RESEARCH STATEMENT - FERNANDO SCHWARTZ

My area of research is the application of techniques of partial differential equations, geometric analysis and calculus of variations to problems in differential geometry.

Specifically, my research focuses on two problems:

1. The Yamabe problem on noncompact manifolds with boundary.
2. The prescribed scalar curvature problem on the sphere, and the prescribed mean curvature problem on the hemisphere.

I addressed Problem 1 when working under the supervision of Prof. Escobar. After he passed away, at the beginning of 2004, I went to Stanford to continue my work under Prof. Schoen. There I finished my answer to Problem 1 and produced [6]. Since then I have been working on an answer to the second problem.

A brief motivation for the above problems together with my main results appear in the Introduction. More details about my current work are in Section 2. I discuss my future research plans in Section 3.

1. Introduction

1.1. **Yamabe problem on noncompact manifolds with boundary.** The well-known Riemann Mapping Theorem states that any simply connected region in the plane is *conformal* to a disk, which is a surface of zero Gauss curvature with constant curvature on the boundary. The generalization of this theorem to higher dimensional manifolds is

Problem Y. *For* $n \geq 3$*, let* (M^n, g) *denote a Riemannian manifold with boundary whose metric is* g*. Does there exist* $\tilde{g}$ *in the conformal class of* g *so that* $(M^n, \tilde{g})$ *has zero scalar curvature and* ∂M *has constant mean curvature?*

More generally, one can study the problem of prescribing *any function* as the mean curvature of the boundary. This is, given a smooth function $f : \partial M \to \mathbb{R}$

Problem Y'. *Does there exist a metric conformally equivalent to* g *that is scalar flat and has mean curvature* f *on* ∂M*?*

In [6] I addressed this last issue. I showed that for a large class of noncompact manifolds with boundary the following holds:

Theorem 1. *Any smooth function on* ∂M *can be realized as the mean curvature of a scalar flat metric conformal to* g*.*

The proof of this theorem follows from a more general PDE theorem that I proved in [6] (see Theorem 2 of Section 2 below).

1.2. **Prescribed scalar curvature.** The following is a natural problem:

Problem P. *Let $(M^n, g), n \geq 3$ be a Riemannian manifold and $K : M \to \mathbb{R}$ smooth. Find a metric $\tilde{g}$ conformal to g with scalar curvature K.*

Let R_g denote the scalar curvature of the metric g. The Kazdan-Warner identity states that for any conformal vector field X on M,

$$\int_M X(R_g) d\mu_g = 0. \tag{1}$$

Any candidate K for scalar curvature on M must itself satisfy equation (1). As long as there are not "too many" conformal vector fields on the manifold, (1) does not pose a serious restriction on K, and the problem of finding a metric with curvature K can be solved using standard methods.

The round sphere is the only manifold for which the Kazdan-Warner identity becomes an issue, since it is the unique compact manifold whose conformal group is noncompact. On the other hand, a consequence of the Yamabe problem is that (S^n, g_0) does not admit a metric of nonpositive scalar curvature. A very challenging case of Problem (P) is:

Problem P'. *Let K be a positive function on S^n. Does there exist $\tilde{g}$ conformal to g_0 with scalar curvature K?*

I believe I will be able to give an answer to this problem in [7], extending the result of Schoen and Zhang in [5] to a nongeneric setting. For more details on my current work see the section below.

2. Background

This section begins with a background that allows a reformulation of Problems (Y') and (P') in terms of PDE. Then, I state a general result that implies Theorem 1 and answers (Y'), followed by a few words on its proof. Later, I mention an interesting construction, and end with a brief discussion of the answer to Problem (P').

