Which Quantifiers are Logical?

A combined semantical and inferential criterion

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Abstract. The aim of logic is to characterize the forms of reasoning that lead invariably from true sentences to true sentences, independently of the subject matter; thus its concerns combine semantical and inferential notions in an essential way. Up to now most proposed characterizations of logicality of sentence generating operations have been given either in semantical or inferential terms. This paper offers a combined semantical and inferential criterion for logicality (improving one originally proposed by Jeffery Zucker) and shows that any quantifier that is to be counted as logical according to that criterion is definable in first order logic.

The aim of logic is to characterize the forms of reasoning that lead invariably from true sentences to true sentences, independently of the subject matter. The sentences involved are analyzed according to their logical (as opposed to grammatical) structure, i.e. how they are compounded from their parts by means of certain operations on propositions and predicates, of which the familiar ones are the connectives and quantifiers of first order logic. To spell this out in general, one must explain how the truth of compounds under given operations is determined by the truth of the parts, and characterize those forms of rules of inference for the given operations that insure preservation of truth. The so-called problem of "logical constants" (Gomez-Torrente 2002) is to determine all such operations. That has been pursued mostly via purely semantical (*qua* set-theoretical) criteria on the one hand—stemming from Tarski (1986)—and purely inferential criteria on the other—stemming from Gentzen (1935) and pursued by Prawitz (1965), among others—even though on the face of it a combination of the two is required.[2] What is offered here is such a combined criterion for quantifiers, whose semantical part is provided by Lindström's (1966) generalization of quantifiers, and whose inferential part

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[2] Some further contributions to the semantical approach are Sher (1991) and McGee (1996), and to the inferential approach is Hacking (1979); Gomez-Torrente (2002) provides a useful survey of both approaches. I have critiqued the semantical approach as given by set-theoretical criteria in Feferman (2000, 2010) where, in conclusion, I called for some combined criterion.

is closely related to one proposed by Zucker (1978).[3] On the basis of this criterion it is shown that any quantifier that is to be counted as logical is definable in classical first order logic (FOL). In addition, part of the proof idea is the same as that provided by Zucker, but his proof itself needs to be corrected in at least one essential respect that will be explained below; fixing that up is my main contribution here in addition to elaborating the criterion for logicality.

One basic conceptual difference that I have with Zucker is that he regards the meaning of a quantifier to be given by some axioms and rules of inference, provided those uniquely determine it on an inferential basis, whereas I assume that its meaning is specified semantically; that is the viewpoint both of workers in model-theoretic logics (cf. Barwise and Feferman 1985) and of workers on quantifiers in natural language (cf. Peters and Westerståhl 2006). For Zucker's point of view, see the Discussion below.

Given a non-empty universe of discourse U and $k \geq 1$, a k-ary relation on U is simply a subset P of U^k; we may also identify such with k-ary "propositional" functions P: $U^k \rightarrow \{t, f\}$, where t and f are the truth values for truth and falsity, respectively. $P(x_1,\ldots,x_k)$ may thus be read as "P holds of $(x_1,\ldots,x_k)$" or as "$P(x_1,\ldots,x_k)$ is true."

Q is called a *(global) quantifier* of type $\langle k_1,\ldots,k_n\rangle$ if Q is a class of relational structures of signature $\langle k_1,\ldots,k_n\rangle$ closed under isomorphism. A typical member of Q is of the form $\langle U,P_1,\ldots,P_n\rangle$ where U is non-empty and P_i is a k_i-ary relation on U. Given Q, with each U is associated the *(local) quantifier* Q_U on U which is the relation $Q_U(P_1,\ldots,P_n)$ that holds between $P_1,\ldots,P_n$ just in case $\langle U,P_1,\ldots,P_n\rangle$ is in Q. Alternatively we may identify Q_U with the associated functional from propositional functions of the given arities on U to $\{t,f\}$.

Examples of such quantifiers may be given in set-theoretical terms without restriction. Common examples are the uncountability quantifier of type $\langle 1\rangle$, the equi-cardinality quantifier of type $\langle 1, 1\rangle$, and the "most" quantifier of type $\langle 1, 1\rangle$. However, even though the definitions of those refer to the totality of relations of a certain sort

[3] An unjustly neglected paper, along with Zucker and Tragesser (1978).

