Stanford University Department of Mathematics

Math 42 — Second Midterm

Examiner: Adrian Butscher

Date: 7 November 2007

Duration: 120 minutes

Family Name: ______________________________

Given Name(s): ______________________________

Student Number: ______________________________

The time of your discussion section: ______________

The name of your TA: ______________________________

Theodora Bourni   Ken Chan   Ben Williams

Your signature: ______________________________

DO NOT OPEN THIS TEST UNTIL INSTRUCTED TO DO SO.

Instructions:

- Your signature above indicates that you have abided by the Stanford Honour Code while writing this test.
- All questions have equal value (20 points). There are six questions and a bonus.
- You may quote theorems from your textbook if you make an appropriate reference.
- Show all your work.
- No electronic devices of any kind (e.g. calculators, cell-phones) are allowed.

Question	Marks
1	
2	
3	
4	
5	
6	
Total (120 points)	

1. Find the arc length of the following curve: the graph of the function $f(x) = \frac{1}{2}(e^x + e^{-x})$ for $x \in [0, \ln(2)]$. (Hint: the integrand will simplify in the end to something that does not involve a square root.)

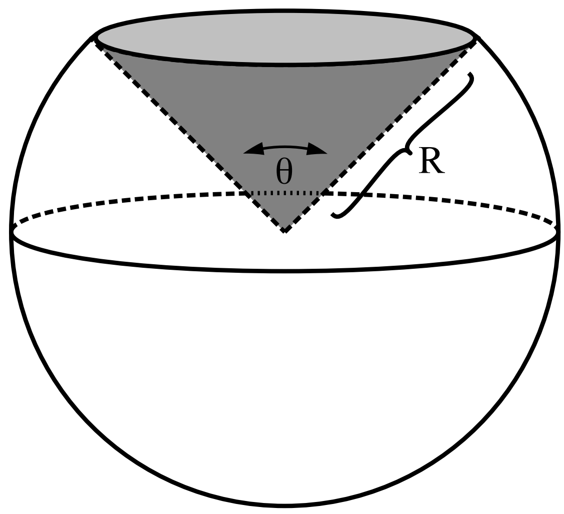
2. An ellipsoid is the surface in $\mathbb{R}^3$ given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

where a, b, c are positive real numbers. Find the volume contained within the ellipsoid. (Hint: note that the cross section at fixed $z \in [-c, c]$ is an ellipse in $\mathbb{R}^2$ whose equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{z^2}{c^2}$. Thus the first thing you should do is find the area of this ellipse.)

2. Continued.

3. Find the volume of what **remains** when a conical hole of central angle θ (the shaded region) has been **removed** from a sphere of radius R as shown.



3. Continued.

4. Use a different method to compute the volume of the solid given in Question 3. (Set up the integral but do not evaluate it... After all, you know what the answer is.)

4. Continued.

5. Consider the differential equation $x' = -xt$.

   (a) Draw a slope field for this differential equation and draw approximate solution curves on your picture starting at initial conditions $x(0) = 2, 1, 0, -1, -2$.

(b) Use Euler's Method with step size $h = \frac{1}{2}$ and initial condition $x(0) = 1$ to find the approximate value of $x(\frac{3}{2})$.

6. The position $x(t)$ of a projectile launched vertically from the surface of the Earth into space subject only to the force of gravity obeys the differential equation

$$m\frac{d^2x}{dt^2} = -\frac{GmM}{(R+x)^2}$$

where m is the mass of the projectile, M is the mass of the Earth, R is the radius of the Earth, and G is the universal constant of gravitation.

(a) Show that the total energy of the projectile, which is given by

$$E \equiv \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 - \frac{GmM}{R+x}, \qquad (*)$$

remains constant throughout the motion. (Hint: show that $\frac{dE}{dt} = 0$.)

(b) Suppose that the initial position and velocity of the projectile are $x(0) = 0$ and $x'(0) = v_0$. Find the maximum height above the Earth which the projectile attains. (Hint: Evaluate $(*)$ at time $t = 0$ and at the time of maximum height.)

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