The development of programs for the foundations of mathematics in the first third of the 20th century

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1 *Legacy of the 19th century: the reductive programs.*

The most prominent "schools" or programs for the foundations of mathematics that took shape in the first third of the 20th century emerged directly from, or in response to, developments in mathematics and logic in the latter part of the 19th century. The first of these programs, so-called *logicism*, had as its aim the reduction of mathematics to purely logical principles. In order to understand properly its achievements and resulting problems, it is necessary to review the background from that previous period.

Following the remarkable but freewheeling *floraison* of mathematics in the 17th and 18th centuries, there was increasing attention to questions of rigorization, conceptual clarification, conceptual reduction, and systematic organization and exposition. In particular, problematic uses of infinitesimals and of infinite sums and products in analysis were eliminated in favor of rigorous use of the limit concept both for real and complex numbers. Moreover, the use of the latter system, generated by the "imaginary" number $i = \sqrt{-1}$, was given a solid foundation by reduction to the real number system. This was simply accomplished (in one way) by representing $x + yi$, for x, y real, by the pair (x, y), with suitable definitions of "addition" and "multiplication" of such pairs.

The process of reduction was carried a substantial step further (independently), by Georg Cantor (1845–1918) and Richard Dedekind (1831–1916), by means of the construction of the real numbers from the system of rational numbers. Cantor did this by representing real numbers as limits of sequences of rational numbers, and Dedekind by representing them as least upper bounds (or, dually, greatest lower bounds) of subsets of the rationals. Finally, Leopold Kronecker (1823-1891) explained the rational numbers in terms of the system of positive integers $(1, 2, 3, \ldots)$, or, alternatively, the natural numbers $(0, 1, 2, \ldots)$.

2 *The general process of reduction.*

Each step in the reductive process consisted in replacing the objects of a given number system S, by representations of them built from objects of a more basic system S_0, where these combinations are identified when they are intended to represent the same object from S. For example, Kronecker's representation of positive rational numbers takes (n, m), with n, m positive integers, to represent n/m, and "identifies" (n, m) with (p, q) just in case we are to have $n/m = p/q$, i.e. just in case $nq = mp$. Cantor's representation of real numbers takes a sequence of rational numbers $r = \langle r_0, \ldots, r_n, \ldots \rangle$ to represent $\lim_{n\to\infty} r_n$ when r satisfies the (internal) Cauchy convergence criterion; then $r = \langle r_0, \ldots, r_n, \ldots \rangle$ is "identified" with $s = \langle s_0, \ldots, s_n \ldots \rangle$ when $\lim_{n\to\infty} (r_n - s_n) = 0$, the latter limit being explained entirely in terms of rational numbers. In general, then, the system S is replaced by a system S^* of combinations of objects built out of S_0, and two members a, b of S^* are "identified" when they are to represent the same object of S. This relation of identification is an *equivalence relation* $\equiv$ between members of S^*, i.e. we have: (i) $a \equiv a$, (ii) $a \equiv b$ implies $b \equiv a$, and (iii) $a \equiv b$ & $b \equiv c$ implies $a \equiv c$, for all a, b, c in S^*. This can be turned into the actual relation of identity by associating with each a in S^* its *equivalence class* $[a]$ which is defined to consist exactly of all those b with $a \equiv b$. Then (assuming that classes are determined entirely by their members), we have

$$[a] = [b] \text{ if and only if } a \equiv b,$$

and the operations of S are mirrored by operations on equivalence classes. Technically speaking, S is in this way replaced up to isomorphism by the system of equivalence classes of members of S^*.

3 *Toward the "final' reduction: the contributions of Dedekind, Peano and Cantor.*

Inspection of the successive reductions from the complex numbers through the reals down to the natural numbers, shows that certain general notions such as those of *pair, set* (or *class), sequence, relation, operation* (or *function*), etc. form the principal ingredients in the construction of each system from a more basic one. These notions are fundamental to all mathematics and, depending on one's point of view, can be regarded as set-theoretical or logical in nature. It appeared from work of Cantor, Dedekind, and Peano (Giuseppe Peano, 1858–1932) toward the end of the 19th century that a "final" reduction could be accomplished, by explaining the system of natural numbers entirely in terms of such more basic notions. The contribution of Dedekind and Peano toward this end was to give an axiomatic characterization of the system of natural numbers (in 1888 and 1889, resp.). In Peano's hands, this essentially takes the following form: the set $\mathbb{N}$ is given (i) with a distinguished element 0 and (ii) a successor operation sc, where $sc(x)$ is interpreted as $x+1$ for any x (though prior to the definition of '+' and '1', it is not written in that form). The axioms further state that (iii) 0 is not the successor of any natural number, (iv) sc is a one–one operation from $\mathbb{N}$ into $\mathbb{N}$, and (v) that $\mathbb{N}$ is the smallest set which contains 0 and is closed under successor. A little more formally, using the symbols '$\in$' for 'is a member of' and '$\Rightarrow$' for 'implies', the *Peano Axioms* are:

I $0 \in \mathbb{N}$

II $x \in \mathbb{N} \Rightarrow sc(x) \in \mathbb{N}$

III $x \in \mathbb{N} \Rightarrow sc(x) \neq 0$

IV $x \in \mathbb{N} \ \& \ y \in \mathbb{N} \ \& \ sc(x) = sc(y) \Rightarrow x = y$

V If X is any subset of $\mathbb{N}$ with $0 \in X$ and $(x \in X \Rightarrow sc(x) \in X)$ for all x, then X contains all members of $\mathbb{N}$.

The last axiom, V, is the basis of *proof by induction on* $\mathbb{N}$: to show that a property $P(x)$ holds for all x in $\mathbb{N}$, it is sufficient to show that $P(0)$ holds and that $P(x) \Rightarrow P(sc(x))$ for all x; this is seen by taking X to be the set of all $x \in \mathbb{N}$ with $P(x)$. Dedekind applied induction to justify *definition by recursion on* $\mathbb{N}$, where a function F is determined on $\mathbb{N}$ by prescribing $F(0)$ and by telling how $F(sc(x))$ is to be defined in terms of $F(x)$ for any x. He used this in turn to show that the axioms for $\mathbb{N}$ are *categorical*, i.e. if $\mathbb{N}, 0, sc$ and $\mathbb{N}', 0', sc'$ are any two realizations of I-V then they are isomorphic: one obtains the required one–to–one correspondence $x \leftrightarrow x'$ by taking $F(0) = 0'$ and $F(sc(x)) = sc'(F(x))$, and then $x' = F(x)$ for any $x \epsilon \mathbb{N}$.

What the (Dedekind or) Peano axioms do not by themselves guarantee is the existence of *some* realization (or "model") of them. Dedekind argued for this via the (potential) infinity of objects of (his) thought, where the successor of any object x is the thought that x can be an object of (his) thought. However, this oddly psychological interpretation was accepted by no one else. Actually, a more mathematical realization was already available via Cantor's *theory of sets*, which he developed in a remarkable series of papers stretching from 1874 to 1897 (cf. the collection 1932). Cantor's main new contribution lay in the coherent extension to arbitrary (finite or infinite) sets of the notions of *ordinal number* and *cardinal number*. The first of these is used to answer questions of the form: How are the elements of a set ranked in a given (well)-ordering of it? The latter is used to answer the question: How many elements does a set X have? which is, in turn, closely related to the question: When do two sets X and Y have the same number of elements? The answer given to that by Cantor is in terms of the relation of *set-theoretic equivalence* $\sim$, where $X \sim Y$ is defined to hold just in case there is a one-to-one correspondence between the elements of X and the elements of Y. He then held that with each set X is associated its cardinal number, $\text{card}(X)$, in such a way that two sets have the same cardinal number just in case they are equivalent in this sense, i.e. $\text{card}(X) = \text{card}(Y)$ if and only if $X \sim Y$. Cantor conceived of $\text{card}(X)$ as arising by a double process of abstraction from X, first by abstracting away from the nature of its individual elements while in some given order, and then by abstracting away from any order whatever; for this reason, he used $\overline{\overline{X}}$ to denote $\text{card}(X)$. However, these processes of abstraction were only vaguely explained and he was not more specific about how $\text{card}(X)$ might be represented in terms of some more basic objects or notions.

If one grants the association of a cardinal number card(X) with each X in some way to satisfy the above condition for equality of cardinals, there are several ways in which a realization of the Peano axioms can be defined set-theoretically; for simplicity, we shall give only one of these. First of all, one defines 0 to be the cardinal number of the empty set. Then, for successor, given $x = \text{card}(X)$, one takes $sc(x) = \text{card}(X')$ where X' is obtained from X by adjoining exactly one new element. (This definition is problematic if a "universal" set X consisting of all objects is admitted; however, in that case, one can replace X by a set Y with $X \sim Y$, for which Y is not universal). Finally, $\mathbb{N}$ can be defined as the smallest set which contains 0 and is closed under successor, i.e.:

$$x \in \mathbb{N} =_{def} \forall X[0 \in X \& \forall y(y \in X \Rightarrow sc(y) \in X) \Rightarrow x \in X]$$

This definition immediately verifies the induction axiom V above, and the other axioms I–IV for 0 and sc are easily checked.