From now on $(M^n, g), n \geq 3$ is a Riemannian manifold and $\tilde{g}$ a metric in the conformal class of g. It is customary to write

$$\tilde{g} = u^{4/(n-2)} g,$$

where u is a smooth positive function on M. The scalar curvature $R_{\tilde{g}}$ of $\tilde{g}$ satisfies

$$\Delta_g u - \frac{n-2}{4(n-1)} R_g u + \frac{n-2}{4(n-1)} R_{\tilde{g}} u^{\frac{n+2}{n-2}} = 0, \tag{2}$$

where Δ_g is the Laplace-Beltrami operator in M.

For the mean curvature $h_{\tilde{g}}$ we have

$$\frac{\partial u}{\partial \eta} + \frac{n-2}{n} h_g u - \frac{n-2}{2} h_{\tilde{g}} u^{\frac{n}{n-2}} \quad = \quad 0.$$

Using this we rewrite Problem (Y') as follows:

Problem Y'. *Given $f : \partial M \to \mathbb{R}$, does there exist positive u on M such that*

$$\begin{cases} \Delta_g u - \frac{n-2}{4(n-1)} R_g u &= 0 & \text{in } M \\ \frac{\partial u}{\partial \eta} + \frac{n-2}{2} h_g u &= \frac{n-2}{2} f u^{\frac{n}{n-2}} & \text{on } \partial M? \end{cases}$$

In [6] I addressed this issue. I was able to prove a more general result for a large class of noncompact manifolds with boundary. Namely, let (M^n, g) be a noncompact manifold with compact boundary and large ends, in the sense of Li and Tam [2]. Assume that (M^n, g) admits a conformally related scalar flat metric of positive mean curvature on the boundary. Then the following holds:

Theorem 2. *Let (M^n, g) be as above. Let f be a smooth function on ∂M and $\beta > 1$. There exist $c > 0$ and smooth u with $c \leq u \leq c^{-1}$ and*

$$\begin{cases} \Delta_g u - \frac{n-2}{4(n-1)} R_g u &= 0 & \text{in } M \\ \frac{\partial u}{\partial \eta} + \frac{n-2}{2} h_g u &= \frac{n-2}{2} f u^{\beta} & \text{on } \partial M. \end{cases} \tag{3}$$

The proof of this theorem is based on the existence of sub- and super-solutions for equation (3) and the convergence of approximations. The technical hypotheses guarantee the existence of barrier functions on the ends of M, which are key to finding global sub- and super-solutions.

By solving equation (3) for any $\beta > 1$, I found solutions to the *supercritical* problems, which is an unexpected phenomena since it does not occur in the compact setting. One readily sees that by letting $\beta = n/(n-2)$ in (3) Theorem 1 follows (together with the positive answer to Problem (Y')).

Escobar studied Problem (Y') on *compact* manifolds in [8]. He showed that provided (M^n, g) admits a scalar flat metric with minimal boundary, a smooth function $f : \partial M \to \mathbb{R}$ is realized as the mean curvature on the boundary of a scalar flat metric if and only if f changes sign and has negative integral over the boundary.

In [6] I found a general way to construct manifolds like those appearing in Theorem 2. Let $(M^n, g), n \geq 3$ be a *compact* manifold with boundary that admits a conformally related scalar flat metric with positive mean curvature.

Lemma 3. *Let (M^n, g) be as above. Let $\Sigma = \cup_i \Sigma_i^{n_i}$ be a finite collection of submanifolds of M of dimensions $0 \leq n_i \leq (n-2)/2$. Then there exists a smooth function u, singular on Σ, such that $(M^n \setminus \Sigma, u^{4/(n-2)} g)$ satisfies the hypothesis of Theorem 2.*

Basically, the above u is obtained from considering appropriate Green's functions for the conformal Laplacian of g together with a construction of Schoen and Yau in [4].

Equation (2) together with the fact that the scalar curvature of the unit n-sphere is $n(n-1)$ implies that an equivalent statement of Problem (P') on the 3-sphere is:

Problem P'. *Given positive $K : S^3 \to \mathbb{R}$, find u on S^3 with*

$$\Delta_{g_0} u - \frac{3}{4}u + \frac{1}{8}Ku^p \;=\; 0, \tag{4}$$

where $p = 5$.

