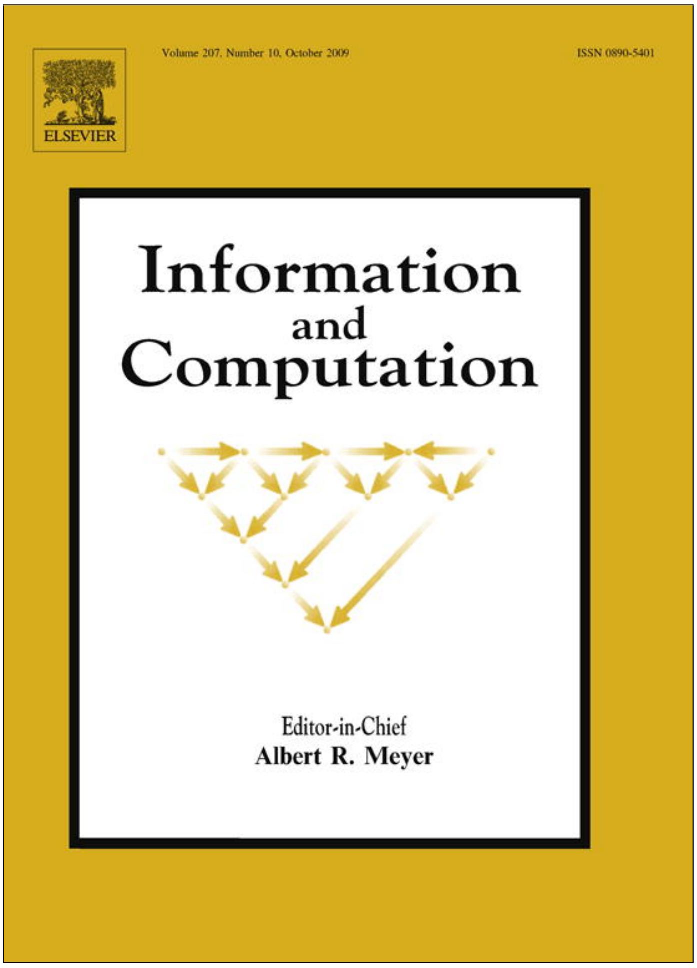
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Operational set theory and small large cardinals

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ABSTRACT

A new axiomatic system OST of operational set theory is introduced in which the usual language of set theory is expanded to allow us to talk about (possibly partial) operations applicable both to sets and to operations. OST is equivalent in strength to admissible set theory, and a natural extension of OST is equivalent in strength to ZFC. The language of OST provides a framework in which to express "small" large cardinal notions—such as those of being an inaccessible cardinal, a Mahlo cardinal, and a weakly compact cardinal—in terms of operational closure conditions that specialize to the analogue notions on admissible sets. This illustrates a wider program whose aim is to provide a common framework for analogues of large cardinal notions that have appeared in admissible set theory, admissible recursion theory, constructive set theory, constructive type theory, explicit mathematics, and systems of recursive ordinal notations that have been used in proof theory.

1. Introduction

"Small" large cardinal notions in the language of ZFC are those large cardinal notions that are consistent with $V = L$. Besides their original formulation in classical set theory, we have a variety of analogue notions in systems of admissible set theory, admissible recursion theory, constructive set theory, constructive type theory, explicit mathematics and recursive ordinal notations (as used in proof theory). On the face of it, it is surprising that such distinctively set-theoretical notions have analogues in such disparate and relatively constructive contexts. There must be an underlying reason why that is possible (and, incidentally, why "large" large cardinal notions have not led to comparable analogues). My long term aim is to develop a common language in which such notions can be expressed and can be interpreted both in their original classical form and in their analogue form in each of these special constructive and semi-constructive cases. This is a program in progress. What is done here, to begin with, is to show how that can be done to a considerable extent in the settings of classical and admissible set theory (and thence, admissible recursion theory).

The approach taken here is to expand the language of set theory to allow us to talk about (possibly partial) operations applicable both to sets and to operations and to formulate the large cardinal notions in question in terms of operational closure conditions; at the same time only minimal existence axioms are posited for sets. The resulting system, called Operational Set Theory, is a partial adaptation to the set-theoretical framework of the explicit mathematics framework in [5]. The specific small large cardinal notions treated here are those of being inaccessible, Mahlo and weakly compact. In the concluding section, it is discussed how these might be extended in a systematic way to stronger notions.

As a general idea, operational set theory may be traced back to von Neumann's theory of sets and functions [21]. That allowed a natural formulation of the Replacement Axiom in operational terms (rather than as a metamathematically formulated axiom scheme as is the case in ZF) as follows: if a is a set and f is an operation that is defined for each $x \in a$ then the

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range of f restricted to a exists. An early version of the present approach was presented in [8], and in fair detail in [9]. The historical notes and references at the conclusion give fuller background.