4 *The logicist program in Frege's hands.*

There remained still one more step in order to achieve the "final" reduction of the system of natural numbers, namely to give a definition of cardinal numbers in terms of more basic set-theoretical or logical notions. Both Gottlob Frege (1848–1925) and Bertrand Russell (1872–1970) advanced this within a more far-reaching "*logicist*" program to reduce mathematics to logical principles. (For Frege this was to include arithmetic and analysis, but not geometry; for Russell this was to comprise all of pure mathematics). Frege's work began somewhat earlier but Russell became aware of it only after having progressed independently some distance along similar lines. Basically, the idea for both was to define card(X) as the equivalence class $[X]$ of X in the relation of $\sim$ between sets, i.e.

$$card(X) =_{def} \text{ the class of all sets } Y \text{ such that } X \sim Y$$

It should be noted that here the notions of "set" and "class" are used interchangeably (unlike the distinction made in modern axiomatic theories of sets and classes). However, the above supposed definition only gives a very crude idea of the actual development of Frege's and Russell's treatments of arithmetic; these are elaborated somewhat in the following.

In order to provide the proper conceptual framework and carry out his program rigorously Frege first began by developing a new system of symbolic logic which he presented in his monograph *Begriffsschrift* (1879). This went far beyond the Aristotelian syllogistic and the then current logic of classes and relations (due to Boole, DeMorgan, Peirce and Schröder). In modern technical terms, Frege's logic was a second order predicate calculus with negation, implication and universal quantification as the basic logical operators, and which included a complete system of the propositional and first-order predicate calculus. These became permanent contributions to modern logic, though in a different symbolic form than that given by Frege, who used an ideographic system of notation that was awkward to handle and contributed to its negative reception. Modern systems of logical symbolism descend instead from the work of Russell who in turn expanded that of Peano.

The first-order or individual variables in Frege's system are to be thought of as ranging over *all* objects in the universe, concrete as well as abstract, and thus among them, all objects of mathematics. The second-order variables are supposed to range over arbitrary predicates or properties (or "concepts" in Frege's terminology) as well as arbitrary relations (predicates of more than one argument). It would get us into too many technicalities to try to describe precisely the principles accepted by Frege for the second-order variables, and so we shall take some liberties with our explanations in the following. (Cf. Dummett 1991 for a thorough analysis).

Basically, Frege assumed that each propositional function $\phi(x)$ of his system determines a property P which holds of any x just in case $\phi(x)$ is true. This is what is nowadays called the *Comprehension Axiom for Properties*:

$$\exists P\, \forall x[P(x) \Leftrightarrow \phi(x)],$$

where '$\exists$' stands for 'there exists', '$\forall$' for 'for all', '$\Leftrightarrow$' for 'if and only if', and where the predicate variable 'P' does not occur in ϕ. Secondly, each propositional function $\phi(x)$ determines a *class* $\hat{x}\phi(x)$ (written $\{x \mid \phi(x)\}$ in more modern symbolism), which is its *extension* (or *Wertverlauf*, in Frege's terminology), and two functions $\phi(x)$ and $\psi(x)$ determine the same class just in case they are co-extensive. That is, the *Axiom of Extensionality* is assumed for classes in the form:

$$\hat{x}\phi(x) = \hat{x}\psi(x) \Leftrightarrow \forall x[\phi(x) \Leftrightarrow \psi(x)].$$

Finally, classes are supposed to be mathematical objects and thus belong to the universe of individuals; speaking formally, each $\hat{x}\phi(x)$ is a first order term which can instantiate properties holding of all individuals. Then the theory of classes may be interpreted in Frege's system, with the membership relation between individuals and classes introduced explicitly by the following definition:

$$(y \in z) =_{def} (\exists P)[z = \hat{x}P(x) \& P(y)].$$

It follows from the above principles that we have a *Comprehension Axiom for Classes*, in the form:

$$y \in \hat{x}\phi(x) \Leftrightarrow \phi(y).$$

Frege carried out his program for the logical foundations of arithmetic in the works *Die Grundlagen der Arithmetik* (1884) and *Grundgesetze der Arithmetik* (Vol. I 1893, Vol. II 1903). His definition of equinumerosity of predicates, $P \sim Q$, is equivalent to Cantor's definition for the associated classes $\hat{x}P(x) \sim \hat{x}Q(x)$. Then the *number of* x's *such that* $P(x)$, in symbols $N_xP(x)$, is defined as the class of extensions of Q's such that $P \sim Q$; this serves the role in Frege's system of $\mathrm{Card}(X)$ for $X = \hat{x}P(x)$. With that as basis, one can proceed to define $0, sc$ and $\mathbb{N}$ just as for sets at the end of the preceding section.

Despite the great care which Frege bestowed on his conceptual framework and the precision with which he carried out his derivations, and despite the superficial plausibility of his program, Frege's system was fundamentally flawed, since it proved to be inconsistent. This was discovered by Russell in 1901 and communicated by him in a letter to Frege in 1902. The *Russell paradox*, as it is now called, is simply obtained by forming the class

$$r =_{def} \hat{x}(x \notin x).$$

Thus by the Comprehension Axiom for Classes, we have

$$\forall x[x \in r \Leftrightarrow x \notin x]$$

In particular,

$$r \in r \Leftrightarrow r \notin r,$$

which immediately results in a contradiction.

Russell's letter to Frege arrived while the second volume of the *Grundgesetze* was at the printer's, and Frege was able to add an appendix reproducing a derivation of Russell's paradox. He also hastily suggested that a restriction of the Axiom of Extensionality might serve to avoid it; however, Frege's "way out" was shown somewhat later to fail. In any case, Frege's plan for his logicist program was totally undermined; he never wrote the projected third volume of the *Grundgesetze* and he eventually abandoned his program.

5 *The logicist program: Russell's salvage operation.*

In his own development of the logicist program, Russell was mainly influenced by the works of Cantor and Peano, the former through his theory of classes, and the latter through his symbolic logic. Russell had met Peano in 1900 and was so impressed by the precision of his approach that he immediately set out to master it. Russell then began in earnest to elaborate his own synthesis of Cantor's and Peano's ideas, which was to appear in 1903 in his book, *The Principles of Mathematics.* Its aim was to demonstrate "...that all pure mathematics deals with concepts definable in terms of a very small number of fundamental logical concepts, and that all its propositions are deducible from a very small number of fundamental logical principles ...," a program subsequently summarized by the thesis that "mathematics and logic are identical." However, the 1903 work is not presented by means of logical symbolism, despite the technical character of the concepts, principles and arguments involved. Rather it is set out in ordinary language for a general philosophical and mathematical audience. Russell's plan was to follow up with a second volume in which all details of the program would be carried out; this was to be co-authored by his teacher, the mathematician and philosopher Alfred North Whitehead (1861–1947) and "addressed exclusively to mathematicians." Late in the preparation for publication of the *Principles* Russell became aware of Frege's prior work, which he recognized largely anticipated his own; in fact, he had seen the *Grundgesetze der Arithmetik*, but said that "...owing to the great difficulty of [Frege's] symbolism, I had failed to grasp its importance or to understand its contents." In recompense, Russell added an Appendix A to his own volume, under the title, "The logical and arithmetical doctrines of Frege."

It was also late in the writing of the *Principles* that Russell addressed the problems raised by paradoxes (also called antinomies) in the theory of classes. His was by no means the first such: among others, Cesare Burali-Forti (1861–1931) had published the paradox of the greatest ordinal number in 1897, and Cantor had written Dedekind in 1899 concerning the paradox of the largest cardinal number. Russell was aware of both, but it was in the process of analyzing Cantor's paradox in order to isolate the crucial difficulty that he was led to his own paradox of the class of all classes which are not members of themselves. Indeed, in comparison with the previous antinomies, Russell's paradox was by far the simplest and most striking, shorn as it was of all special notions and arguments. Regarding his own, Russell recognized–in contrast to Frege–that the crux of the problem lay in the unrestricted assumption of the Comprehension Axiom, according to which each property $\phi(x)$ determines as an object a class $\hat{x}\phi(x)$ of all objects x satisfying ϕ. In Appendix B to *The Principles of Mathematics*, Russell made his first stab at a solution to the contradictions, under the title, "The doctrine of types." This was to the effect that every propositional function $\phi(x)$ has a prior range of significance, the *type* of its variable x, only within which are questions of its truth or falsity meaningful. Under a suitable ordering of types, then, the type of a class is higher than the type of any argument x, and for this reason it would not be meaningful to ask whether or not $x \in x$ is true. Thus, according to this line of reasoning, neither the property $\phi(x) = (x \in x)$ nor its negation $(x \notin x)$ is meaningful.