(namely 1-1 functions), all quantifiers in Lindström's sense satisfy the following principle:

Locality Principle. Whether or not $Q_U(P_1,\ldots,P_n)$ is true depends only on U and $P_1,\ldots,P_n$, and not on what sets and relations exist in general over U.

As shown by Lindström, given any first-order language L with some specified vocabulary of relations, functions and constant symbols, we may add Q as a formal symbol Q to be used as a new constructor of formulas φ from given formulas ψ_i, $i = 1,\ldots,n$. For each i, let $\underline{x}_i$ be a k_i-ary sequence of distinct variables such that $\underline{x}_i$ and $\underline{x}_j$ are disjoint when $i \neq j$, and let $\underline{y}$ be a sequence of distinct variables disjoint from all the $\underline{x}_i$. The syntactical construction associated with Q takes the form

$$\varphi(\underline{y}) = Q\underline{x}_1\ldots\underline{x}_n(\psi_1(\underline{x}_1,\underline{y}),\ldots,\psi_n(\underline{x}_n,\underline{y}))$$

where the $\underline{x}_i$ are all bound and the free variables of φ are just those in $\underline{y}$. The satisfaction relation for such in a given L-model $\mathcal{M}$ is defined recursively: for an assignment $\underline{b}$ to $\underline{y}$ in U, $\varphi(\underline{b})$ is true in $\mathcal{M}$ iff $(U, P_1,\ldots,P_n)$ is in Q when each P_i is taken to be the set of k_i-tuples $\underline{a}_i$ satisfying $\psi_i(\underline{a}_i,\underline{b})$ in $\mathcal{M}$.

Next what is needed to bring inferential considerations into play is to explain which quantifiers have axioms and rules of inference that completely govern its forms of reasoning. It is here that we connect up with the inferential viewpoint, beginning with Gentzen (1935). Remarkably, he showed how *prima facie* complete inferential forms could be provided separately for each of the first-order connectives and quantifiers, whether thought of constructively or classically, via the *Introduction* and *Elimination Rules* in the calculi NJ and NK, resp., of natural deduction. In addition, he first formulated the idea that the *meaning* of each of these operations is given by their characteristic inferences. Actually, Gentzen claimed more: he wrote that "the [Introduction rules] represent, as it were, the 'definitions' of the symbols concerned." (Gentzen 1969, p. 80). Prawitz put teeth into this by means of his Inversion Principle

(Prawitz 1965, p. 33): namely, it follows from his normalization theorems for NJ and NK that each Elimination rule for a given operation in either calculus can be recovered from the appropriate one of its Introduction rules when that is the last step in a normal derivation.

As I have stated above, in my view the meaning of given connectives and quantifiers is to be established semantically in one way or another *prior* to their inferential role. Their meanings may be the primitives of our reasoning in general, including "and", "or", "not", "if…then", "all", "some"—or they may be understood informally like "most", "has the same number as", etc., in a way that may be explained precisely in basic mathematical terms. What is taken from the inferentialists (or Zucker) is not the thesis as to meaning but rather their formal analysis of the essential principles and rules which are in accord with the prior semantical explanations and that govern their use in reasoning. And in that respect, *the Introduction and Elimination Rules for each logical operation of first-order logic implicitly characterize it in the sense that any other operation satisfying the same rules is provably equivalent to it*.[4] That unicity will be a key part of our criterion for logicality in general.

To illustrate, since I will be dealing here only with classical truth functional semantics, I consider schematic axioms and rules of inference for sequents $\Gamma \vdash \Delta$ as in LK, but in the case of each connective or quantifier, show only those formulas in Γ and Δ directly needed to characterize the operation in question. That may include possible additional side formulas (or parameters), to which all further formulas can be adjoined by thinning. In LK, the *Right* and *Left Introduction Rules* take the place of the *Introduction* and *Elimination Rules*, resp., in NK. I shall then show how unicity is expressed for the corresponding Hilbert-style axioms and rules.

Consider for illustrative purposes the (axioms and) rules for $\rightarrow$ and $\forall$. For notational simplicity, $\Rightarrow$ is used for inference from one or more sequents as hypotheses, to a sequent as conclusion.