2. The system OST of operational set theory

L°, the language of OST, extends the language $L(=,\in)$ of ZF by a binary operation symbol $^\circ$ for application, a unary relation symbol $\downarrow$ for definedness and various constants to be specified (for which boldface letters are used). The terms $r, s, t \ldots$ of L° are generated from the variables $(a, b, c \ldots f, g, h \ldots x, y, z)$ and constants (distinguished by Roman boldface letters or expressions) by closing under application.We write st or $s(t)$ for $^\circ(s,t)$ and $st_1 \ldots t_n$ or $s(t_1, \ldots t_n)$ for the result of associating application to the left, as $(\ldots(st_1)\ldots t_n)$. The underlying logic of OST is the (classical) *logic of partial terms* LPT due to [3, pp. 97–99]. The atomic formula $t\downarrow$ expresses that t is defined; compound terms such as st may or may not be defined even when s and t are both defined. The distinctive modification of ordinary first-order logic in LPT lies in the scheme for universal instantiation:

$$\forall x \varphi(x) \wedge t \downarrow \rightarrow \varphi(t)$$

and the dual scheme for existential instantiation. In addition, we assume the strictness property for definedness in the sense that $(st)\downarrow$ implies both $s\downarrow$ and $t\downarrow$ and $(s = t)$ and $(s \in t)$ imply the same.[1] Partial equality of terms is defined by:

$$s \simeq t := (s\downarrow \vee t\downarrow \rightarrow s = t)$$

Informally speaking, operations are regarded as *intensional objects* given by *representations* (or *codes*) in the universe V of all sets of extensional operations whose domain is all or part of V. Thus any set can serve to represent (or code) an operation. In particular, xx is admitted as a term, though for any given x, we may not have $(xx)\downarrow$. Indeed, given the combinatory axioms and the axiom for the logical operation of negation that will be introduced below, we can produce a Russellian term obtained from the term xx that is not defined. It is for this reason that operations are considered to be possibly partial. In any case it is natural that not all operations are total, just as is the case in arithmetic, analysis and recursion theory. The advantage of our setup is that operations may be applied to operations, and thus the use of higher types is built in. We shall show in OST how functions in the set-theoretical sense determine operations and how, conversely, operations regarded extensionally determine functions.

The axioms of OST divide into five groups:

(1) Applicative axioms;
(2) Basic set-theoretic axioms;
(3) Logical operation axioms;
(4) Operational set-theoretic axioms;
(5) Induction on sets.

2.1. Applicative axioms

In axiom group 1, we have two constants **k** and **s** for the (partial) combinators for *constant operations* and *substitution*, respectively.

(i) $\mathbf{k} \neq \mathbf{s}$
(ii) $\mathbf{k}xy \neq x$
(iii) $\mathbf{s}xy\downarrow \wedge\ \mathbf{s}xyz \simeq x(z, yz)$

As usual from (i)–(iii) we can introduce for each term t a term $\lambda x{\cdot}t$ whose variables are those of t other than x and is such that

$$\lambda x{\cdot}t\downarrow \wedge\ (\lambda x{\cdot}t)y \simeq t(y/x)$$

and then a *recursor* **rec** (or *fixed point operator*) with

$$\mathbf{rec}(f)\downarrow \wedge\ [\mathbf{rec}(f) = g \rightarrow gx \simeq fgx]$$

For the constructions of $\lambda x{\cdot}t$ and **rec**, see [5, pp. 95–96].

2.2. Basic set-theoretic axioms

These consist of the axiom of extensionality, the existence of the empty set, closure under unordered pairs, closure under unions, and existence of the first infinite ordinal, all as usually formulated in ZF.

[1] This corresponds to the E^+ logic with equality and strictness of [20, pp. 52–53], where $E(t)$ is written instead of $t\downarrow$.

On the basis of these axioms we make free use of ordinary set-theoretic notions and notations in the following. In addition, we shall treat classes $A, B, C \dots$ formally as given by abstracts $\{x \mid \varphi(x)\}$ where φ is an arbitrary formula of L°; we write $t \in \{x \mid \varphi(x)\}$ for $\varphi(t)$. When a class is extensionally equivalent to a set we identify it with that set, in particular, every set a determines the class $a = \{x \mid x \in a\}$. But we do not assume (as in the Bernays–Gödel system) that sets are those classes that are elements of other classes or that subclasses of sets are sets. The class of all sets is $V := \{x \mid x = x\}$, and the class of all ordinal numbers is denoted *ORD*. The truth values $\mathbf{1}$ (true) and $\mathbf{0}$ (false) are identified with the sets $\{0\}$ and 0, respectively, so the set of Boolean values is simply the set $\{\mathbf{0}, \mathbf{1}\}$.