Russell later introduced the term *predicative* to distinguish those properties $\phi(x)$ which determine classes from those, called *impredicative*, which do not, but he was unsettled as to the proper criteria to decide between the two in each case. Between 1903 and 1908 he wrestled with this problem and seriously considered a variety of solutions other than the use of type distinctions, among them the so-called "no classes" theory about which we will speak below. In the same period he was also fending off attacks on the entire logicist program by Henri Poincaré (1854–1912). Ironically, it was by combining Poincaré's *vicious-circle principle* for the analysis and solution of the antinomies, with the doctrine of types and the no classes theory, that Russell eventually settled on his own approach in his landmark article "Mathematical logic as based on the theory of types" (1908).

Poincaré's attention had centered on the so-called Richard paradox of 1905 (due to Jules Richard, 1862–1956), which concerned not classes, but an unrestricted notion of definability. In his writings on the subject of the paradoxes in 1906 and thereafter, Poincaré identified the source of the difficulty always to lie in the presence of a vicious circle. According to his analysis, in each such case there is a purported definition of an object in terms implicitly involving the object itself; for example, in the case of the Richard paradox the definition is that of a certain real number in terms of the supposed totality of all definable real numbers, while in the paradox of Burali–Forti it is that of the largest ordinal in terms of the supposed totality of all ordinals. Adopting Russell's terminology to this more specific analysis, Poincaré called *predicative* those definitions which do not involve a vicious circle and *impredicative* those which do (implicitly or otherwise) and which are thus to be banned. Russell, in his 1908 article, adopted Poincaré's proscription of impredicative definitions in his own formulation of the vicious-circle principle: "No totality can contain members defined [only] in terms of itself." As indicated above, this was embodied in a modified version of the doctrine of types, more complicated than originally envisaged in order also to encompass the no classes theory. We now turn to an explanation of the ideas behind the resulting formalism.

First of all, the idea of the no classes theory was that, instead of speaking of the class of all objects having a given property, one speaks only of that property itself, as given explicitly by means of a formula $\phi(x)$ for a propositional function. Under this interpretation, '$\hat{x}\phi(x)$' is read as "the property which holds just of those x satisfying $\phi(x)$." Now, since distinct formulas $\phi(x)$ and $\psi(x)$ may express different properties and yet be satisfied by exactly the same objects, the Axiom of Extensionality is no longer to be assumed. For this reason, the no classes theory is to be regarded as an *intensional theory of properties*, as opposed to an *extensional theory of classes*.

Next, special attention is given to (what Russell called) the *apparent variables* of a formula ϕ, i.e. those that occur bound in some sub-formula or sub-expression, such as an individual or property variable 'y' in $(\forall y)\psi(y)$ or $(\exists y)\psi(y)$ or $\hat{y}\psi(y)$. In the case that 'y' is a property variable of given type, the totality of those properties is implicitly used in defining the property $\hat{x}\phi(x)$. Thus we would have a violation of the vicious-circle principle if we allowed $\hat{x}\phi(x)$ also to be of the same type. Indeed, Russell reformulated his vicious-circle principle in the form: "Whatever contains an apparent variable must not be a possible value of the variable"; in a suitable sense, it must of of *higher* type than that variable.

In order to give specific meaning to these ideas, one must have a systematic means of assigning types to variables and to property expressions of the form $\hat{x}\phi(x)$, as well as an explanation of the relation of being of higher type. In modern treatments one distinguishes two such relations, but Russell conflated them in a way that can be confusing. Thus we shall explain instead the modern approach, which uses independent notions of *type* and *level*. The numerals $0, 1, 2, \ldots$ are used for both type and level assignments and then the relation of being of higher type or level is just that of being numerically larger.

The objects of type 0 are taken to be the *individuals* in the universe of discourse; these are supposed to be "simples," not subject to further analysis. Then the objects of type 1 are understood to be properties of objects of type 0, and, in general, the objects of type $n+1$ are conceived of as the properties of objects of type n. Each variable is assigned a specified type; if $\phi(x)$ is a well-formed formula with 'x' of type n, then $\hat{x}\phi(x)$ is assigned type $n+1$. If s and t are any terms (either variables or property expressions), then $s \in t$ is allowed as a well-formed formula only when s is of some type n and t is of type $n+1$. Formulas in general are built up from such *atomic* formulas by propositional operations (such as negation '$\neg$' and conjunction '&') and by applying the quantifiers ('$\forall$' and '$\exists$') to variables. If no further distinctions as to level (to be described next) are made, the resulting symbolism is said to be that of the *simple theory of types* (STT). Now the comprehension Axiom is restricted in STT to well-formed formulas $\phi(x)$, to give:

$$y \in \hat{x}\phi(x) \Leftrightarrow \phi(y), \text{ when '}y\text{' is of the same type as '}x\text{'.}$$

Already the simple theory of types excludes Russell's paradox: $\hat{x}\neg(x \in x)$ is not a legitimate term in its language, no matter what the type of 'x' is.

However, the effect of Russell's revision of the vicious-circle principle, that "whatever contains an apparent variable must not be a possible value of that variable" required the further division into levels, which correspond to stages of definition in any type n greater than 0. Each (predicate) variable of type n is assigned a natural number k as level; then the level of a term of the form $\hat{x}\phi(x)$ must be greater than the levels of all bound variables which occur in ϕ. Under this restriction, if a property $(\forall P)\psi(P)$ holds with bound variable 'P' of level k and if the level of $\hat{x}\phi(x)$ is larger than k, we *cannot* instantiate this universal statement to conclude that $\psi(\hat{x}\phi(x))$. As will be seen, the inability to do so creates genuine difficulties in the foundations of arithmetic.

The logical system incorporating Russell's vicious circle principle in the ways indicated is called the *ramified theory of types* (RTT), in which each type is ramified according to level. The plan for the definition of the natural numbers in RTT was essentially the same as Frege's, namely via the notion of cardinal numbers as equivalence classes for the relation $\sim$ of equinumerosity. However, in type theory (simple or ramified), there is no single such relation; rather, one can only define $P \sim Q$ for classes *(qua* properties) of the same type $n(\neq 0)$. Then the equivalence classes under $\sim$ are objects of the next type $m = n + 1$. Thus there is no single notion of cardinal number, but rather one that is duplicated in each such type m. Russell made light of this multiplicity of notions of cardinal numbers in his theory: he argued that each class of a given type is equinumerous with the class of singletons of its members, and that by successive applications of this observation, classes of any two types can be compared as to cardinality. However, this idea cannot be stated formally in STT, let alone RTT.

Matters become more complicated with the definition of natural numbers in the ramified theory of types. Given any type $n \neq 0$, define 0_n and sc_n to be the zero and successor operation for cardinal numbers of type $n + 1$. Then one can only define the property of being a natural number in the form: $Nat_{n,k}(x) =_{def} \forall P[0_n \in \ P \ \& \ \forall y[y \in P \Rightarrow sc_n(y) \in P] \Rightarrow x \in P\}$, where '$P$' is a predicate variable of type $n + 1$ and level k. That is, we have a *double* multiplicity of notions of natural number depending on both type and ramification level. Russell observed that this creates problems for proofs by induction on the natural numbers of properties ϕ which themselves involve $Nat_{n,k}$; any such property must have level *greater* than k, and so the variable 'P' of level k in the above definition cannot be instantiated by $\hat{x}\phi(x)$. In order to get around such difficulties, Russell introduced an *ad hoc* assumption called the *Axiom of Reducibility*; informally, this expresses that each property $\hat{x}\phi(x)$ is co-extensive with a property of the lowest level. In effect, this "Axiom" obliterates the distinctions according to levels and deeply compromises the vicious-circle principle in the very specific form stated by Russell. Though he had initially tried to give some arguments in its favor, he eventually stated that "it is not the sort of axiom with which we can rest content" and that its justification is purely pragmatic: "it leads to the desired result and to no others."

There were other troublesome hypotheses that Russell had to make, namely the so-called *Axiom of Infinity* and the *Axiom of Choice*. The first, that there exist infinitely many individuals, is needed to assure that each finite cardinal number has a successor. The second had emerged in the theory of sets as necessary to establish many results in the arithmetic of transfinite cardinals; it asserts the existence of simultaneous choices from arbitrary collection of non-empty sets, even where no means is available to define those choices.

