Stanford University Department of Mathematics

Math 42 — Final Exam

Examiner: Adrian Butscher

Date: 10 December 2007

Duration: 180 minutes

FAMILY NAME: ______________________________

GIVEN NAME(S): ______________________________

STUDENT NUMBER: ______________________________

THE TIME OF YOUR DISCUSSION SECTION: ______________________________

THE NAME OF YOUR TA: ______________________________

Theodora Bourni   Ken Chan   Ben Williams

YOUR SIGNATURE: ______________________________

DO NOT OPEN THIS TEST UNTIL INSTRUCTED TO DO SO.

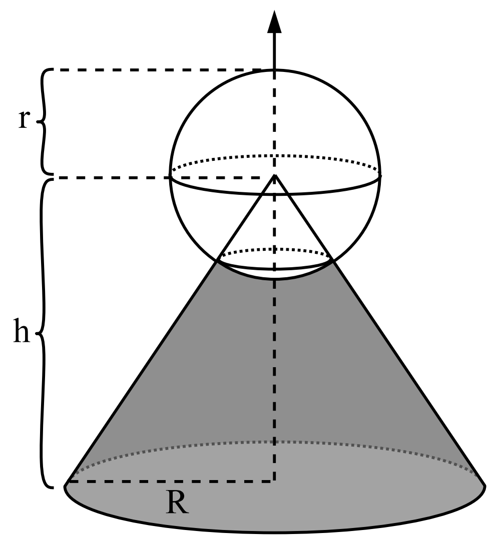
INSTRUCTIONS:

- Your signature above indicates that you have abided by the Stanford Honour Code while writing this test.
- All questions have equal value (20 points). There are eight questions.
- You may quote theorems from your textbook if you make an appropriate reference.
- Show all your work.
- No electronic devices of any kind (e.g. calculators, cell-phones) are allowed.

Question	Marks
1	
2	
3	
4	
5	
6	
7	
8	
Total (160 points)	

1. Prove that the volume of a sphere of radius R is $\frac{4}{3}\pi R^3$.

2. Consider a circular cone of base R and height h, as well as a sphere of radius $r \leq h$ centered at its vertex. Find the volume of the shaded region pictured below.



2. Continued.

3. Do the following series converge or diverge? Prove your assertions.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1+2n^2}{2+n+3n^2}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n+1}}{4^n + 1}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1}$

4. A sequence is defined recursively by $a_1 = 1$ and $a_{n+1} = \frac{1}{3}(a_n + 4)$.

   (a) Show that $a_n < 2$ for all n. (Hint: use induction — i.e. it is clear that $a_1 < 2$, so assume that $a_n < 2$ and deduce that $a_{n+1} < 2$.)

   (b) Show that $a_{n+1} - a_n > 0$. (Hint: the fact that $a_n < 2$ for all n is helpful.)

   (c) Why can you conclude that $\lim_{n\to\infty} a_n$ exists?

5. Solve the initial value problem:

$$\begin{aligned} x'' &= -1 - x' \\ x(0) &= 1 \\ x'(0) &= 0 \end{aligned}$$

6. Consider the ODE $x' = 2t(1 + x^2)$.

   (a) Make a rough sketch of the slope field for this ODE in the region of the plane given by $t > 0$ and $x > 1$.

   (b) Find the exact solution of this initial value problem.

(c) Let $x(t)$ be the solution of the above ODE that satisfies the initial condition $x(0) = 1$. Use Euler's Method with step size $\Delta t = \frac{1}{2}$ to approximate $x(\frac{3}{2})$.

7. Consider a population of wolves and rabbits in an ecosystem. Let $W(t)$ be the number of wolves and $R(t)$ be the number of rabbits at time t. We have decided to model the evolution of the ecosystem according to the equations

$$\frac{dR}{dt} = \left(k - \frac{R}{K} - aW\right) R$$
$$\frac{dW}{dt} = (-r + bR)\, W$$

where a, b, k, r, K are positive parameters.

(a) Explain what features this model is trying to capture.

(b) What are the equilibrium solutions of this system?

(c) Write down an equation for the approximate behaviour of very small deviations from equilibrium.

8. Consider an object of mass $m = 1$ moving horizontally on a surface. The object is attached by a spring (with spring constant $k = 1$) to a wall. Suppose the effects of air resistance are not neglected. If we denote by $x(t)$ the displacement from equilibrium of the mass as a function of time, then $x(t)$ solves the ODE

$$x'' + 2\beta x' + x = 0$$

where β is the coefficient of air resistance. Assume $\beta < 1$.

(a) Show that $x(t) = e^{-\beta t}\cos(\omega t)$ is a solution of this equation, where $\omega = \sqrt{1-\beta^2}$.

(b) The *energy* of an arbitrary solution $x(t)$ is given by

$$E(t) \equiv \frac{1}{2}\left((x')^2 + (x)^2\right) .$$

Show that

$$\frac{d}{dt}E(t) = -2\beta\,(x')^2$$

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