2.3. Logical operations

In the axiom groups 3 and 4 we write:

$$(f : A \to B) := (\forall x \in A)(fx \in B)$$

$$(f : A^n \to B) := (\forall x_1 \dots x_n \in A)(f(x_1 \dots x_n) \in B)$$

As special cases, for sets a and b, $(f : a \to V)$ means that f is total on a, and $(f : V \to b)$ means that f maps all sets into b, while $(f : V \to V)$ means that f is a total operation; similarly for V^n in place of V. Note that under our definition, if $f : A \to B$ and $A' \subseteq A$ and $B \subseteq B'$ then $f : A' \to B'$. When $f : a \to \{\mathbf{0}, \mathbf{1}\}$, we may regard f as the characteristic function of a *definite property* (or *predicate*) when restricted to the set a; similarly with V, V^n in place of a.

In axiom group 3, we have constants **el**, **cnj**, **neg**, $\mathbf{uni}_b$, respectively, for the definite predicate of *elementhood*, the Boolean operations of *conjunction* and *negation*, and the operation of *bounded universal quantification*.

(i) $(\mathbf{el} : V^2 \to \{\mathbf{0}, \mathbf{1}\}) \wedge \forall x, y[\mathbf{el}(x, y) = 1 \leftrightarrow x \in y]$
(ii) $(\mathbf{cnj} : \{\mathbf{0}, \mathbf{1}\}^2 \to \{\mathbf{0}, \mathbf{1}\}) \wedge (\forall x, y \in \{\mathbf{0}, \mathbf{1}\})[\mathbf{cnj}(x, y) = 1 \leftrightarrow x = \mathbf{1} \wedge y = \mathbf{1}]$
(iii) $(\mathbf{neg} : \{\mathbf{0}, \mathbf{1}\} \to \{\mathbf{0}, \mathbf{1}\}) \wedge (\forall x \in \{\mathbf{0}, \mathbf{1}\})[\mathbf{neg}(x) = 1 \leftrightarrow x = \mathbf{0}]$
(iv) $(f : a \to \{\mathbf{0}, \mathbf{1}\}) \to \mathbf{uni}_b(f, a) \in \{\mathbf{0}, \mathbf{1}\} \wedge [\mathbf{uni}_b(f, a) = 1 \leftrightarrow (\forall x \in a)(fx = \mathbf{1})]$

2.4. Operational set-theoretic axioms

The axiom group 4 makes use of three new (functional) operation constants, **S** for *separation*, **R** for *replacement* (or *range*) and **C** for universal choice.

(i) Separation for definite properties
$(f : a \to \{\mathbf{0}, \mathbf{1}\}) \to \mathbf{S}(f, a) \downarrow \wedge \forall x[x \in \mathbf{S}(f, a) \leftrightarrow x \in a \wedge fx = \mathbf{1}]$
(ii) Replacement
$(f : a \to V) \to \mathbf{R}(f, a) \downarrow \wedge \forall y[y \in \mathbf{R}(f, a) \leftrightarrow (\exists x \in a)(y = fx)]$
(iii) Choice
$\exists x(fx = \mathbf{1}) \to \mathbf{C}f \downarrow \wedge f(\mathbf{C}f) = \mathbf{1}$

2.5. Induction on sets

$$\forall x((\forall y \in x)\varphi(y) \to \varphi(x)) \to \forall x \varphi(x)$$

for *all* formulas $\varphi(x \dots)$ of the language L°.

This is called the $\text{Ind}_\in$ schema. An interesting restriction of this scheme is obtained by taking $\varphi(x, f)$ to be $fx = \mathbf{1}$ for $f : V \to \{\mathbf{0}, \mathbf{1}\}$, i.e., for f a definite property. By the system OST^r is meant OST with $\text{Ind}_\in$ replaced by this special case.

Note that OST does not contain the power set operation and that we do not have a logical operation corresponding to *unbounded universal quantification*. These may be considered separately by introducing the new constants **P** and **uni** with the following axioms:

(Pow) $(\mathbf{P} : V \to V) \wedge \forall x \forall a(x \subseteq a \leftrightarrow x \in \mathbf{P}(a))$
and
(Uni) $(f : V \to \{\mathbf{0}, \mathbf{1}\}) \to \mathbf{uni}(f) \in \{\mathbf{0}, \mathbf{1}\} \wedge [\mathbf{uni}(f) = \mathbf{1} \leftrightarrow \forall x(fx = \mathbf{1})]$

Below we shall consider the systems OST $\pm$ (Pow) $\pm$ (Uni), with Ind possibly restricted to definite properties.[2]

3. First consequences of OST

The notions and results in this section assume only OST or inessential extensions of its language L° by the adjunction of constant symbols.

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2 Jäger [11–13] has continued the work initiated in this paper. Some of his symbolism differs from mine. In particular, he uses blackboard font for certain of the constant symbols, including $\mathbb{S}$, $\mathbb{R}$, and $\mathbb{C}$; he also uses $\mathbb{P}$ for **P**. Finally, Jäger uses an unbounded existential operator **E** in place of my unbounded universal operator **uni**; these are interdefinable using the negation operator.