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[4] The observation that the natural deduction Introduction and Elimination rules for the operations of FOL serve to uniquely specify each such operation is, I think, well known. At any rate, one can find it stated in Zucker and Tragesser (1978) p. 509. In apparent agreement with Gentzen that the Introduction rules provide the meaning of each operation, they say that the related Elimination rules serve to "stabilize" or "delimit" it.

$$(\mathrm{R}\!\to)\quad r, p \vdash q \Rightarrow r \vdash p\to q \qquad (\mathrm{L}\!\to)\quad p, p\to q \vdash q$$

$$(\mathrm{R}\forall)\quad r \vdash p(a) \Rightarrow r \vdash \forall x\, p(x) \qquad (\mathrm{L}\forall)\quad \forall x\, p(x) \vdash p(a).$$

Given an operation $\to'$ satisfying the same rules as for $\to$ we can infer from the left rule $p\to q, p \vdash q$ the conclusion $p\to q \vdash p\to' q$ by the substitution of $p\to q$ for r in $(\mathrm{R}\!\to')$; the reverse holds by symmetry. In the case of the universal quantifier, given $\forall'$ that satisfies the same rules as $\forall$, we can derive $\forall x\, p(x) \vdash \forall' x\, p(x)$ by substituting $\forall x\, p(x)$ for r in $(\mathrm{R}\forall')$. What is crucial in these proofs of uniqueness is the use of substitution of the principal formula ($p\to q$ and $\forall x\, p(x)$ and their $'$ versions, resp.) for a side formula (parameter) r.

If we accept $\to$ as a basic fully understood operator, we can pass to the Hilbert-style axioms and rules for the universal quantifier by simply replacing the turnstile symbol by '$\to$', as follows:

$$(\mathrm{R}\forall)^{\mathrm{H}}\quad r \to p(a) \Rightarrow r \to \forall x\, p(x) \qquad (\mathrm{L}\forall)^{\mathrm{H}}\quad \forall x\, p(x) \to p(a).$$

Then in a suitable metatheory for axioms and rules in which we take *all* the connectives and quantifiers of FOL for granted, we can represent this rule and axiom by the following single statement in which we treat universal quantification as a quantifier Q of type $\langle 1\rangle$:

$$\mathrm{A}(\mathbf{Q}) \qquad \forall p\, \forall r\, \{[\forall a(r \to p(a)) \to (r \to \mathbf{Q}(p))] \wedge [\forall a(\mathbf{Q}(p) \to p(a))]\},$$

where 'r' ranges over arbitrary propositions and 'p' over arbitrary unary predicates. Then, as above, we easily show that

$$(\mathrm{A}(\mathbf{Q}) \wedge \mathrm{A}(\mathbf{Q}')) \to (\mathbf{Q}(p) \leftrightarrow \mathbf{Q}'(p)).$$

Our question now is: Which quantifiers Q in general have formal axioms and rules of inference that uniquely characterize it in the same way as for universal quantification? The answer to that will initially be treated via a *second-order* language L_2 of individuals, propositions and predicates, first without and then with a symbol for Q.

L_2 is specified as follows:

Individual variables: a, b, c,…, x, y, z

Propositional variables: p, q, r,…

Predicate variables, k-ary: $p^{(k)}$, $q^{(k)}$, …; the superscript k may be dropped when determined by context.

Propositional terms: the propositional variables p, q, r,… and the $p^{(k)}(x_1,\ldots,x_k)$ (any sequence of individual variables)

Atomic formulas: all propositional terms

Formulas: closed under $\neg, \wedge, \rightarrow, \forall$ applied to individual, propositional and predicate variables. (Other connectives and quantifiers defined as usual.)

Next, models $\mathcal{M}_2$ of L_2 are specified as follows:

(i) Individual variables range over a non-empty universe U

(ii) Propositional variables range over $\{t, f\}$ where $t \neq f$.

(iii) Predicate variables of k arguments range over $Pred^{(k)}(\mathcal{M}_2)$, a subset of $U^k \rightarrow \{t, f\}$.

Clause (iii) is in accord with the Locality Principle, according to which predicate variables may be taken to range over any subset of the totality of k-ary relations on U.

