

# THE ZERO SCALAR CURVATURE YAMABE PROBLEM ON NONCOMPACT MANIFOLDS WITH BOUNDARY

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*Dedicated to the memory of Prof. José Fernando Escobar, Chepe.*

ABSTRACT. Let  $(M^n, g)$ ,  $n \geq 3$  be a noncompact complete Riemannian manifold with compact boundary and  $f$  a smooth function on  $\partial M$ . In this paper we show that for a large class of such manifolds, there exists a metric within the conformal class of  $g$  that is complete, has zero scalar curvature on  $M$  and has mean curvature  $f$  on the boundary.

The problem is equivalent to finding a positive solution to an elliptic equation with a non-linear boundary condition with critical Sobolev exponent.

## 1. INTRODUCTION

The celebrated Riemann Mapping Theorem states that any simply connected region in the plane is conformally diffeomorphic to a disk. This theorem is less successful in higher dimensions since very few domains are conformally diffeomorphic to the ball. Nevertheless, we can still ask whether a manifold with boundary is conformally diffeomorphic to a manifold that resembles the ball, namely to one that has zero scalar curvature and constant mean curvature on its boundary. Escobar studied this problem in [1]. He showed that most compact manifolds with boundary admit such conformally related metrics.

A generalization of this problem is the so-called *prescribed mean curvature problem*: given a manifold with boundary  $(M^n, g)$ ,  $n \geq 3$  and a smooth function  $f : \partial M \rightarrow \mathbb{R}$ ,

**Problem 1.1.** *Does there exist a metric conformally equivalent to  $g$  that is complete, scalar flat and has mean curvature  $f$  on  $\partial M$ ?*

Escobar and Garcia [2] studied this problem on  $(B^3, \delta_{ij})$ . They proved that a Morse function is the mean curvature of a scalar-flat metric  $g \in [\delta_{ij}]$  if it satisfies some Morse inequalities. They paralleled Schoen and Zhang's [6] blow-up analysis for the prescribed scalar curvature problem on  $S^3$ . In both cases, a general solution is unexpected because of the Kazdan-Warner obstruction [3].

In this paper we address Problem 1.1 on a large class of **noncompact** manifolds with boundary  $(M^n, g)$ ,  $n \geq 3$ . As a corollary of Theorem 2.4 about PDEs we get:

**Theorem 1.2.** *Any smooth function  $f$  on  $\partial M$  can be realized as the mean curvature of a complete scalar flat metric conformal to  $g$ .*

In contrast with the compact case, no topological obstructions on  $f$  arise. This is a surprising phenomena.

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## 2. PRELIMINAIRES

Let  $(M^n, g)$ ,  $n \geq 3$  be a complete,  $n$ -dimensional Riemannian manifold with boundary  $\partial M \neq \emptyset$ . Denote by  $\tilde{g} = u^{4/(n-2)}g$  a metric conformally related to  $g$ , where  $u > 0$  is a smooth function.

It is a standard fact that the relation between the scalar curvature  $R(g)$  of the metric  $g$  and the scalar curvature  $R(\tilde{g})$  of the metric  $\tilde{g}$  is given by

$$R(\tilde{g}) = -\frac{4(n-1)}{n-2} \frac{L_g u}{u^{(n+2)/(n-2)}}, \quad (1)$$

where  $L_g = \Delta_g - \frac{n-2}{4(n-1)}R(g)$ , and  $\Delta_g$  is the Laplacian calculated with respect to the metric  $g$ .

The relation between the mean curvature of the boundary  $h(g)$  of the metric  $g$ , and the mean curvature of the boundary  $h(\tilde{g})$  of the metric  $\tilde{g}$  is given by

$$h(\tilde{g}) = \frac{2}{n-2} \frac{B_g u}{u^{n/(n-2)}}, \quad (2)$$

where  $B_g = \frac{\partial}{\partial \eta} + \frac{n-2}{2}h(g)$  and  $\partial/\partial \eta$  is the outward-pointing normal derivative on  $\partial M$  calculated with respect to the metric  $g$ .