Definition 1. Write $\mathrm{App}(x, y, z)$ for $xy \simeq z$. The ess-$\Sigma(\mathrm{App}^+)$ (essentially existential, positive in App) formulas $\varphi, \psi, \ldots$ are generated as follows:

$$\psi := (x = y) \mid \neg(x = y) \mid (x \in y) \mid \neg(x \in y) \mid \mathrm{App}(x, y, z) \mid$$

$$\mid \psi \wedge \chi \mid \psi \vee \chi \mid (\forall y \in x)\psi \mid (\exists y \in x)\psi \mid \exists y \psi$$

The Δ_0 formulas are those generated without App and unrestricted $\exists$; they are thus (equivalent to) the Δ_0 formulas in the usual sense of L. The Σ_1 formulas of L are (up to equivalence) those generated without App. A formula is in e-Σ^+ form if it is provably equivalent to one in ess-$\Sigma(\mathrm{App}^+)$ form allowing substitution of constant symbols for one or more free variables. Thus for any terms s and t, the formulas $s = t$ and $s \in t$ are in e-Σ^+ form while, in general, $s \neq t$ and $s \notin t$ are in that form only for s and t variables or constants. If ψ is in e-Σ^+ form and t is a term that does not contain the variable y, $(\exists y \in t)\psi$ is in that form while, in general, $(\forall y \in t)\psi$ is in that form only for t a variable or constant.

In the following, $\psi(\underline{x})$ indicates a formula with free variables contained in $\underline{x} = x_1, \ldots, x_n$, and $t(\underline{x})$ is written for $t(x_1, \ldots, x_n)$.

Lemma 1.

(1) *With each* Δ_0 *formula* $\psi(\underline{x})$ *is associated a closed term* t_ψ *such that*

$$t_\psi \downarrow \wedge (t_\psi : V^n \to \{\mathbf{0}, \mathbf{1}\}) \wedge \forall \underline{x}(\psi(\underline{x}) \leftrightarrow t_\psi(\underline{x}) = \mathbf{1})$$

(2) *With each* e-Σ^+*formula* $\psi(\underline{x})$ *is associated a closed term* t_ψ *such that*

$$t_\psi \downarrow \wedge \forall \underline{x}(\psi(\underline{x}) \leftrightarrow t_\psi(\underline{x}) = \mathbf{1})$$

Proof. First define a characteristic function **eq** of equality using the axioms for logical operations and the equivalence $x = y \leftrightarrow (\forall z \in x)(z \in y) \wedge (\forall z \in y)(z \in x)$. Then the rest of (1) follows by induction using those axioms. For (2), first define $\mathbf{ap} = \lambda x \cdot \lambda y \cdot xy$; then $\mathrm{App}(x, y, z) \leftrightarrow \mathbf{eq}(\mathbf{ap}(x, y), z) = \mathbf{1}$. The only new thing that has to be considered in (2) is unrestricted $\exists$. Given $\psi(\underline{x}) = \exists y \chi(\underline{x}, y)$ and t_χ for $\chi(\underline{x}, y)$, we can take $t_\psi = \lambda x \cdot t_\chi(\underline{x}, \mathbf{C}(\lambda y \cdot t_\chi(\underline{x}, y)))$. □

Corollary 2. *We have closed terms* 0 *for the empty set,* ω *for the first infinite ordinal,* $\boldsymbol{p}$ *for unordered pair, and* $\bigcup$ *for union.*

Proof. Each is given by an axiom of the form $\exists y \psi$ where ψ is in Δ_0 form, and where y is the unique set specified in terms of the parameters of ψ. Then apply **C** to choose that y.[3] □

Define, as usual in L, $\langle x, y\rangle = \{\{x\}, \{x, y\}\}$, $\mathrm{dom}(a) = \{x \mid \exists y(\langle x, y\rangle \in a)\}$, $\mathrm{rng}(a) = \{y \mid \exists x(\langle x, y\rangle \in a)\}$, and $a \times b = \{z \mid (\exists x \in a)(\exists y \in b)(z = \langle x, y\rangle)\}$. Each set a determines the binary relation $a \cap \{\mathrm{dom}(a) \times \mathrm{rng}(a)\}$. We write $\mathrm{Fun}(a)$ if $(\forall x \in \mathrm{dom}(a))\exists! y(\langle x, y\rangle \in a)$, and, if this holds, $a(x)$ for $(\iota y)\langle x, y\rangle \in a$. The following two lemmas provide closed terms corresponding to these operations.

Lemma 3. *We have a closed term* ***prod*** *such that for each* a, b, $\boldsymbol{prod}(a, b) \downarrow$ *and* $\boldsymbol{prod}(a, b) = a \times b$.

Proof. Let f be such that for each $x, y, f(x, y) = \langle x, y\rangle$, and let $f_x = \lambda y \cdot f(x, y)$. Then for each a, b and $x \in a$, $f_x : b \to \{x\} \times b$ and $\mathbf{R}(f_x, b) = \{x\} \times b$. The operation $g = \lambda b.\lambda x \cdot \mathbf{R}(\lambda y \cdot f(x, y), b)$ thus has $R(g(b), a) = \{\{x\} \times b \mid x \in a\}$ and so $a \times b = \bigcup R(g(b), a)$. □

Lemma 4.