The detailed execution of this greatly modified logicist program sketched in Russell's 1908 article was carried out in the monumental work of Whitehead and Russell, *Principia Mathematica*, published in three volumes in the period 1910–1913. Whitehead's name was listed first, as the senior author, and he certainly contributed a great deal to its writing, but the over-all plan and major part of the work was Russell's. This is what had originally been intended as the second volume of *The principles of mathematics.* Although the passage from the *Principles* to the *Principia* through the Ramified Theory of Types had required the explicit introduction of assumptions (such as the Axiom of Reducibility, the Axiom of Infinity and the Axiom of Choice) whose logical status was questionable, Russell still asserted many years later, in his introduction to the second edition of the *Principles*: "The fundamental thesis of the following pages, that mathematics and logic are identical, is one which I have never seen any reason to modify." But in this respect, Russell had hardly any followers and his extraordinary effort to salvage the logicist program is generally deemed a failure.

In 1925, Russell's student Frank P. Ramsey (1903–1930) pointed out (as we have already seen for Russell's paradox) that to block the paradoxes of classes it was not necessary to use the full force of the vicious-circle principle and that the distinctions according to types already served this purpose. The choice, then, between STT and RTT would have to be on different grounds, with STT interpreted as a theory of classes (for which Extensionality would be appropriate) and RTT a theory of predicative properties. In philosophical terms, the former fits a Platonic realist view of mathematics, while the latter is allied with nominalistic tendencies. In the transition from the *Principles* to *Principia*, Russell himself had consciously made this philosophical passage. The problem for those who, following Poincaré and the later Russell, would seek predicative foundations of mathematics, would be how to carry such a program out in a reasonable way without compromising oneself as Russell had done with the Axiom of Reducibility. The problem for those who were ready to accept a Platonic conception of logical and mathematical entities would be how to set up a formal system in which informal practice could be more readily mirrored while still avoiding the paradoxes of classes. First answers to both these problems would be given, respectively, by Hermann Weyl (1885–1955) and Ernst Zermelo (1871–1953), as will be described below.

While most of the actual details of the work in *Principia Mathematica* and its answer to the question — What is mathematics? — did not have lasting value, the fact of its existence *did* have enormous impact. What the *Principia* showed was that it is possible to define many mathematical notions in precise symbolic terms from a very few basic notions and to carry out extended tracts of mathematical argument in a completely rigorous step by step form, from a few basic principles and rules of reasoning. Whether or not the thesis that mathematics is logic was justified, the exercise of *Principia Mathematica* certainly penetrated to the very logical roots of mathematics and broadened to all its parts the ideal of formal rigor whose previous exemplar had been that of Euclidean geometry, two millenia earlier.

6 *Predicativism: Poincaré and Weyl.*

Henri Poincaré, one of the foremost mathematicians of his time, is generally considered to be a precursor of both the predicativist and the intuitionistic programs for the foundations of mathematics. Noted for his prolific contributions to all branches of mathematics, but especially of analysis and mathematical physics, Poincaré also wrote many popular books and essays on the philosophy of science and mathematics in a lively and vivid style. In his writings on the philosophy of mathematics, beginning in 1893 and continuing until his death in 1912, Poincaré gave primary concern to the role of intuition as against that of logic. In particular, he considered the natural number system to be directly understood and the associated principle of proof by induction to be thereby sanctioned by intuition and thus not to require reduction to anything (purporting to be) more basic. Beginning in 1905, Poincaré mounted an unremitting attack on set-theoretical and logical programs for the foundations of mathematics, and especially on the logicist program. Besides seeing no need for the efforts to reduce the notion of natural number to logical concepts, he argued that a *petitio* was involved in whatever way one would attempt to carry out such a reduction. For example, the theory of types presumes an understanding of the natural numbers in the general description of its syntax. Poincaré's criticism in this respect is justified if one views the theory of types from the outside, i.e., in metatheoretical terms, rather than the inside, as Russell would presumably have it (cf., however, Goldfarb 1988).

Poincaré's attack on Cantorian set theory rested on a fundamentally contrary view as to the nature of mathematics. According to him, all mathematical notions have their source in human conceptions which are either given directly in intuition or obtained from such by explicit definition. Mathematical objects do not have an independent Platonic existence, as seems to be required to justify set-theoretic principles (see the next section), and in particular there are no completed infinite totalities. Through this *definitionist* philosophy of mathematics, Poincaré was led to his analysis of the paradoxes and to the proscription of impredicative definitions: for, one must be careful to distinguish apparent definitions from those which are proper. Impredicative definitions characteristically purport to single out an object from a totality of objects by essential reference (either explicit or implicit) to that totality. It this is viewed as the "creation" of such an object by definition, one violates the requirement that the *definiens* must, in all respects, be prior to the *definiendum*.

Poincaré did not elaborate his ideas in any systematic way; in particular, he said nothing about the choice of underlying symbolism for definitions or of underlying logic for mathematical reasoning. The predicativist program followed out his definitionist philosophy without restriction on the logic involved, i.e., it took the classical predicate calculus for granted. The intuitionist program (to be discussed later in this article) took more seriously his views as to the subjective source of mathematics and abandoned the logic of truth and falsity in favor of a logic of what can be established by intuitively evident, constructive means.

Though Russell was the first to formulate predicative principles explicitly in his ramified theory of types RTT, as we saw he compromised these fatally by the introduction of the Axiom of Reducibility. The next substantial advance in the predicativist program was made by Hermann Weyl in 1918, in his monograph *Das Kontinuum*. An extremely broad and deep mathematician of the Hilbert school, Weyl also had long-standing interests in the philosophy of science and mathematics, but the 1918 work was his single foray into a detailed foundational development. Apparently he arrived at a predicativist position independently of Poincaré and Russell. Like Poincaré and unlike Russell, Weyl accepted the natural number system and the associated principles of proof by induction and definition by recursion as basic. He credited Russell with formulation of the vicious-circle principle, while referring to Poincaré's "very uncertain remarks" concerning impredicative definitions. On the other hand, Weyl said that he is separated by a "veritable abyss" from Russell in his (Russell's) attempt to define the natural numbers as equivalence classes under the equinumerosity relation and with his assumption of the Axiom of Reducibility.

For Weyl, the essential first task was to see how much mathematical analysis could be carried out on a strictly predicativist basis, *given* the natural numbers. Since the rational number system can be reduced to the system of natural numbers and since the "continuum" of real numbers can be identified with sequences or sets of rational numbers (à la Cantor or Dedekind), the appropriate principles to consider are those that govern sets or sequences of natural numbers. Thus Weyl erected in *Das Kontinuum* a second-order axiomatic system whose variables of type 0 are interpreted as ranging over the natural numbers, and those of type 1 as ranging over predicatively definable sets, relations and functions of natural numbers. However, unlike Russell, he did not further ramify the objects of type 1 into levels, for he saw that this would lead to difficulties of the same sort that Russell had previously run into for arithmetic (and tried to overcome by resort to the predicatively unjustified Axiom of Reducibility). Speaking informally, one can say that the type 1 variables in Weyl's system range only over level 0 objects of that type in the ramified hierarchy, and the principles of definition of type 1 objects give conditions under which the level 0 objects are closed.

Actually, Weyl's axiom system does not meet modern standards of formalization and there are certain ambiguities that one finds when attempting to reconstruct it in those terms. An exegesis carried out by the author (Feferman 1988) has led to two systems, one of which can be interpreted as just suggested and a second one (still predicatively justified) that requires the introduction of objects of type 2 and yields stronger principles at type 1. We shall content ourselves here with an indication of the weaker system (which is the one customarily ascribed to Weyl).

Beginning with the constant 0 and the successor operation, one may introduce further specific (type 1) functions by explicit and recursive definition. A formula ϕ built up from equations and membership statements without any bound second-order (function, set or relation) variables is said to be *arithmetical*. Then the *Arithmetical Comprehension Axiom* (ACA) for sets of numbers is given by the scheme:

$$\exists X \forall x[x \in X \Leftrightarrow \phi(x)],$$

where ϕ is any arithmetical formula which does not contain 'X' free. (More generally, this scheme is also formulated for relations of any number of arguments). The set X defined by ACA is denoted $\{x \mid \phi(x)\}$.

By the usual type restrictions, set variables in formulas occur only to the right of the membership relation symbol '$\in$'. Then if we substitute for the set variable in such an occurrence, say $t \in Y$, an arithmetical definition $\{x \mid \psi(x)\}$ of Y, the result is there equivalent to $\psi(t)$; it follows that arithmetical formulas are closed under substitution for set (or relation) variables by arithmetical formulas. (This is the closure condition on objects of type 1 and level 0 indicated above.)

Since Weyl assumes classical logic in his system, quantification over the natural numbers is regarded as being truth-functionally determinate. This may be considered to constitute implicit acceptance of the set $\mathbb{N}$ of natural numbers as a *completed infinite totality* (contrary to Poincaré). But there is no acceptance in Weyl's system, either implicitly or explicitly, of a completed totality of subsets of $\mathbb{N}$, contrary to Cantorian set theory.