**Remark 2.1.** *The exponent  $n/(n-2)$  of equation (2) is called a **critical exponent** since the Sobolev trace embedding  $H^1(M) \hookrightarrow L^q(\partial M)$  ceases to be compact for  $q \geq 2(n-1)/(n-2)$ . This condition rules out the direct method of minimization to prove existence of solutions.*

It follows directly from the above discussion that finding a conformally related metric  $\tilde{g} = u^{4/(n-2)}g$  on  $M$  that is scalar flat (i.e. has zero scalar curvature) and has prescribed mean curvature  $f$  on the boundary is equivalent to finding smooth  $u > 0$  on  $M$  that satisfies equation (1) with  $R(\tilde{g}) \equiv 0$  and equation (2) with  $h(\tilde{g}) \equiv f$ .

In this paper we find such  $u$  for a more general problem, the so-called **super-critical** equation, in which the critical exponent  $n/(n-2)$  of (2) is replaced by an arbitrary number  $\beta > 1$ .

**Definition 2.2.** *Let  $(M^n, g)$  be a complete, noncompact Riemannian manifold. On each end  $E$  of  $M$ , consider the volume of the set obtained by intersecting  $E$  with the geodesic ball of radius  $t$  centered at some fixed  $p \in M$ , and denote it by  $V_E(t)$ . We say that the end  $E$  is **large** if*

$$\int_1^\infty \frac{t}{V_E(t)} dt < \infty.$$

On a large end  $E$  of  $M$ , Theorem 4.3 of Li and Tam's [4] assures the existence of a harmonic, non-negative *barrier* function  $v_E$ , which is asymptotic to 1 and it is exactly zero on the boundary of a large ball intersected with the end.

**Definition 2.3.** We say that  $M$  is **positive** if it has a positive first eigenvalue for the problem

$$\begin{cases} L_g u = 0 & \text{in } M, \\ B_g u = \lambda u & \text{on } \partial M. \end{cases}$$

It is easy to see that  $M$  is positive if and only if it admits a conformally related scalar flat metric with positive mean curvature on the boundary.

We now state our main

**Theorem 2.4.** *Let  $(M, g)$  be a noncompact complete positive Riemannian manifold with compact boundary and finitely many ends, all of them large. Let  $f$  be a smooth function on  $\partial M$  and  $\beta > 1$ . There exists  $\epsilon, \delta > 0$  and a smooth function  $\epsilon \leq u \leq \epsilon + \delta$  on  $M$  with*

$$\begin{cases} L_g u = 0 & \text{in } M, \\ B_g u = \frac{n-2}{2} f u^\beta & \text{on } \partial M. \end{cases} \quad (3)$$

When  $\beta$  is the critical exponent  $n/(n-2)$ ,  $\tilde{g} = u^{4/(n-2)}g$  is a complete, scalar flat metric on  $M$ , with mean curvature  $h(\tilde{g}) \equiv f$ .

**Remark 2.5.** For the  $\beta = n/(n-2)$  case, the bound  $\epsilon \leq u \leq \epsilon + \delta$  guarantees a complete metric  $\tilde{g} = u^{4/(n-2)}g$ .

**Remark 2.6.** As we mentioned before, the assumption that  $(M, g)$  is positive is equivalent to the existence a conformally related metric  $\tilde{g}$  on  $M$  that is scalar-flat and has positive mean curvature on the boundary. We may assume that  $g$  is precisely that metric to get that

$$L_g \equiv \Delta_g.$$

A very important class of examples of positive *noncompact* manifolds with boundary is obtained by removing submanifolds of large codimension out of positive *compact* manifolds with boundary. We refer the reader to the Appendix for more details on the construction.

The proof of Theorem 2.4 is divided in two parts. The first one, inside Section 3, shows that a method of sub- and super-solutions converges when trying to solve equations (3). Later, in Section 4, we construct the appropriate sub- and super-solutions. Theorem 1.2 follows by choosing  $\beta = n/(n-2)$  in Theorem 2.4.

### 3. METHOD OF SUB- AND SUPER-SOLUTIONS

In this section we adapt a method of sub- and super-solutions to our setting. We begin by proving a form of maximum principle.