(1) *We have closed terms* $\boldsymbol{p}_0$ *and* $\boldsymbol{p}_1$ *such that for each* x, y, $\boldsymbol{p}_0\langle x, y\rangle = x$ *and* $\boldsymbol{p}_1\langle x, y\rangle = y$.
(2) *We have closed terms* ***dom*** *and* ***rng*** *such that for each* a, $\boldsymbol{dom}a = \mathrm{dom}(a)$ *and* $\boldsymbol{rng}a = \mathrm{rng}(a)$.
(3) *We have a closed term* ***op*** *such that for each* a, $\boldsymbol{op}a \downarrow$ *and if* $\mathrm{Fun}(a)$ *and* $f = \boldsymbol{op}a$ *then for each* $x \in \mathrm{dom}(a), fx = a(x)$.

Proof. (1) is obtained using Lemma 1, Corollary 2, and the choice operator **C**. (2) is obtained from the fact that $\mathrm{dom}(a)$ and $\mathrm{rng}(a)$ are included in the double union of a, and we then apply the separation operator **S**. For (3) we can take $\mathbf{op}a = \lambda x \cdot \mathbf{C}(\lambda y \cdot t(x, y, a))$ where $t(x, y, a) = \mathbf{1} \leftrightarrow \langle x, y\rangle \in a$.[4] □

Note that by (3) every function in the set-theoretical sense is represented by an operation (in a uniform way). The following gives a partial converse, namely that the restriction of an operation to a set is extensionally equivalent to such a function.

[3] If it is desired to avoid the choice operator **C** for such simple conclusions, one could of course assume constants $0, \omega, \mathbf{p}, \bigcup$ with their defining properties as axioms in group 2.

[4] Again, use of the choice operator **C** can be avoided by modification of the group 2 axioms at the outset.

Lemma 5. *There is a closed term* **fun** *such that for each* f, a *if* $f : a \to V$ *then* $\mathbf{fun}(f, a)\downarrow$ *and if* $c = \mathbf{fun}(f, a)$ *then* $\mathrm{Fun}(c)$ *and for each* $x \in \mathrm{dom}(c)$, $c(x) = fx$.

Proof. Let $b = \mathbf{R}(f, a)$, so $f : a \to b$. We want $c = \{\langle x, y\rangle \mid x \in a \wedge y \in b \wedge fx \simeq y\}$. This is given by $c = \{z \mid z \in a \times b \wedge \mathbf{eq}(\mathbf{ap}fx)y = \mathbf{1}\}$, which is constructed using **prod** and separation, **S**. □

Lemma 6. *The Axiom of Choice, AC, holds.*

Proof. The operation $g = \lambda x{\cdot}\mathbf{C}(\lambda y{\cdot}\mathbf{el}(y, x))$ is such that for each $x \neq 0$, $gx \in x$. By the preceding, given a set a such that each $x \in a$ is nonempty, g restricted to a determines a choice function on a in the usual sense. □

4. The consistency strength of OST and some extensions

Recall the system KPω of Kripke–Platek (or admissible) set theory with the axiom of infinity (see [2]). It is formulated in the language L of ZF and its axioms are those for extensionality, empty set, unordered pair, union, infinity, Δ_0-Separation, Δ_0-Collection, and the $\mathrm{Ind}_\in$ scheme. As usual we write AC for the Axiom of Choice and $V = L$ for the Axiom of Constructibility. As is well known, the systems KPω, KPω + AC, and KP$\omega + (V = L)$ (which proves AC) are all of the same consistency strength; moreover, KP$\omega + (V = L)$ is conservative over KPω for formulas which are absolute (i.e., provably Δ_1) w.r.t. KPω.

Theorem 7. (Strength of OST)

(1) KPω + AC $\subseteq$ OST.
(2) OST *is interpretable in* KP$\omega + (V = L)$.

Proof. (1) follows from the results of the preceding section. In particular, we use Lemma 1 and the separation operator, **S**, to establish Δ_0-Separation, while the choice operator **C** is employed, in addition, in the proof of Δ_0-Collection. For (2) we interpret the applicative structure in the codes for functions that are Σ_1 definable in parameters, obtained by uniformizing the Σ_1 predicates. This proceeds as in [2, pp. 164–167], which is applicable since under the assumption $V = L$, the universe is recursively listed in the sense given there. The treatment in Barwise must be modified slightly to account for parameters; this is done as follows. First one constructs a Σ_1 formula $\psi(w, x, y, z)$ such that for each Σ_1 formula $\theta(x, y, z)$ one can effectively find an $e \in \omega$ such that $\theta(x, y, z)$ is equivalent to $\psi(e, x, y, z)$. Then one uniformizes ψ with respect to y, i.e., produces a Σ_1 formula $\psi^*(w, x, y, z)$ that satisfies:

$$\psi^*(w, x, y, z) \to \psi(w, x, y, z)$$

and

$$\exists y \psi(w, x, y, z) \to \exists! y \psi^*(w, x, y, z)$$

Given a set parameter p, one takes $\langle e, p\rangle$ to be the code of the partial function

$$\langle e, p\rangle(x) = y \leftrightarrow \psi^*(e, x, y, p)$$

One can then define generalized “S-n-m” functions in a straightforward way, and from those give a model of the applicative axioms of OST. The rest of the interpretation proceeds in a straightforward way. □

