$$\begin{aligned}\int \frac{1}{(x^2+1)^2}dx &= \int \frac{\sec^2(\theta)}{\sec^4(\theta)}d\theta \\ &= \int \cos^2(\theta)d\theta \\ &= \frac{\theta}{2} + \frac{\sin(2\theta)}{4} + C \\ &= \tfrac{1}{2}\arctan(x) + \tfrac{1}{2}\sin(\theta)\cos(\theta) + C\end{aligned}$$

Now the equation $x = \tan\theta$ can be represented geometrically by a right triangle having opposite side x, adjacent side 1 and thus hypotenuse $\sqrt{1+x^2}$. Hence $\sin(\theta) = x/\sqrt{1+x^2}$ and $\cos(\theta) = 1/\sqrt{1+x^2}$. This gives a final answer

$$\int \frac{1}{(x^2+1)^2}dx = \frac{1}{2}\arctan(x) + \frac{x}{2(1+x^2)} + C\,.$$