Let  $u \in C^2(M) \cap C^1(\bar{M})$ , and define the operators

$$\begin{aligned} L_\lambda u &:= \Delta_g u - \lambda u && \text{in } M, \\ B_\gamma u &:= \frac{\partial u}{\partial \eta} + \left(\frac{n-2}{2} h_g + \gamma\right) u && \text{on } \partial M, \end{aligned}$$

for  $\lambda, \gamma > 0$  fixed large numbers.

**Proposition 3.1** (Maximum Principle). *Suppose that  $u \in C^2(M) \cap C^1(\bar{M})$  satisfies  $L_\lambda u \geq 0$ ,  $B_\gamma u \leq 0$ . Then  $u \leq 0$  on  $M$ .*

*Proof.* Put  $w(x) = \max\{0, u(x)\}$ ,  $B_R = \{x : \text{dist}(x, \partial M) \leq R\}$ . Then

$$\begin{aligned} 0 &\leq \int_M (L_\lambda u)w - \int_{\partial M} (B_\gamma u)w \\ &\leq - \int_M \nabla u \cdot \nabla w - \lambda \int_{B_R} uw - \gamma \int_{\partial M} uw \\ &= - \int_M |\nabla w|^2 - \lambda \int_{B_R} w^2 - \gamma \int_{\partial M} w^2. \end{aligned}$$

Hence  $w = 0$ , and so  $u \leq 0$ .  $\square$

**Definition 3.2.** *A sub-solution (resp. super-solution) of equation (3) is a function  $u_-$  (resp.  $u^+$ ) in  $C^2(M) \cap C^1(\bar{M})$  with*

$$\begin{cases} \Delta_g u_- &\geq 0 & \text{in } M, \\ B_g u_- - \frac{n-2}{2} f u_-^\beta &\leq 0 & \text{on } \partial M, \end{cases}$$

respectively

$$\begin{cases} \Delta_g u^+ &\leq 0 & \text{in } M, \\ B_g u^+ - \frac{n-2}{2} f (u^+)^\beta &\geq 0 & \text{on } \partial M. \end{cases}$$

**Theorem 3.3.** *If there exist sub- and super-solutions  $0 \leq u_- \leq u^+ \leq c_0$  for equation (3), then there exists a smooth solution  $u$  with  $u_- \leq u \leq u^+$ .*

*Proof.* Fix  $\lambda, \gamma > 0$  big enough. Let  $u_0 = u^+$ , and define inductively  $u_i \in C^2(M) \cap C^1(\bar{M})$ ,  $i = 1, 2, \dots$ , to be the unique solution to

$$\begin{cases} L_\lambda u_i &= -\lambda u_{i-1} & \text{in } M, \\ B_\gamma u_i &= \frac{n-2}{2} f u_{i-1}^\beta + \gamma u_{i-1} & \text{on } \partial M. \end{cases} \quad (4)$$

*Claim.* We have  $u_- \leq \dots \leq u_i \leq u_{i-1} \leq \dots \leq u^+$ .

To prove the claim, we'll use induction twice. First, to show that the sequence  $\{u_i\}$  is non-increasing and bounded by  $u^+$ . Then, to prove that it is bounded below by  $u_-$ .

To check the first induction step, we see that  $L_\lambda(u_1 - u_0) = (\Delta u_1 - \lambda u_1) - (\Delta u_0 - \lambda u_0) = -\lambda u_0 - \Delta u_0 + \lambda u_0 = -\Delta u_0 \geq 0$ , because  $u_0 = u^+$  is a super-solution.

On the other hand, one has

$$\begin{aligned} B_\gamma(u_1 - u_0) &= \frac{\partial u_1}{\partial \eta} + \left(\frac{n-2}{2} h_g + \gamma\right) u_1 - \frac{\partial u_0}{\partial \eta} - \left(\frac{n-2}{2} h_g + \gamma\right) u_0 \\ &= \frac{n-2}{2} f u_0^\beta + \gamma u_0 - \frac{\partial u_0}{\partial \eta} - \left(\frac{n-2}{2} h_g + \gamma\right) u_0 \\ &= \frac{n-2}{2} f u_0^\beta - \frac{\partial u_0}{\partial \eta} - \frac{n-2}{2} h_g u_0 \\ &\leq 0 \end{aligned}$$

since  $u_0 = u^+$  is a super-solution.