Rational numbers are introduced in Weyl's system as 4-tuples of natural numbers (x, y, z, w), with $y \neq 0$, $w \neq 0$, representing $(\frac{x}{y} - \frac{z}{w})$, and called "equal" under the expected equivalence relation. Then real numbers are taken to be lower Dedekind sections in the rational numbers $\mathbb{Q}$, i.e. suitable sets of such 4-tuples. By the one-one correspondence between $\mathbb{Q}$ and $\mathbb{N}$, real numbers correspond to certain subsets of $\mathbb{N}$. Then a set S of real numbers is given by a property which corresponds to the property $\phi(Y)$ of subsets Y of $\mathbb{N}$. Now the l.u.b. (least upper bound) of S (when it is bounded above) is in Dedekind's model simply the union of the lower sections which are members of S. Membership in this union corresponds to membership in the union of sets Y satisfying $\phi(Y)$, i.e. it should be a set X satisfying:

$$\forall x[x \in X \Leftrightarrow \exists Y(\phi(Y) \;\&\; x \in Y)].$$

However, the existence of such X is not guaranteed by ACA; in fact, this is *prima facie* a definition of the subset X of $\mathbb{N}$ from the totality of subsets of $\mathbb{N}$, hence is impredicative, and its use cannot in general be justified in Weyl's system. Since the l.u.b. axiom for real numbers is the fundamental principle of analysis, the inability to assert the above would appear to constitute an insuperable obstacle to Weyl's predicativist program for analysis. However, the l.u.b. principle for *sequences of reals* can be derived in his system, and this turns out to be sufficient for most applications. For, a sequence of real numbers corresponds to a sequence $\langle X_0, \ldots, X_n, \ldots \rangle$ of subsets of $\mathbb{N}$, and that is given by a single binary relation R with $x \in X_n \Leftrightarrow R(x, n)$. Then the union of the sets X_n in this sequence is defined by

$$\forall x[x \in X \Leftrightarrow \exists n R(x, n)];$$

the union set X thus exists by ACA. On the basis of this restricted l.u.b. principle, Weyl was able to show in *Das Kontinuum* that the full standard theory of continuous functions of real numbers (which are determined entirely by their values at rational numbers) can be developed in a straightforward way on the basis of his system. What Weyl did not do was show how to deal predicatively with the more modern theories of integration (such as that of Lebesgue) applying to much wider classes of functions, as needed for various applications. This was to be left for later developments of the predicative approach, which was not taken up again in a systematic way until the 1950's. (Cf. e.g. Feferman 1964). Weyl himself shifted his views two years after the publication of *Das Kontinuum*, in the direction of the still more radical approach of the intuitionist L. E. J. Brouwer (1887–1966); however, in later years he became pessimistic about the prospects for the Brouwerian revolution, though he remained sympathetic with its underlying philosophy. On the other hand, Weyl never completely disavowed his own program for predicative foundations, which he continued to mention in his articles on the philosophy of mathematics over the years.

7 Zermelo's axioms for set-theoretical foundations.

The remainder of this article concerns the three other most significant programs for the foundations of mathematics which were developed in the first third of the 20th century: set-theoretical foundations via axiomatic set theory, the intuitionist program of constructive mathematics, and Hilbert's finitist consistency program. Because of their continued importance, these are all treated at length in separate chapters of the Encyclopedia. Thus we content ourselves here with a relatively brief introduction to each, with emphasis on their philosophical differences.

According to Cantor, any two sets A and B can be compared as to their cardinality, i.e. $\mathrm{card}(A) \leq \mathrm{card}(B)$ or $\mathrm{card}(B) \leq \mathrm{card}(A)$. His argument for this made use of the so-called *Well-ordering Principle* (WO), according to which the elements of any set can be laid out in a transfinite sequence $\langle X_0, X_1, \ldots, X_\omega, X_{\omega+1}, \ldots \rangle$ in which each non-empty subset has a first element. Cantor also demonstrated that for each set A there is a set B whose cardinality is larger than that of A, i.e. $\mathrm{card}(A) < \mathrm{card}(B)$, namely the *set of all subsets* X of A, written $\{X \mid X \subseteq A\}$. This is also called the power set of A, and denoted $\mathcal{P}\ (A)$, since in the arithmetic of cardinals, card($\{X \mid X \subseteq A\}$) $= 2^{card(A)}$ for arbitrary A. Now by WO, we have a complete transfinite scale of infinite cardinals, for which Cantor used the Hebrew letter $\aleph$ (*aleph*) with subscripts $0, 1, \ldots, \omega, \omega + 1, \ldots$; $\aleph_0 = \text{card}(\mathbb{N})$ is the cardinal of any countably infinite set (e.g. also the set of rational numbers), and $\aleph_1$ is the least uncountable cardinal. Since $\aleph_0 < 2^{\aleph_0}$ it follows that $\aleph_1 \leq 2^{\aleph_0}$. The number $2^{\aleph_0}$ is not only the cardinality of $\mathcal{P}(\mathbb{N})$, it is also the cardinality of the set $\mathbb{R}$ of all real numbers (via their correspondence with Dedekind subsets of the rational numbers); thus $2^{\aleph_0}$ is also called the *cardinality of the continuum.* An immediate question to raise from the above is whether $2^{\aleph_0} = \aleph_1$ is true; Cantor's conjecture that that is the case is called the *continuum hypothesis* (CH):

$$2^{\aleph_0} = \aleph_1$$

To this day CH remains the basic unsettled question in Cantor's theory of cardinal numbers.

Evidently, Cantor took for granted that for any infinite set A, the power set $\mathcal{P}(A) = \{X \mid X \subseteq A\}$ is a definite completed totality with a definite cardinal number. Moreover, according to his well-ordering principle, there must be some way of setting up a well-ordering of $\mathcal{P}(A)$ as $\langle X_0, X_1, \ldots, X_\omega, X_{\omega+1}, \ldots \rangle$, by making a transfinite sequence of arbitrary choices of subsets of A until $\mathcal{P}(A)$ is exhausted. There is no known procedure for well-ordering $\mathcal{P}(\mathbb{N})$, let alone $\mathcal{P}(A)$ for any other infinite set (e.g., $\mathcal{P}(\mathbb{R})$ or $\mathcal{P}(\mathcal{P}(\mathbb{N}))$, etc.). Despite the intuitive appeal to many mathematicians of Cantor's notions and arguments, even those sympathetic to his ideas found the well-ordering principle to be problematic.

The emergence of the set-theoretical paradoxes added fuel to the concerns about the security of the subject, though one might dismiss the sets involved in these cases (of "all sets," or "all cardinals," or "all ordinals") as somehow marginal rather than central like the continuum. In addition, there was a group of influential mathematicians, beginning with Kronecker, who were extremely critical of Cantor's theory because of its "metaphysical" basis, i.e., because of the underlying Platonistic philosophy of mathematics that it took for granted: sets are supposed to be entities existing independently of human constructions and definitions.

No wonder, then, that in the list of twenty-three major unsolved problems presented by David Hilbert (1862–1943) in his famous lecture for the 1900 International Congress of Mathematicians, the first two concerned the foundations of mathematics: namely, the continuum problem and the consistency of an axiom system for the real numbers. The second problem was presumably motivated by the aim to ensure the security of mathematical analysis, no matter what inconsistencies one might meet at the fringes of set theory. In his discussion of the first problem, Hilbert said he was not convinced by Cantor's argument for the well-ordering principle WO, and he reformulated CH in a way that did not depend on it, namely: if $S \subseteq \mathbb{R}$ and S is infinite, then $S \sim \mathbb{N}$ or $S \sim \mathbb{R}$. Hilbert thought there should also be a "direct proof of this remarkable statement [WO]" perhaps by "actually giving" a well-ordering of $\mathbb{R}$.

At the turn of the century the analyst Ernst Zermelo accepted a position at the university in Göttingen, where he came under Hilbert's influence. Before long he had shifted his attention to the foundations of set theory and in particular to WO. In 1904, Zermelo published an article in which he introduced a new principle called the *Axiom of Selection* or *Axiom of Choice* (AC), from which he proved the well-ordering principle WO. Zermelo's 1904 statement of AC was that for any set A there is a function f on the collection of all non-empty subsets of A which chooses an element of each such subset, i.e., for which $f(X) \in X$ for each non-empty $X \subseteq A$. In effect, AC postulated the existence of simultaneous choices which could be used to make the required successive choices in setting up a well-ordering of A, a step that could be deemed progressive only if AC were regarded as evident in a way that WO is not.