Conservation of OST over KPω for absolute formulas is a direct consequence. A different proof of Theorem 7(2) is given in [11] by a method using a special inductive definition to interpret the applicative structure in a way that is adaptable to various extensions of OST, such as dealt with in the next statement. This provides a system of operational set theory of strength exactly ZFC, thus confirming a conjecture made by Thomas Strahm.

Theorem 8 [13].

(1) ZFC $\subseteq$ OST^r + (Pow) + (Uni).
(2) OST^r + (Pow) + (Uni) *is interpretable in* ZF + $(V = L)$.

Proof. Extending Lemma 1, in OST^r + Uni every formula ψ of L determines a closed t_ψ satisfying condition (i) of that lemma. Thus we obtain full separation and full reflection. In addition, well-foundedness of the $\in$ relation follows from the restricted $\mathrm{Ind}_\in$ axiom. Finally, the power set axiom is a consequence of (Pow). This proves (1). The proof of (2) requires a rather special inductive definition of the applicative structure that is given in [13]. □

Again, this implies a conservation result, in this case of $\text{OST}^r + (\text{Pow}) + (\text{Uni})$ over ZFC for absolute formulas.

Correction to [9]: Theorem 4 there stated that (1) $\text{ZFC} \subseteq \text{OST} + (\text{Pow})$ and (2) $\text{OST} + (\text{Pow})$ is interpretable in $\text{ZFC} + (V = L)$. Of these, (2) is correct, but not (1), as pointed out to me by Michael Rathjen.[5]

Questions: What is the strength of each of the following?

(1) $\text{OST} + (\text{Pow})$
(2) $\text{OST} + (\text{Uni})$

The system of (1) has been investigated in [13] with the following results in analogy to Theorem 8. Let $\text{KP}\omega + (\text{Pow})$ be the system $\text{KP}\omega$ enlarged by the power set axiom in its usual set-theoretical formulation, with an associated constant symbol for the power set operation. Then we have:

(a) $\text{KP}\omega + (\text{Pow}) + \text{AC} \subseteq \text{OST} + (\text{Pow})$
(b) $\text{OST} + (\text{Pow})$ is interpretable in $\text{KP}\omega + (\text{Pow}) + (V = L)$.

What is not known is whether the system $\text{KP}\omega + (\text{Pow})$ is of the same strength as the system with $V = L$; curiously, the usual argument for interpreting $V = L$ does not apply without the use of stronger principles.

Jäger [12] has gone on to determine an interesting extension of the Bernays–Gödel theory of sets and classes that is of the same strength as $\text{OST} + (\text{Pow}) + (\text{Uni})$.

5. Operational formulation of some large cardinal axioms

In the following, we use lower case Greek letters $\alpha, \beta, \gamma, \ldots, \kappa, \lambda, \ldots, \xi, \eta, \zeta$ to range over the ordinals, defined as usual. Ω is also written here for the class *ORD* of all ordinals.

Definition 9.

(1) $\text{Reg}(\kappa) := (\kappa > 0) \wedge \forall\alpha, f[\alpha < \kappa \wedge (f : \alpha \to \kappa) \to \exists\beta < \kappa(f : \alpha \to \beta)]$.
(2) $\text{Inacc}(\kappa) := \text{Reg}(\kappa) \wedge (\forall\alpha < \kappa)(\exists\beta < \kappa)[\text{Reg}(\beta) \wedge \alpha < \beta]$.
(3) $\text{Reg}_1(\kappa) := (\kappa > 0) \wedge \forall f[(f : \kappa \to \kappa) \to \exists\alpha < \kappa(0 < \alpha \wedge f : \alpha \to \alpha)]$.
(4) $\text{Mahlo}(\kappa) := (\kappa > 0) \wedge \forall f[(f : \kappa \to \kappa) \to (\forall\xi < \kappa)(\exists\alpha < \kappa)(\xi < \alpha \wedge \text{Reg}(\alpha) \wedge f : \alpha \to \alpha)]$.
(5) The statements Reg, Inacc, Reg_1, and Mahlo are obtained by replacing κ in the definitions on the r.h.s. by Ω.