The maximum principle implies  $u_1 \leq u_0$  and the first step of the induction follows.

Assume, by induction, that  $u_i \leq u_{i-1}$ .

Then,  $L_\lambda(u_{i+1} - u_i) = \Delta u_{i+1} - \lambda u_{i+1} - \Delta u_i + \lambda u_i = -\lambda u_i + \lambda u_{i-1} = \lambda(u_{i-1} - u_i) \geq 0$ .

On  $\partial M$ ,

$$\begin{aligned} B_\gamma(u_{i+1} - u_i) &= \frac{n-2}{2} f u_i^\beta + \gamma u_i - \frac{n-2}{2} f u_{i-1}^\beta - \gamma u_{i-1} \\ &= \frac{n-2}{2} f(u_i^\beta - u_{i-1}^\beta) + \gamma(u_i - u_{i-1}). \end{aligned}$$

If  $f$  is nonnegative, then the above quantity is nonpositive by induction hypothesis.

On the other hand, if there exists  $x \in \partial M$  with  $f(x) \leq 0$ , then by choosing  $\gamma > \frac{n-2}{2} \beta \|f\| \|u^+\|_{\partial M}$  we get

$$\frac{n-2}{2} f(u_i^\beta - u_{i-1}^\beta) + \gamma(u_i - u_{i-1}) \leq 0,$$

so the inequality  $B_\gamma(u_{i+1} - u_i) \leq 0$  follows from the fact that

$$u_{i-1}^\beta - u_i^\beta \leq \beta(u^+)^{\beta-1}(u_{i-1} - u_i).$$

This way,  $u_i$  is non-increasing.

We now show that  $u_- \leq u_i$ .

By hypothesis,  $u_- \leq u^+ = u_0$ . Assume, by induction, that  $u_- \leq u_{i-1}$ . Then  $L_\lambda(u_- - u_i) = \Delta u_- - \lambda u_- - \Delta u_i + \lambda u_i = \Delta u_- + \lambda(u_{i-1} - u_-) \geq 0$ , by induction hypothesis and the fact that  $\Delta u_- \geq 0$ .

On  $\partial M$  we have

$$\begin{aligned} B_\lambda(u_- - u_i) &= \frac{\partial u_-}{\partial \eta} + \left(\frac{n-2}{2} h_g + \gamma\right) u_- - \left(\frac{n-2}{2} f u_{i-1}^\beta + \gamma u_{i-1}\right) \\ &= B(u_-) + \frac{n-2}{2} f(u_-^\beta - u_{i-1}^\beta) + \gamma(u_- - u_{i-1}) \\ &\leq \frac{n-2}{2} f(u_-^\beta - u_{i-1}^\beta) + \gamma(u_- - u_{i-1}). \end{aligned}$$

Should  $f$  be positive, this last term would be non-positive by induction hypothesis. Otherwise,  $\gamma > \beta \frac{n-2}{2} \|u^+\|_{\partial B} \|f\|$  guarantees  $B_\lambda(u_- - u_i) \leq 0$  since  $u_k^\beta - u_-^\beta \leq \beta(u^+)^{\beta-1}(u_k - u_-)$  for  $k = 1, \dots, i-1$ . Apply the maximum principle and the claim follows.

The inequality  $u_- \leq u_i \leq u^+$  implies that the sequence  $u_i$  is uniformly bounded. From the first equation in (4) we conclude that  $|\Delta u_i|$  is uniformly bounded as well. Standard elliptic estimates imply that  $\|u_i\|_{2,p}$  is locally uniformly bounded for any  $p > 1$ , and hence the Sobolev embedding implies that there is a local uniform

bound for the sequence  $u_i$  in the  $C^{1,\nu}(\bar{M})$ -norm. Differentiating the first equation in (4) we find that  $|\nabla\Delta u_i|$  is uniformly bounded, and  $L^p$  elliptic estimates imply that  $\|u_i\|_{3,p}$  is locally uniformly bounded for any  $p > n$ . The compactness of the embedding  $H^{3,p}(M) \hookrightarrow C^{2,\nu}(\bar{M})$ ,  $0 < \nu < 1 - \frac{n}{p}$ ,  $p > n$ , guarantees the existence of a subsequence of functions  $u_{i_k}$  converging to a function  $u \in C^{2,\nu}(\bar{M})$ . Because the sequence of functions  $u_i$  is monotone we conclude that the whole sequence converges to  $u$ . It is now straightforward to check that the function  $u$  is a solution to equation (3), and a standard argument shows that  $u \in C^\infty(\bar{M})$ .  $\square$