Zermelo's 1904 paper provoked intense criticism, both in his assumption of AC and in his use of other set-theoretical concepts and principles. Some critics mistakenly thought that a form of the Burali-Forti paradox was involved in Zermelo's argument. But the main criticism of AC was that a function f is asserted to *exist* without any means available to explicitly define or construct it (except for those A which are already well-ordered). Indeed, it is easily seen that WO implies AC, so they are equivalent principles. Later, Zermelo found a modified but equivalent form of AC which appears intuitively clearer. It asserts that if S is any collection of disjoint non-empty sets, then there is a function f on S which assigns to each X in S a member of X; this form of AC is more readily visualized than the original one and is considered by many to be intuitively evident, despite the fact that it is no more constructive. (Cf. Moore 1982 for a thorough history of the Axiom of Choice).

In order to respond to the various objections to AC and his proof of WO from it, as well as to make clear what other set-theoretical principles were involved, Zermelo published in 1908 the first axiomatic system of set theory. Intuitively speaking, the universe of discourse of this theory consists of some basic underlying objects, called *urelements*, and sets "built up" out of these objects. Sets are supposed to exist independently of any means of defining them, so the *Axiom of Extensionality* is assumed, i.e. $\forall A \forall B[\forall x(x \in A \Leftrightarrow x \in B) \Rightarrow A = B]$. Zermelo's axioms also assert the existence of various sets, such as the empty set 0, and the *singleton* of any $x, \{x\}$. Further axioms allow one to form the *union* of any two sets, $A \cup B$, and the *power set* of any set, $\mathcal{P}(A)$. The existence of an infinite set is postulated by having an A which contains 0 and which satisfies $\forall x(x \in A \Rightarrow \{x\} \in A)$; by taking $sc(x) = \{x\}$ one can identify the set $\mathbb{N}$ of natural numbers with the smallest subset of A containing 0 and closed under this successor operation. More generally, Zermelo posited a restricted form of the comprehension axiom, called the *Axiom of Separation (Aussonderungsaxiom)*, which says that for any set A and any "definite property" $P(x)$ of elements of A we can form the subset B of A consisting exactly of those $x \in A$ satisfying $P(x)$; in symbols, $B = \{x \in A \mid P(x)\}$.

An informal model of these axioms was first explained by Zermelo in 1930. Let V_0 be the set of urelements and let for each $n < \omega, V_{n+1} = V_n \cup \mathcal{P}(V_n)$; then take $V_\omega = \bigcup_{n<\omega} V_n$ and continue again with $V_{\alpha+1} = V_\alpha \cup \mathcal{P}(V_\alpha)$ for $\alpha = \omega + n$. The model is given finally as $V_{\omega+\omega} = \bigcup_{n<\omega} V_{\omega+n}$. Note that $0 \in V_1$ and if $x \in V_n$, then $\{x\} \in V_{n+1}$. Hence $\mathbb{N} = \{0, \{0\}, \{\{0\}\}, \ldots\} \subseteq V_\omega$ and so $\mathbb{N} \in V_{\omega+1}$. This informal model is called the *cumulative hierarchy* since at each stage α we combine everything obtained up to that stage, V_α, with all subsets of everything so far obtained, $\mathcal{P}(V_\alpha)$. This contrasts with the informal model for simple type theory where one takes $S_0 = V_0$ and $S_{n+1} = \mathcal{P}(S_n)$ for each n. There is no natural way to carry that further into the transfinite without taking $S_\omega = \bigcup_{n<\omega} S_n$ and thus cumulating the objects together; if that is to be done, one might as well do it at each stage.

Zermelo's system is *type-free* in the sense that it is meaningful to consider whether $x \in y$ for any objects x and y; in particular, one can ask whether $x \in x$ holds. However, there is no (evident) way to derive Russell's paradox in his system because of the restricted nature of his comprehension axiom, the Axiom of Separation. From it we can only conclude that for any set A we can form the set $B = \{x \in A \mid x \notin x\}$; but in this case we merely draw the perfectly consistent conclusion that $B \notin A$ (for if $B \in A$, then $B \in B \Leftrightarrow B \notin B$). It follows that there is no set of *all* sets in Zermelo's system. Similarly there is no set of all ordinals or of all cardinals, so the paradoxes of Burali-Forti and of Cantor are also blocked. Of course, it is still conceivable that Zermelo's axiom system is inconsistent; what the preceding shows is that there is no obvious inconsistency to be obtained by reproducing one of the familiar antinomies. On the other hand, there is no means in Zermelo's system to treat cardinal numbers as equivalence classes under $\sim$ à la Frege and Russell, since one cannot establish the existence of the set of *all* sets equivalent to a given set. In Zermelo-style axiomatic systems alternative means must therefore be prescribed for identifying cardinal numbers (and, similarly, ordinal numbers) with specific sets; that was accomplished in the simplest way in the late 1920's by John von Neumann (1903–1957).

Later improvements and extensions of Zermelo's set theory were given by Thoralf Skolem (1887–1963) and Abraham Fraenkel (1891–1965), Paul Bernays (1888–1977) and Kurt Gödel (1906–1978). Thus one speaks nowadays of the axiom systems ZF (Zermelo–Fraenkel) and BG (Bernays–Gödel). With the Axiom of Choice added, these provide extremely flexible systems in which all of Cantorian set theory can be represented without any apparent source of inconsistency. In that respect, they are far superior to theories of types and have superseded them as a foundation for that part of modern mathematical practice which makes essential use of set-theoretical concepts and principles. (See the chapter on set theory in this Encyclopedia for further detail on these systems).

Zermelo's axiomatization and its descendants brought out more clearly than ever the grounding of set theory in a Platonistic conception of mathematics. The ingredients of the latter include the views that: (i) Sets are entities existing independently of human thoughts and constructions, which, though abstract, are supposed to be part of an external, objective reality; (ii) infinite sets such as the natural numbers and the real numbers are supposed to exist as actual, complete objects; (iii) for each set, the totality of arbitrary subsets of that set exists as a definite, completed set; (iv) every proposition about sets has a definite truth value (true or false), independent of any means we may have to verify it.

It should be noted that acceptance of the general Platonistic position, under which mathematical objects are viewed as independently existing (abstract) entities, does not necessarily commit oneself to any or all of (i)–(iv). It is for this reason that acceptance of (i)–(iv) is called *set-theoretical Platonism.* Among the statements readily granted on this position are the Axioms of Extensionality, Infinity, Power Set, Separation and Choice; moreover, in logic one accepts the *Law of Excluded Middle*, according to which for any proposition ϕ, either ϕ *or* $\neg\phi$ holds, in symbols $\phi \vee \neg\phi$. As we have seen in Section 6, the use of the Power Set Axiom applied to $\mathbb{N}$, together with the Axiom of Separation for arbitrary properties formulated in the language of set theory, leads to impredicative conclusions. The acceptance of classical logic embodied in the Law of Excluded Middle leads directly to non-constructive existence results by the method of proof by contradiction. This was one of the features seized on in the constructivist critique of set-theoretical Platonism and in the development of an opposing foundational program, which we take up next.

8 *Brouwer's intuitionism.*

The earliest critic of Cantorian set theory from the constructivist standpoint was Leopold Kronecker, Cantor's former teacher. Beyond what is finite, he would admit only *potentially* infinite sets to mathematics and, of such, only those reducible to the natural number sequence $0, 1, 2, \ldots$; furthermore, he would countenance only constructive existence proofs. His attacks on Cantor's work in the latter part of the 19th century were quite severe and had a deleterious personal effect on Cantor. Henri Poincaré was, as we already mentioned, another critic of set theory and logicism who is considered to be a precursor of the intuitionistic brand of constructivism in certain respects, though not consistently so, nor in any systematic way. More or less contemporary with him were the so-called "semi-intuitionists", comprising the French mathematicians Emile Borel (1871–1956), René Baire (1874-1932), and Henri Lebesgue (1875–1941) among others. They accepted countably infinite sets and constructions and the transfinite iteration of such up to the least uncountable ordinal. The semi-intuitionists did not accept the Axiom of Choice, though some forms of this were unwittingly involved in their mathematics (cf. Moore 1982). And while they thought all existence proofs should be obtained by explicit construction or definition, they did not argue for the restriction of the logic employed.

It was the Dutch mathematician L. E. J. Brouwer who first focused attention on the question of justifiability of the *Law of Excluded Middle* (LEM) for a constructivist view of mathematics. In his 1908 article "The unreliability of logical principles," Brouwer argued that LEM for infinite sets is based on an unjustified extension of that principle from finite sets. In his doctoral dissertation of 1907, he had already insisted on the subjective origin of mathematics in human intuition, and on the necessity to restrict questions of truth in mathematics to those statements which can be verified or disproved. Of course, for a finite set A and decidable P we can verify $\exists x P(x) \vee \forall x \neg P(x)$ by testing each $x \in A$ in turn to see whether or not $P(x)$ holds. But in general there is no way to carry out such a verification when A is infinite, even for decidable P.