Satisfaction of a formula φ of L_2 in $\mathcal{M}_2$ at an assignment σ to all variables, $\mathcal{M}_2 \vDash \varphi[\sigma]$, is defined inductively as follows:

(1) For $\varphi \equiv p$, a propositional variable, $\mathcal{M}_2 \vDash \varphi[\sigma]$ iff $\sigma(p) = t$

(2) For $\varphi \equiv p(x_1,\ldots,x_k)$, p a k-ary predicate variable, $\mathcal{M}_2 \vDash \varphi[\sigma]$ iff $\sigma(p)(\sigma(x_1),\ldots,\sigma(x_k)) = t$

(3) Satisfaction is defined inductively as usual for formulas built up by $\neg, \wedge, \rightarrow, \forall$, given the specified ranges in (ii) and (iii) for the propositional and predicate variables when it comes to quantification.

Now, given a quantifier Q of arity $\langle k_1,\dots,k_n\rangle$, the language $L_2(\mathbf{Q})$ adjoins a corresponding symbol **Q** to L_2. This is used to form propositional terms $\mathbf{Q}(p_1,\dots,p_n)$ where p_i is a k_i-ary variable. Each such term is then also counted as an atomic formula of $L_2(\mathbf{Q})$, with formulas in general generated as before. A model $(\mathcal{M}_2, Q|\mathcal{M}_2)$ of $L_2(\mathbf{Q})$ adjoins a function $Q|\mathcal{M}_2$ as the interpretation of **Q**, with $Q|\mathcal{M}_2$: $Pred^{(k1)}(\mathcal{M}_2) \times \dots \times Pred^{(kn)}(\mathcal{M}_2) \rightarrow \{t, f\}$.

Axioms and rules for a quantifier Q as in LK can now be formulated directly by a sentence $A(\mathbf{Q})$ in the language $L_2(\mathbf{Q})$, as was done above for the universal quantifier, by using the associated Hilbert-style rules as an intermediate auxiliary. To formulate the translation in general if we start with rules in the sequent calculus, suppose those for a formal quantifier $\mathbf{Q}(p_1,\dots,p_n)$ of the sort we are considering are $\text{Rule}_1,\dots, \text{Rule}_m$, where each Rule_j has 0 or more sequents $\Gamma_{j,v} \vdash \Delta_{j,v}$ in the hypothesis and one sequent $\Gamma_j \vdash \Delta_j$ as conclusion. Some of these will be Right rules and some Left rules for **Q**.[5] Consider any such Rule_j. If there is more than one term in the antecedent of one of the sequents in the hypothesis, replace that by their conjunction, and if in the succeedent by their disjunction. Replace an empty antecedent by $\forall p(p\rightarrow p)$ and an empty succeedent by $\neg\forall p(p\rightarrow p)$. Finally, replace $\vdash$ by $\rightarrow$. Next, for each j, take the conjunction of the translations of the $\Gamma_{j,v} \vdash \Delta_{j,v}$, and universally quantify that by all the individual variables that occur in it; call that H_j. Similarly, replace the conclusion $\Gamma_j \vdash \Delta_j$ by the universal quantification C_j over the individual variables of its translation. Finally, replace the inference sign $\Rightarrow$ from the hypotheses to the conclusion by $\rightarrow$. Let $B_j = H_j \rightarrow C_j$ be the translation of Rule_j thus obtained. Finally, take $A(\mathbf{Q})$ to be the sentence

$$\forall p\forall q\forall r\dots(B_1\wedge\dots\wedge B_m),$$

[5] Zucker and Tragesser (1978) pp. 502-503 make further assumptions about the nature of the rules in a natural deduction calculus for a candidate operator. Since our criterion will be formulated under much looser assumptions, we don't have to invoke those here.

where p, q, r, … are all the propositional and predicate variables that appear in any of the B_j. Now the criterion for accepting a quantifier Q given by such rules is that they implicitly define Q_U in each model of A(**Q**) (more precisely, the restriction of Q_U to the predicates of the model).

We need not restrict to such specific descriptions of axioms and rules of inference for a global quantifier Q in formulating the following more general partial criterion for acceptance of Q as logical. The reason this is not claimed to be a necessary and sufficient condition for logicality will be discussed below.

Semantical-Inferential Necessary Criterion for Logicality. A global quantifier Q of type $\langle k_1,\ldots k_n\rangle$ is *logical* only if there is a sentence A(**Q**) in L_2(**Q**) such that for each model $\mathcal{M}_2 = (U,\ldots)$, Q_U is the unique solution of A(**Q**) when restricted to the predicates of $\mathcal{M}_2$.