#### 4. EXISTENCE OF SUB- AND SUPER-SOLUTIONS

We construct an appropriate harmonic function that we will use as a base for our sub- and super-solutions.

**Lemma 4.1.** *There exists  $\mu > 0$ , and a positive smooth function  $\mu \leq v \leq 1 + \mu$  on  $M$ , with*

$$\begin{cases} \Delta_g v = 0 & \text{in } M, \\ B_g v < 0 & \text{on } \partial M, \\ v \sim 1 + \mu & \text{near infinity.} \end{cases}$$

*Proof.* Let  $R > 0$  be large. There always exists a positive solution  $v_R$  of the homogeneous problem

$$(P_R) \quad \begin{cases} \Delta v_R = 0 & \text{in } \{x : d(x, \partial M) < R\}, \\ v_R = 0 & \text{on } \partial M, \\ v_R = 1 & \text{on } \{x : d(x, \partial M) = R\}. \end{cases}$$

A standard argument shows that as  $R_i \rightarrow \infty$ , the sequence  $v_{R_i}$  converges uniformly on compact sets to a harmonic function  $0 \leq v_\infty \leq 1$ .

*Claim.*  $v_\infty \sim 1$  on each end's infinity.

Let  $E$  be an end and  $0 \leq v_E \leq 1$  be a harmonic barrier function that vanishes on the boundary of a large ball intersected with  $E$  and is asymptotic to 1 (see Theorem ??). By the maximum principle,  $v_E$  is smaller or equal than  $v_\infty$ . This way,  $v_\infty$  is non-zero and asymptotic to 1 on all ends.

We get that  $Bv_\infty = \partial/\partial\eta(v_\infty)$ , but  $\partial/\partial\eta(v_\infty) < 0$  by Hopf's principle, since  $v_\infty$  attains its minimum along the boundary (recall that  $\eta$  is the outward-pointing normal of the boundary).

Pick  $\mu > 0$  so that  $v := v_\infty + \mu$  still satisfies  $Bv < 0$ . This way,  $v \geq \mu > 0$  and  $v$  is asymptotic to  $1 + \mu$ , as desired.  $\square$

**Proposition 4.2.** *For appropriately small constants  $\epsilon, \delta > 0$ ,  $u_- := \epsilon v$  is a sub-solution, and  $u^+ := \epsilon v + \delta$  is a super-solution.*

*Proof.* Let  $v$  be as before. Note that, since  $v$  is positive on the boundary, it makes sense to write  $\epsilon v = O(\epsilon)$  on  $\partial M$ . This way, for  $\epsilon, \delta > 0$ ,  $\beta > 1$ , one has

$$(\epsilon v + \delta)^\beta = O(\epsilon^\beta) + O(\delta^\beta) \text{ on } \partial M.$$

By definition,  $u_- \leq u^+$ , and both are harmonic. In order for them to be sub- and super-solutions, we just have to check their behavior on the boundary.

*Claim 1.*  $Bu_- - \frac{n-2}{2}f(u_-)^\beta \leq 0$ .

Recall that by construction,  $Bv < 0$  on the boundary. Hence

$$\begin{aligned} Bu_- - \frac{n-2}{2}f(u_-)^\beta &= \epsilon Bv - \frac{n-2}{2}f(\epsilon v)^\beta \\ &\leq -\epsilon \min_{\partial M} |Bv| + \frac{n-2}{2} \max_{\partial M} |f|(\epsilon v)^\beta \\ &= -O(\epsilon) + O(\epsilon^\beta) \\ &\leq 0 \end{aligned}$$

by taking  $\epsilon > 0$  small enough.

*Claim 2.*  $Bu^+ - \frac{n-2}{2}f(u^+)^\beta \geq 0$ .