Schoen and Zhang studied this problem in [5]. A consequence of their work is the following:

Let K be a positive Morse function with $\Delta K \neq 0$ at its critical points, and so that K is a regular value for the scalar curvature map $g \mapsto R_g$ (a generic K is a regular value of this functional because of the Sard-Smale theorem).

For $\mu = 0, 1, 2$ let D_μ denote the number of critical points of K on S^3 at which $\Delta K < 0$ and at which the Morse index of $-K$ is μ.

Theorem 4 (Schoen and Zhang)**.** *Assume K is as above. Then the number of solutions of equation (4) is bounded below by $N(K)$, where*

$$N(K) = |1 - \sum_{\mu=0}^{2} (-1)^\mu D_\mu|.$$

Schoen and Zhang proved this theorem by studying the subcritical version of equation (4) (i.e. (4) with $p < 5$) and understanding the limiting phenomena $p \to 5$. They showed that the behavior of the subcritical solutions u_p is governed, in the worst case, by simple blow-up as p approaches the critical exponent. Their argument shows that these u_p either form "simple bubbles" at the critical points of K of negative Laplacian and disappear in the limit, or they remain bounded and become actual solutions as $p \to 5$. Then they use Morse inequalities to count the total number of subcritical solutions: those that are lost via blow-up are matched with the critical points of K of negative Laplacian, and the remaining subcritical solutions converge to solutions of the original problem.

My current work [7] aims to remove the genericity assumptions on K, and to provide multiple solutions to equation (4) with $p = 5$. I believe I will be able to prove the following

Conjecture 5. *Let K be a positive Morse function on S^3 so that $\Delta K \neq 0$ at its critical points. If $N(K) > 1$ then equation (4) (with $p = 5$) has at least two solutions.*

Escobar and Garcia [1] use a parallel approach to the above by Schoen and Zhang [5] in order to find scalar flat metrics with prescribed mean curvature on the boundary of the round hemisphere (S^3_+, g_0). They use Morse theory to show existence of solutions which requires genericity assumptions as well.

I believe that my technique will also work to remove the genericity assumptions for the parallel result on the round hemisphere. I believe I will be able to prove the following

Conjecture 6. *Let h be a positive Morse function on ∂S^3_+ so that $\Delta h \neq 0$ at its critical points. If $N(h) > 1$ then there are at least two conformally related scalar flat metrics on S^3_+ of mean curvature h.*

3. Further research

There is a natural way to combine the techniques of Theorem 2 and Lemma 3 in order to approach the general question of prescribed scalar curvature *and* singularities. A similar problem was addressed by Mazzeo and Pacard in [3], where they studied the existence of metrics with constant scalar curvature and prescribed point-singularities in domains of $\mathbb{R}^n$. I intend to further develop the techniques in [6] to work on the following problem:

- Let (M, g) be a compact manifold with boundary and $\{\Sigma_t : t \in I\}$ a family of submanifolds of M that approach the boundary. Study the limiting behavior of scalar flat metrics that are singular along Σ_t and have prescribed mean curvature.

In the near future I also intend to apply the tools I developed in the noncompact setting to study the problem:

- Let (M, g) be a noncompact manifold with compact boundary. Let f be a function on ∂M and λ a constant. Find a metric whose scalar curvature is λ and whose boundary has mean curvature f.

From the proof of Theorem 2 in [6] it is easy to see that there are many solutions of the equation (3). A question that I would like to address is:

- Describe the moduli space of solutions to equation (3).

I am interested in expanding my research into areas of geometry and PDE that I did not cover in this review. In particular I am interested in inverse mean curvature flow, which I believe is key to solving the following two open problems:

- Classify compact manifolds with boundary whose Sobolev quotient is near to that of the hemisphere.

- Prove the Riemannian Penrose inequality on asymptotically hyperbolic manifolds.

I have made some preliminary exploration into the above problems and I hope to continue with those lines of research.

References

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