From 1908–1913 Brouwer turned away from foundational questions to the subject of topology, to which he made deep and important contributions that established his credentials as a mathematician of the first rank. Then from 1918 until the last decade before his death (in 1966) Brouwer set out to redevelop mathematics entirely on *intuitionistic* grounds (the term he used for his form of constructivism). Not only was the Law of Excluded Middle to be rejected, but also all forms of the completed infinite. To verify $\exists x P(x)$ for *'x'* ranging over an infinite set S, one must actually produce some $a \in S$ for which $P(a)$ is proved; on the other hand, to verify $\forall x P(x)$ one must have a proof which shows how, given any $a \in S$, one may produce a proof of $P(a)$. The latter justifies proof by induction on the natural numbers: given $P(0)$ and $\forall x(P(x) \Rightarrow P(sc(x))$ one recognizes that one has a procedure which, given any $n \in \mathbb{N}$, establishes $P(n)$, first from $P(0) \Rightarrow P(1)$, then from

$P(1) \Rightarrow P(2), \ldots$, until we reach $P(n-1) \Rightarrow P(n)$.

The first obstacle to the straightforward redevelopment of mathematics according to intuitionistic tenets arises not in the theory of arithmetic (i.e., of $\mathbb{N}$), but in the theory of real numbers. To be sure, real numbers could be identified with potentially infinite Cauchy sequences of rationals, but what would it mean to deal with "arbitrary" such sequences? It turns out that if one restricts attention to just those sequences which are determined by effective laws, all sorts of anomalies appear. Brouwer introduced instead a novel conception, that of (free) *choice sequences*, such as might be determined in non-lawlike ways by a series of free choices or random acts, and of which one would have only finite partial information at any stage. (Brouwer was anticipated in this conception of the continuum by Borel). With real numbers viewed as Cauchy choice sequences of rationals, a function from reals to reals can be determined using only a finite amount of information about any argument, in order to determine its value to any given degree of precision. Following this line of reasoning, Brouwer came to the conclusion that any such function must be *continuous*, in direct contradiction to the classical existence of discontinuous functions. Increasingly, Brouwer found himself forced to introduce unconventional notions and to reach strange conclusions, which, despite his stature as a mathematician, were received with general incomprehension and/or rejection by the mathematical community at large. Nevertheless, the metamathematical study of intuitionism, initiated by his student and disciple, Arend Heyting (1898–1980) sparked later development of other systematic forms of constructivism. These are detailed in the chapter, "History of constructivism in the 20th century," in this Encyclopedia. (cf. also Heyting 1971, and Troelstra and van Dalen, 1988).

9 *Hilbert's finitist consistency program.*

When Hilbert delivered his address "Mathematische Probleme" at the International Congress in 1900 he was already acknowledged to be one of the world's greatest mathematicians of his time. At mid-career, he had, by then, made fundamental contributions to algebra, number theory, geometry and analysis; in the years to come he would make further important contributions to analysis, mathematical physics and the foundations of mathematics. The list of twenty-three problems in his 1900 address covered considerable parts of mathematics and have been the source of a great deal of effort and attention up to the present day; those who solved or made substantial progress on one or another of Hilbert's problems were guaranteed fame in the mathematical community (cf. Browder 1976).

As has already been mentioned, the first two of Hilbert's problems directly concerned the foundations of mathematics. Hilbert's program to establish the consistency of axiomatic systems was first expressed in more specific terms in Problem 2 with its call for a proof of the consistency of a system of axioms for the real numbers. This had its origins in Hilbert's work in earlier years on the axiomatic foundations of geometry. In that project, Hilbert had returned to Euclid's axioms, first to improve them to meet modern standards of rigor, and then to raise new, metatheoretical questions, such as their independence and consistency. He demonstrated the consistency of Euclidean plane geometry by its interpretation in the Cartesian plane, given by pairs of real numbers, and for the Euclidean geometry of space, in the three-dimensional real number coordinate system.

Hilbert's call for a proof of the consistency of the real number system thus went a step beyond that for geometry. His statement of Problem 2 shows why he thinks this is necessary: According to him, the foundations of any science must provide an *exact* and *complete* system of axioms; moreover, a mathematical concept *exists* if, and only if, such a system of axioms can be shown to be *consistent*. In particular, "the proof of the compatibility of the axioms [for real numbers] is at the same time the proof of the mathematical existence of the complete system of real numbers". What Hilbert did not provide at the time was any suggestion as to how such a consistency proof might be carried out. Evidently, he believed that it would be straightforward, and that one would then go on to demonstrate the existence of Cantor's transfinite number classes by a consistency proof, "just as that of the continuum." But in a 1904 article he presented a less optimistic prospect for that extension, in view of the set-theoretical paradoxes.

During the subsequent years, through World War I, Hilbert's attention was partially drawn aside by work on integral equations and mathematical physics. However, he returned in print to foundational questions in his 1918 address "Axiomatic thought", where he began to lay out his mature program for the foundations of mathematics; this was then elaborated in a succession of publications through 1931. Hilbert's ultimate aim was to justify the use of Cantorian notions and methods in mathematics. The avenue for doing this would be provided by a consistency proof of a Zermelo-style axiom system for set theory. But Hilbert was sensitive to the criticisms of Kronecker and Brouwer and recognized that a genuine justification of this sort of the actual infinite would have to be based on methods from which all infinitary ingredients were drained. Hilbert did not think this could be accomplished all at once, but would have to be carried out step by step, first for an axiomatic theory of arithmetic, then for analysis and eventually for set theory.

In more detail, Hilbert's program was conceived of as follows. A given informal body of mathematics is to be treated as formally represented in an axiomatic theory T, which is to be specified precisely within a formal language L. This language is to be given by a stock of basic symbols and by rules for building up from them the well-formed formulas (or statements) of L as finite sequences of basic symbols. Certain of those formulas are then to be specified as axioms, including both those of a general logical character and those concerning the intended subject matter of T; in addition, there are to be given rules of inference for constructing formal proofs (or derivations) from the axioms, as finite sequences of formulas. Under these hypotheses, it can be effectively decided, for each finite sequence σ of finite sequences of basic symbols and each formula ϕ, whether or not σ is a proof of ϕ. Once a formal axiomatic theory T is presented in such a way, the set of provable formulas of T is determined as the end results of such proofs, and T is consistent if no formal contradiction $\phi \ \& \neg\phi$ is provable from T. The statement of the consistency of T is purely *finitary* in the sense that it refers only to possible finite configurations σ given as finite sequences of terms, each of which is a finite sequence of symbols of L. Hilbert's program to produce uncontrovertible justification of such T required that the consistency proof of T be carried out by purely finitary methods.

In addition to his general program, Hilbert also proposed some specific proof-theoretical techniques to carry it out. As he saw the matter, the actual infinite already arises in classical arithmetic in formulas such as $\exists x R(x) \vee \forall x \neg R(x)$, where R is decidable, since this implicitly involves the possibility of surveying the totality of natural numbers (just as Brouwer had argued). Purely finitary statements would be those of the form $R(x)$ with R decidable and 'x' an open free variable, which could be verified for each specific instance $R(a)$ in a domain of finite objects such as the natural numbers. Hilbert's idea was that a proof π of such $R(x)$ should somehow be transformed by the succession elimination of quantified statements in π into a proof $\pi\prime$ of $R(x)$, all of whose statements are of this finitary form. Hilbert's *Ansatz* showed how this might be carried out in relatively simple cases, but he left it to his assistants and co-workers in Göttingen — especially to Paul Bernays, Wilhelm Ackermann (1896–1962), and John von Neumann — to carry out the *Beweistheorie*, or theory of proofs, for his program. The initial target would be a consistency proof for a first-order version PA of Peano Arithmetic. But after some missteps by Ackermann (who thought he had obtained not only a consistency proof for PA but one for a theory of analysis as well), von Neumann realized that only the consistency of a fragment of PA had actually been established by clearly finitary methods.

That this was no temporary failure emerged from the stunning results of Kurt Gödel in 1931 (cf. Gödel 1986), whereby if T is any finitarily presented formal theory T which includes PA, the consistency of T cannot be proved by methods that can be formalized in T, unless T is already inconsistent. Moreover, it appeared that all finitary methods of the sort that had been employed in the Hilbert school could readily be formalized in PA. Thus, if one could not come up with essentially new finitary methods that go beyond PA one could not hope to give a finitary consistency proof of PA (assuming that it is, indeed, consistent–which could hardly be doubted). Just such methods were later proposed by Gerhard Gentzen (1909–1945) in 1936, by the use of transfinite induction up to certain finitely described countable ordinals, applied to statements of purely finitary form. However, Gentzen's extension of finitary mathematics is controversial and it is by no means generally agreed whether his proof of the consistency of PA meets Hilbert's criteria. At any rate, it can hardly be doubted that all finitary methods can be formalized in a (relatively weak) part T of Zermelo set theory, and for such T Gödel's theorem on the unprovability of consistency constitutes a definitive obstacle to Hilbert's program. For the further history of this subject, see the chapters on Gödel's incompleteness theorems and the theory of proofs (or demonstrations) in this Encyclopedia.