We see that

$$\begin{aligned} Bu^+ - \frac{n-2}{2}f(u^+)^\beta &= \epsilon Bv - \frac{n-2}{2}f(\epsilon v + \delta)^\beta \\ &\geq -\epsilon \max_{\partial M} |Bv| + \delta \left( \frac{n-2}{2} h_g \right) \\ &\quad - \frac{n-2}{2} \max_{\partial M} |f|(\epsilon v + \delta)^\beta \\ &= -O(\epsilon) + O(\delta) - O(\epsilon^\beta) - O(\delta^\beta). \end{aligned}$$

The above line can be made nonnegative by choosing  $\epsilon$  smaller than  $\delta$ , and  $\delta$  small (notice the plus sign next to  $O(\delta)$ ).

This way,  $0 < \mu\epsilon \leq u_- \leq u^+ \leq \epsilon + \mu\epsilon + \delta$  are sub- and super-solutions respectively.  $\square$

*Proof of Theorem 2.4.* The existence of  $u$  satisfying (3) is granted by applying the above Proposition 4.2 and Theorem 3.3. For the critical case, i.e.  $\beta = n/(n-2)$ . The completeness of the metric  $\tilde{g} = u^{4/(n-2)}g$  follows from the lower bound  $u \geq u_- \geq \mu\epsilon > 0$ .  $\square$

#### APPENDIX A. CONSTRUCTION OF POSITIVE MANIFOLDS

We show how to construct a large class of noncompact complete positive manifolds with boundary. Basically, these examples come from removing "small" submanifolds from positive *compact* manifolds with boundary.

**Remark A.1.** *One easily sees that any compact manifold with boundary of positive Yamabe constant is positive, since it admits a conformally related metric that is scalar flat and has constant positive mean curvature on the boundary (see [1]).*

Let  $(N^n, \bar{g})$ ,  $n \geq 3$ , be a positive compact manifold with boundary. Consider a collection of submanifolds  $\Sigma = \cup_{i=1}^k \Sigma_i^{n_i}$ , where each  $\Sigma_i$  is a submanifold in the

interior of  $N$  of dimension  $0 \leq n_i \leq \frac{n-2}{2}$ ; put  $\Sigma_i = \{p_i\}$  whenever  $n_i = 0$ .

We will construct a metric  $g = u^{4/(n-2)}\bar{g}$  on  $M = N \setminus \Sigma$ , that is complete, scalar flat and has positive mean curvature on the boundary. Also,  $(M, g)$  will have large ends and will remain positive.

For  $p \in \text{int}(N)$  let  $G_p > 0$  denote the Green's function for the conformal Laplacian on  $(N, \bar{g})$ , which always exists and satisfies  $L_{\bar{g}}G_p = \delta_p$  and  $B_{\bar{g}}G_p = 0$ . This way, for  $c > 0$ ,  $G_p + c$  satisfies

$$L_{\bar{g}}(G_p + c) = \delta_p, \quad B_{\bar{g}}(G_p + c) = c \frac{n-2}{2} h_{\bar{g}} > 0$$

since  $(N, \bar{g})$  is positive.

By a construction on the Appendix of Schoen and Yau's paper [5] which involves the Green's function, one can find, for each  $\Sigma_i$  of positive dimension, positive functions  $G_i$  that are singular on  $\Sigma_i$  and satisfy  $L_{\bar{g}}G_i = 0$  on  $N \setminus \Sigma_i$ .

A simple argument like that of Proposition 4.2 shows that for appropriate coefficients  $a_i > 0, c > 0$  the function

$$u = \sum_{\{i|n_i>0\}} a_i G_i + \sum_{\{i|n_i=0\}} a_i G_{p_i} + c$$

is singular on  $\Sigma$  and satisfies  $L_{\bar{g}}u = 0$  and  $B_{\bar{g}}u > 0$ . Therefore  $g = u^{4/(n-2)}\bar{g}$  remains positive.

The large codimension of the  $\Sigma_i$  guarantees, via the standard argument in [5], that the singularities of  $u$  are strong enough to make  $g = u^{4/(n-2)}\bar{g}$  complete with large ends.

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