ADDITIVE COMBINATORICS

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1. INTRODUCTION

This is a sketch for a course on additive combinatorics to be given in Spring 2010. The primary sources will be Sound’s notes (available from his webpage) and the book of Tao and Vu. We will also try to cover one or more recent papers, depending on Sound’s suggestions and the interest of the audience. As of March 2, I have prepared roughly the first five lectures.

2. FOUNDATIONS

We will cover the following topics in some detail.

2.1. Introduction. What is additive combinatorics about? Give some notes from Tao-Vu and Sound’s notes. Also, discuss a little bit about the probabilistic method, and prove a couple of the theorems in Tao-Vu Ch.1.

2.2. Basic Fourier analysis. We will review the basic elements of Fourier analysis over finite fields. This is covered briefly in Sound/Gowers and in more detail in Tao-Vu (Chapter 4). I will plan to give basics somewhat more attention than Sound/Gowers, without taking too much time.

2.3. Roth’s theorem. Sound/Gowers then proceed on to an interesting and short proof of Roth’s theorem based on Fourier analysis, as well as an interesting counterexample of Behrend, and an example that Fourier coefficients do not control four-term APs.

   We will cover all of this. There is additional related material which I may cover in the second half of the course, or may skip.

2.4. Vinogradov’s three-primes theorem. Sound/Gowers spend some effort discussing Vinogradov’s result that all large odd integers are the sum of three primes. We will give their (and Hardy-Littlewood’s) proof, conditional on GRH. This introduces the circle method, which is a nice application of Fourier analytic techniques to additive problems.

   I also plan on one additional lecture on the circle method which briefly explains its other applications: Vinogradov without GRH, partitions, Waring’s problem. One lecture on all three. The details will be omitted.

2.5. Freiman’s theorem. We will present the proof of Freiman’s theorem in some detail, essentially following Sound’s (excellent) notes, and perhaps borrowing from elsewhere as appropriate.

2.6. Gowers’ proof of Szemeredi’s theorem for four term progressions. We will present at least part of this, and perhaps all of it, both for its own sake and as a further window into how the results are further generalized.

3. ADDITIONAL TOPICS

We will cover as many of these and other additional topics as possible, depending on time, student interest, and suggestions from Sound or anyone else.
3.1. **Further proofs of Roth’s theorem.** There is a wealth of interesting mathematics in Chapters 10 and 11 of Tao-Vu. Further proofs of Roth’s theorem are given in Chapter 10.5 and 10.7 (an ergodic theory argument and an elementary argument, respectively). It would be quite interesting to cover these.

In Chapter 11 Szemeredi’s theorem for \( k > 3 \) is discussed. This is related to the discussion at the end of Sound/Gowers, but much more is presented in the Tao-Vu book. It would be quite interesting to cover as many of these topics as time permits.

3.2. **Nilmanifolds (unlimited).** It would be interesting to discuss Green-Tao’s recent work on nilmanifolds. For example, see the following links:


In particular, this leads to a proof of the Möbius and nilsequence conjectures. A brief look at the proof indicates that it generalizes the circle method, so if we go this route, perhaps it is desirable not to skip Vinogradov’s three primes theorem.

One reason *not* to cover this is that apparently Tamar Ziegler might visit in a couple of years, and so students would have the opportunity to learn this material from one of the pioneers of the subject.

3.3. **Dvir’s proof of the finite field Kakeya conjecture (1 day).** This is interesting, and could be covered in a single day. Indeed, the proof is given in its entirety in a blog post of Terry Tao:


3.4. **Croot and Sisask’s approach (2-5+ days).** See here:


Sound recommended this paper, and it looks like an extremely interesting complement to the work above. They present a different approach to convolutions, on arbitrary groups (i.e. not just abelian), based on an elementary, probabilistic approach. (Here “elementary” means “not using Fourier analysis”.) Their approach yields, among other results, non-commutative analogues of results that say that sumsets are structured, and a probabilistic proof of Roth’s theorem.

3.5. **Bateman and Hawk, New bounds on cap sets (2-5 days).** A set \( A \subseteq \mathbb{F}_3^N \) is called a cap set if it contains no lines. This paper proves a bound on the largest possible size of a cap set.

The proof is largely Fourier-analytic, and studies additive structure in the spectrum of cap sets.


3.6. **Comments, questions, errata in Sound’s notes.** P.4, Freiman’s theorem should say “contains in” somewhere.

Normalization of Fourier transform. See MO 55787.

Vinogradov three primes. Several typos (see my notes); how to get good bound on \( \sum_{q>Q} \phi(q)^{-2} \), curious Fourier transform formula (what’s the obvious proof?)

Lemma 1 on p. 39, need some bound on \(|f|\).
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