The notions (1)–(4) make sense in the language L of ZFC when we take the 'f' variables to range over functions in the usual set-theoretical sense. In ZFC, $\text{Inacc}(\kappa)$ holds if κ is weakly inaccessible. Under a hypothesis such as $V = L$ which implies GCH, that is equivalent to being strongly inaccessible. Turning to the language L°, the notions (1)–(4) make sense when both the 'f' variables are interpreted in the extensional set-theoretical sense as well as in the intensional operational sense. Moreover, by Lemmas 4 and 5 the results are equivalent when read in both senses. The notions in (5) only make sense in the operational language L°.

Lemma 10. OST *proves the following*:

(1) $\text{Reg}_1(\kappa) \leftrightarrow \text{Reg}(\kappa) \wedge \kappa > \omega$.
(2) $\text{Reg}_1 \leftrightarrow \text{Reg}$.

Proof. We begin with (2). Define normal operations as usual (i.e., continuous and strictly increasing), show that every such operation has arbitrarily large ω-cofinal fixed points, and show that every f is majorized by a normal g. Then to show $\text{Reg} \to \text{Reg}_1$, given $f : \Omega \to \Omega$, using such g, find $\alpha > 0$ with $g\alpha = \alpha$, so that then $g : \alpha \to \alpha$, hence also $f : \alpha \to \alpha$. Conversely, given $\alpha > 0 \wedge (f : \alpha \to \Omega)$, choose normal g majorizing f with $g0 = \alpha$, and find $\beta > 0$ with $g : \beta \to \beta$. Then $\alpha < \beta$ and so $f : \alpha \to \beta$.

To prove (1), one relativizes the argument to κ. $\square$

Remark. The statement corresponding to Lemma 10(1) in ZFC, with functions in the set-theoretical sense instead of operations as here was stated in [1]. This was used by them to motivate a definition of Reg_2, again with set-theoretical functions. Here, we do the same with operations instead of functions.

Definition 11.

$$(f \equiv g) :\leftrightarrow \forall x(fx \simeq gx)$$

$$(f{\restriction}\alpha \equiv g{\restriction}\alpha) :\leftrightarrow \forall\xi < \alpha(f\xi \simeq g\xi)$$

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[5] Personal communication.

Write $f \in \kappa^\kappa$ if $f : \kappa \to \kappa$, and $F : \kappa^\kappa \to \kappa^\kappa$ if

$$\forall f \in \kappa^\kappa (Ff \in \kappa^\kappa) \wedge \forall f, g \in \kappa^\kappa [f \equiv g \to Ff \equiv Fg]$$

We say F is *κ-bounded* if

$$(\forall f \in \kappa^\kappa)(\forall \xi < \kappa)(\exists \gamma < \kappa)(\forall g \in \kappa^\kappa)[f \restriction \gamma \equiv g \restriction \gamma \to Ff\xi = Fg\xi]$$

α is a *κ-witness* for F if

$$0 < \alpha < \kappa \wedge \forall f \in \kappa^\kappa [f \in \alpha^\alpha \to Ff \in \alpha^\alpha]$$

Similarly, define $f \in \Omega^\Omega$, $F : \Omega^\Omega \to \Omega^\Omega$, F is *bounded*, and α is a *witness* for F, by replacing κ with Ω throughout.

NB. 'F' here is an operation variable, like 'f'.

Definition 12. $\text{Reg}_2(\kappa) :\leftrightarrow \forall F\ [F\ \kappa\text{-bounded} \to \exists \alpha(\alpha \text{ is a } \kappa\text{-witness for } F)]$
$\text{Reg}_2 :\leftrightarrow \forall F\ [F \text{ bounded} \to \exists \alpha\ (\alpha \text{ is a witness for } F)]$.

Aczel and Richter [1] state—and Richter and Aczel [18, pp. 329–331], prove—that if we use set-theoretic functions in place of operations, then in ZFC, κ is Reg_2 iff κ is weakly compact. By Lemmas 4 and 5, the set-theoretical interpretation of $\text{Reg}_2(\kappa)$ is equivalent to its definition above, since κ^κ can be replaced by the set of all functions from κ to κ in the set-theoretical sense, and then F can be replaced by a function on that set to itself. On the other hand, it is not clear if the operational sentence Reg_2 has a set-theoretical interpretation.

6. Connections of regularity statements with reflection principles and analogues of small large cardinals on admissible sets

The two Aczel and Richter papers cited above also give an analogue formulation of these notions in terms of recursion theory on admissible sets. If κ is an admissible ordinal and we interpret $fx \simeq y$ as $\{f\}(x) \simeq y$ in the sense of the Σ_1 recursion theory on κ (or L_κ) then each statement φ translates into a statement φ^{Ad} which gives the analogue notion. In the case of Reg_2 the analogue notion is proved in their paper [18] to be equivalent to Π_3-reflection (see below). Formalizing the arguments of Aczel and Richter, one should arrive at the following, though I have not checked the details.

Theorem 13.

(1) OST + (Inacc) *is interpretable in* KPi + $(V = L)$.
(2) OST + (Mahlo) *is interpretable in* KPM + $(V = L)$.
(3) OST + (Reg_2) *is interpretable in* KPω + $(\Pi_3 - \text{Reflection})$ + $(V = L)$.