Hilbert himself refused to concede (in his Preface to Hilbert and Bernays 1934) that Gödel's theorem signaled the breakdown of his program. But he should already have taken cognizance of another result which undermined his overall views. Namely, the theorem of Löwenheim–Skolem, established by 1920, showed that any theory T couched in first-order logic which has an infinite model must have a denumerable model. Thus no such T could be a categorical axiom system for the real numbers, i.e., would uniquely determine the real number concept up to isomorphism. To obtain a categorical system of axioms for the reals one would have to use essentially infinitary higher-order axioms such as the statement of Cauchy completeness; but this would take us out of the realm of finitary formal axiomatic systems, contrary to the requirements of Hilbert's program.

What Hilbert did achieve in his teachings on logic and foundations (see Hilbert and Ackermann 1928) was the final transformation of the subject — whose initial aim, in the hands of Frege and Russell, had been to set up a global, universal system of logic encompassing all of mathematics — to one in which the objects of interest would be various individual "local" axiomatic theories T for different parts of mathematics, to be treated by mathematically rigorous methods in the emerging and many-faceted discipline of *mathematical logic* or *metamathematics*. Within this, moreover, the theory of proofs initiated by Hilbert has developed into a substantial and technically sophisticated part of mathematical logic, in which *relativized* forms of Hilbert's program have been successfully pursued (cf. Feferman 1988, Schütte 1977, and Takeuti 1987).

References

Brouwer, L. E. J., *Collected Works*, vol. 1, edited by A. Heyting, North–Holland, Amsterdam, 1975.

Browder, F. (edited by), *Mathematical developments arising from Hilbert problems*, Proc. of Symposia in Pure Mathematics, vol. 28, American Math. Soc., Providence RI, 1976.

Cantor, G., *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts*, edited by E. Zermelo, Springer, Berlin, 1932.

Dedekind, R., *Was sind und was sollen die Zahlen*? Brunswick, 1888.

Dummett, M., *Frege: Philosophy of Mathematics*, Duckworth, London, 1991.

Feferman, S., Systems of predicative analysis, *J. Symbolic Logic* 29, 1964, 1–30.

Feferman, S. Hilbert's program relativized: Proof-theoretical and foundational reductions, *J. Symbolic Logic* 53, 1988, 364–384.

Feferman, S., Weyl vindicated: *Das Kontinuum* 70 years later, in *Temi e prospettive della logica e della filosofia contemporanee I*, CLUEB, Bologna, 1988, pp. 59–93.

Fraenkel, A. and Y. Bar–Hillel, *Foundations of set theory*, North–Holland, Amsterdam, 1958.

Frege, G., *Begriffsschrift, eine der arithmetischen nachgebildete Formel-sprache des reinen Denkens*, Nebert, Halle, 1879; English translation by S. Bauer–Mengelberg in (van Heijenoort, 1967), pp 1–82.

Frege, G., *Die Grundlagen der Arithmetik, eine logisch-mathematische Untersuchung über den Begriff der Zahl*, Breslau, 1884; English translation (with the German text) by J. L. Austin in *The foundations of arithmetic. A logico-mathematical inquiry into the notion of number*, Blackwell, Oxford, 1950.

Frege, G., *Grundgesetze der Arithmetik, begriffsschriftlich abgeleitet*, vol. I, Olms, Hildesheim, 1893, and vol. II, Pohle, Jena, 1903; partial English translation by M. Furth in *The basic laws of arithmetic. Exposition of the system*, Univ. of California Press, Berkeley, 1964.

Gödel, K., *Collected Works*, vol. I. Publications 1928–1936, edited by S. Feferman, J. W. Dawson, Jr., S. C. Kleene, G. H. Moore, R. M. Solovay and J. van Heijenoort, Oxford Univ. Press, New York, 1986.

Goldfarb, W., Poincaré against the logicists, in *History and Philosophy of Modern Mathematics*, edited by W. Aspray and P. Kitcher, Univ. of Minnesota Press, Minneapolis, 1988, pp. 61–81.

Hallett, M., *Cantorian set theory and the limitation of size*, Oxford Univ. Press, Oxford, 1984.

Heyting, A., *Intuitionism. An Introduction* (3d edn.) North-Holland, Amsterdam, 1971.

Hilbert, D., Axiomatisches Denken, *Mathematische Annalen*, vol. 78, 1918, pp. 405–415.

Hilbert, D., Über das Unendliche, *Mathematische Annalen*, vol. 95, 1926, pp. 161–190; English translation by S. Bauer-Mengelberg, in (van Heijenoort, 1967), pp. 464–479.

Hilbert, D. and W. Ackermann, *Grundzüge der theoretischen Logik*, Springer, Berlin, 1928; English translation by L. M. Hammond *et al.*, *Principles of mathematical logic*, Chelsea, New York, 1950.

Hilbert, D. and P. Bernays, *Grundlagen der Mathematik*, vol. I, Springer, Berlin, 1934; second edition, 1968.

Moore, G., *Zermelo's axiom of choice: Its origins, development and influence*, Springer, New York, 1982.

Parsons, C., Mathematics, foundations of, in *Encyclopedia of Philosophy*, vol. 5, edited by P. Edwards, Macmillan, New York, 1967, pp 188–213.

Peano, G., *Arithmetices principia, nova methodo exposita*, Turin, 1889, English translation in (van Heijenoort, 1967), pp. 83-97.

Poincaré, H., *La science et l'hypothèse*, Flammarion, Paris, 1902.

Poincaré, H., *Dernières pensées*, Flammarion, Paris, 1913.

Ramsey, F. P., *The foundations of mathematics, and other logical essays*, edited by R. B. Braithwaite, Paul, Trench,Trubner, London, 1931, reprinted 1950 (Humanities Press, New York).

Richard, J., Les principes des mathématiques et le problème des ensembles, *Revue générale des sciences pure et appliquées*, vol. 16, p. 541; English translation in (van Heijenoort, 1967), pp. 142–144.

Russell, B., *The principles of mathematics,* Cambridge Univ. Press, Cambridge, 1903; second edition, Norton, New York (no date).

Russell, B., Mathematical logic as based on the theory of types, *Amer. J. of Mathematics*, vol, 30, 1908, pp 222–262, reprinted in (van Heijenoort, 1967), pp. 150–182.

Schütte, K. *Proof Theory*, Springer Verlag, Berlin, 1977.

Takeuti, G., *Proof Theory* (2nd edn.), North-Holland, Amsterdam, 1987.

Troelstra, A. S., and D. Van Dalen, *Constructivism in Mathematics. An introduction*, Volume I, North-Holland, Amesterdam, 1988.

Van Heijenoort, J. *From Frege to Gödel: A source book in mathematical logic*, 1879–1931, Harvard Univ. Press, Cambridge MA, 1967.

Von Neumann, J., Zur Hilbertschen Beweistheorie, *Mathematische Zeitschrift*, vol 26, 1927, pp. 1–46.

Weyl, H., *Das Kontinuum. Kritische Untersuchungen über die Grundlagen der Analysis*, Veit, Leipzig, 1918; Italian translation by B. Veit, *Il continuo. Indagine critiche sui fondamenti dell'analisi*, Bibliopolis, Napoli, 1977; English translation by S. Pollard and T. Bole, *The continuum. A critical examination of the foundations of analysis*, Thomas Jefferson Univ. Prss, Kirksville, 1987.

Weyl, H., *Philosophy of mathematics and natural science*, Princeton Univ. Press, Princeton, 1949.

Whitehead, A. N. and B. Russell, *Principia Mathematica*, vols 1/2/3, Cambridge Univ. Press, Cambridge, 1910/12/13; second edition 1925/27/27.

Zermelo, E., Beweis, dass jede Menge wohlgeordnet werden kann, *Mathematische Annalen*, vol. 59, 1904, pp. 514–516; English translation in (van Heijenoort, 1967), pp. 139–141.

Zermelo, E., Untersuchungen über die Grundlagen der Mengenlehre I, *Mathematische Annalen*, vol. 65, 1908, pp. 261–281; English translation in (van Heijenoort, 1967), pp. 199–215.

Zermelo, E. Über Grenzzahlen und Mengenbereiche, *Fundamenta Mathematicae*, vol. 16, 1930, pp. 29–47.