Exam Review Notes, Review Problems, and Extra Credit Problems

**Extra credit:** After Friday, December 3, any extra credit problems are to be done (on the honor code) by yourself. Do as many problems as you like. Any starred problems from the Bondy-Murty sections indicated, or any problems [2+] or above, are good for extra credit. Also, the extra credit problems from the Midterm and from Homework 8 are still in play.

There are hints/solutions in the various books. Please avoid looking at them unless you must. On the honor code, you must say if you read the hints; then solutions will be good for half credit.

The deadline for the extra credit is Friday, December 10, at 2:30. No exceptions.

**Important disclaimer.** I have listed some topics that I think are important, but I prepared this rather quickly. Omission of a topic here in no way guarantees that it won’t appear on the exam.

1. Eulerian circuits: Understand which graphs have them, and how to prove it. Also understand various basic properties of trees and graphs. Review exercises: VL-W Ch.1, all.

2. Counting trees and Pr"ufer codes: These proofs had an enumerative flavor, so please review them. The second proof should make more sense now!
   
   **Exercise.** Do and generalize Problem 2B, and give two proofs of your answer: one by counting Pr"ufer codes, the other by the enumerative argument discussed in class; “Proof 1” which starts on p. 13 of VL-W.

3. Planarity: Understand how to prove graphs are nonplanar. Know how to prove, use, and apply Euler’s formula.
   
   **Exercises:** Ch. 9.3 of Bondy-Murty, all.

4. Ramsey theory: You should know how to prove Ramsey’s theorem and bound Ramsey numbers.
   
   **Exercises:** Ch. 7.2 of Bondy-Murty, all.

5. Vertex coloring intro. You should understand these types of arguments. You probably don’t need to memorize the theorems but they are useful for producing examples. Exercises: Ch. 8.1 of Bondy-Murty, all.

6. Chromatic polynomials: These are very nice, and you have two ways to compute them. Know both of them. Exercises: 8.4 of Bondy-Murty.

   **Extra credit.** Consider the set $S$ of integers $n$ for which there is some graph $G$ which can be 10-colored in exactly $n$ ways.

   How many elements $n$ of $S$ satisfy $n \leq 10000$? (The answer is much smaller than 10000.) An exact formula would be great, but a nontrivial upper bound would also be an excellent answer.

7. You should know how to prove the five color theorem. See Theorem 9.11 of B-M or Soifer’s book.

   **Extra credit.** Read Ch. 9.4-9.5 of Bondy-Murty and write a page or so explaining what is going on.
8. Turan’s theorem. Know at least one of the proofs, preferably all, and know counterexamples that show that the theorem can’t be sharpened.

**Exercises.** VL-W 4A, B, C, and B-M Ch. 7.3.

9. The marriage theorem. Know it, and try Problem 5A. Also try 5.2.1 of B-M. (Look familiar?) Flows in networks: Know this. Exercise: Look up sample networks in B-M, VL-W, Google, wherever, and find (with proof) maximal flows for them.

10. Picard groups: Understand the theoretical underpinnings of Picard groups, and be able to generalize the construction to abelian varieties over an arbitrary base scheme.

Just kidding, this lecture was mostly fun (and that is even scarier than it sounds). But do the extra credit problem from the last midterm for a good challenge. You don’t need to “know” anything.

11. Intro to enumerative combinatorics. This is foundational material, **know it cold**.

12. Intro to EC2, ditto above.

**Exercises in Stanley:** Here are some recommended exercises (from Chapter 1) covering these and later topics. **Any** exercises labeled [2+] or above from Chapter 1 are good for extra credit.

2, 3, 7, 8, 12, 13, 17, 21, 23-28, 30, 32, 34-37, 41 (if you dare!!), 42-45, 61-66, 68, 71-72, 77-78, 165-166.

If you do 93d, please submit it to the Annals of Mathematics and tell people that I was your thesis advisor.

13. Inclusion-exclusion. Definitely know this. Be able to prove standard problems (such as the derangement problem). Review the homework problems. The exercises in VL-W are kind of obnoxious. Do 10B if you know what $F_p$ is. From Chapter 2 of Stanley, try 3, 4, 8, 10, 19, 20, 1 (?!)

14. More PIE, see above.

15. Recurrences: Know how to solve linear recurrences. (Exercise: Write some down at random and solve them.) Know Stirling numbers and be able to prove the recurrence formulas. Try the problems in VL-W Ch. 13.

16. Exponential generating functions: Know how to multiply and “why we care”. Try 14E, review Example 14.11, review and generalize the homework exercises on this topic.

17. Catalan numbers. Know a proof for the formula, and how to obtain the recurrence relation. Review the generating function proof too. **Exercises:** See the last homework and keep going where you stopped. See also Problem 14B (although you will have to read the discussion beforehand).

**Exercise.** Define $u_n = C_{n+1}$. Now write down the recurrence for $u_n$, and prove an exact formula (without “cheating”, this defeats the point of the exercise) for $u_n$ using the generating function approach.
18. **Partitions.** Know the diagrams, the various bijections, and the generating functions. **Exercise:** Write down the generating function for the number of partitions with (1) all parts even, (2) all parts odd, (3) all parts unequal, (4) all parts divisible by 5, (5) invent your own conditions!

**Extra credit:** Consider the partition function for partitions whose parts are all at least 5 and unequal. Derive an upper bound for the number of such partitions, using the approach given in class or in Theorem 15.7 of VL-W.

19. **Twelvefold way.** This is basically review, know it cold.

20. **WZ method:** Find the book *A = B* online for more.

  Crazy stuff about partitions: If you think this is cool, (then you are right, and) check out Ken Ono’s REU program (Google it) and feel free to ask me for more information.