In each case, we interpret the theory on the left in the theory on the right using the translation of φ as φ^{Ad}. While it is not obvious that the theories on the right are contained in those on the left, it is hard to believe that they are any stronger. In terms of the relation $\equiv$ of consistency equivalence, I thus make the following:

Conjecture 14.

(1) OST + (Inacc) $\equiv$ KPi.
(2) OST + (Mahlo) $\equiv$ KPM.
(3) OST + (Reg_2) $\equiv$ KPω + $(\Pi_3 - \text{Reflection})$.[6]

Aczel and Richter [1] indicate a generalization Reg_n of Reg_2, called n-regularity for each $n \geq 2$, defined in the language L of ZFC. This uses a notion of boundedness (a form of continuity) and of witness extended to higher types. They state the following (op. cit. p. 7):

(1) κ is 1-regular iff κ is Π^1_0-indescribable.
(2) For $n > 0$, κ is $n + 1$-regular iff κ is strongly Π^1_n-indescribable.

[6] The referee has suggested that related conjectures should hold if we replace OST on the l.h.s of (1)–(3) by OST^r + (Pow) + (Uni). One should also consider the same with the adjunction of the existence of a κ such that Inacc(κ), or Mahlo(κ), or $\text{Reg}_2(\kappa)$, respectively. For example, the resulting system in this last case should be equivalent to ZFC plus the existence of a weakly compact cardinal. For the notions in ZFC of Mahlo and weakly compact cardinals, cf. [15, Ch. 1].

Aczel and Richter go on to formulate an analogue notion on admissible sets as follows:

> Roughly speaking, the notion of *n-admissible* is obtained from that of *n-regular* by replacing in the definition of the latter, *bounded* by *recursive* [in the sense of admissible recursion theory] and replacing the functions by their Gödel numbers (op. cit., p. 8).

Thus Aczel and Richter propose that the ordinal $|\Pi^0_{n+1}|$ associated with non-monotone inductive definitions generated by operators Γ in the class Π^0_{n+1} are the appropriate recursive analogue of the first Π^1_n-indescribable cardinal. But there is no obvious direct connection of this choice of analogue with the classical notion.

The paper [18] proves (2) for the case $n = 1$, but unfortunately does not give the general definition of n-regularity and refers to the earlier publication for the definition of n-admissibility. I asked Wayne Richter if he would supply me with the former, and he did so for 3-regularity, but the details are somewhat more complicated than would be suggested by a reading of [1], and I have still not seen the general definition. Modulo that, the following is plausible to me: let $\mathrm{Reg}_n(\kappa)$ express in the language of OST that κ is n-regular, where we use operations satisfying suitable hereditary extensionality conditions in place of functionals of higher type. Then $\mathrm{Reg}_n(\kappa)^{\mathrm{Ad}}$ is equivalent to κ being n-admissible. To prove this, I expect one would make use of a generalization of the Myhill–Shepherdson theorem to finite types in admissible recursion theory; that is applied in ordinary recursion theory to show the equivalence of hereditarily extensional operations and hereditarily continuous functionals (cf. [16, p. 117]).

The primary aim here would be to formulate a general abstract reflection principle in the language of OST covering both classical and admissible set theory, from which the above small large cardinal principles and others follow. This should further have some intuitive justification and follow from syntactic reflection principles (indescribability properties) in the theory of small large cardinals, including those that use higher type class variables (cf. [15, pp. 57–67]). Steps in that direction were taken in the conclusion of [9], where it was sketched how Inacc and Mahlo follow from a certain operational reflection principle. But already to obtain Reg_2, stronger principles, yet to be formulated, will be needed.

7. Historical notes

(1) The axiomatization by [21] of a theory of sets and functions is a precursor in spirit of OST. Von Neumann's functions are of type 1 over the universe of sets and are closed under combinatory and logical axioms; it would be of interest to re-examine that work in the light of OST. For improvements of von Neumann's formulation see [19].

(2) My use of operational theories of various kinds dates back to [5], "A language and axioms for explicit mathematics". Models of extensions of set theory by relatively weak operational axioms were produced there (pp. 109–110), by adaptation of the notion of prime computability over abstract structures due to [17]. Further uses of such models were made in various subsequent publications including Feferman [6,7]. The germs of the present program are to be found in [8], with the first full presentation in the unpublished notes [9].

(3) The paper [4] presents a system of operational set theory that has some overlap with OST + (Pow) + (Uni); its consistency is proved by a model construction like that of my paper [5]. The purpose of his system is to provide a computation system that can make fuller use of the expressive power of set theory.

(4) As mentioned above, Jäger [11] gives full details of a proof of Theorem 7 different from the one sketched here, and uses that method in [13] to obtain the results about the strength of OST + (Pow) and OST^r + (Pow) + (Uni) stated in Section 4.

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