Remark. I spoke above of the use of axioms and rules of inference for a quantifier Q that completely govern its forms of reasoning. One should be careful to distinguish completeness of a system of axioms in the usual sense from completeness of a sentence A(**Q**) for Q in the sense that it meets the above criterion. For example, let Q_α be the type $\langle 1\rangle$ quantifier which holds of a subset P of U just in case P is of cardinality at least $\aleph_\alpha$. Keisler (1970) has proved completeness of a system of axioms for first-order logic extended by Q_1. But it is easily seen that those same axioms are satisfied by Q_α for any α greater than 1 (cf. ibid, p. 29). Hence a sentence A(**Q**) formally expressing Keisler's axioms does not meet the above criterion.

Main Theorem. Suppose Q is a quantifier that satisfies the preceding partial criterion for logicality. Then Q is equivalent to a quantifier defined in FOL.

The sketched proof of the related theorem in Zucker (1978) pp. 526 ff makes use of a different second order language than here, and claims to apply Beth's definability theorem to obtain an equivalence of **Q** with a formula in FOL. The first problem with that is the question of the applicability of Beth's theorem to a second-order language.

That may be possible for certain languages such as L_2 whose semantics is not the standard one but rather is "Henkin" or "general". So far as I know a Beth theorem for such has not been established in the literature, even though that is quite plausible. In order to do that, one might try to see how the extant model-theoretic or proof-theoretic proofs can be adapted to such languages. But even if one has done that, all that the corresponding Beth theorem would show is that Q is definable by a formula in L_2; in order to obtain a definition in FOL, one would still have to eliminate the propositional and predicate variables, and that requires a further argument, not considered at all by Zucker. It is shown here how to take care of both difficulties by *simulating* the languages L_2 and $L_2(\mathbf{Q})$ and their models in corresponding *first-order* languages L_1 and $L_1(\mathbf{Q})$ in which the proposition and predicate variables are taken to be two new sorts of variables at type level 0 besides the individual variables.

Here is the specification of this first-order language L_1:

Individual variables: a, b, c, …, x, y, z

Propositional variables: p, q, r,…

Propositional constants: **t**, **f**

Predicate variables $p^{(k)}$ of k arguments for $k \geq 1$; where there is no ambiguity, we will drop the superscripts on these variables.

Predicate constants $\mathbf{t}^{(k)}$ of k arguments for each $k \geq 1$.

In addition, L_1 has for each k a k+1-ary function symbol App_k for application of a k-ary predicate variable $p^{(k)}$ to a k-termed sequence of individual variables $x_1,\ldots,x_k$; we write $p^{(k)}(x_1,\ldots,x_k)$ for $\mathrm{App}_k(p^{(k)}, x_1,\ldots,x_k)$.

The terms of L_1 are the variables and constants of each sort, as well as the terms $p^{(k)}(x_1,\ldots,x_k)$ of propositional sort for each k-ary predicate variable $p^{(k)}$. The atomic formulas are just those of the form $\pi_1 = \pi_2$, where π_1 and π_2 are terms of propositional sort. Formulas in general are built up from these by means of the first-order connectives and quantifiers over each of the sorts of variables as usual.

By the language $L_1(\mathbf{Q})$ is meant the extension of L_1 by a function symbol **Q** taking a sequence $(p_1,\ldots,p_n)$ of predicate variables (not necessarily distinct) as arguments, where p_i is k_i-ary, to a term $\mathbf{Q}(p_1,\ldots,p_n)$ of propositional sort. For any term π of propositional sort, whether in the base language or this extension, we write T(p) for p = **t**, to express that p is true.

The following is a base set S of axioms for L_1:

(i) $\neg(\mathbf{t} = \mathbf{f})$

(ii) $\forall p(p = \mathbf{t} \vee p = \mathbf{f})$, ('p' a propositional variable)

(iii) $\forall x_1\ldots\forall x_k(\mathbf{t}^{(k)}(x_1,\ldots,x_k) = \mathbf{t})$ for each $k \geq 1$

(iv) $\forall p\forall q\,[\forall x_1\ldots\forall x_k(p(x_1,\ldots,x_k) = q(x_1,\ldots,x_k)) \rightarrow p = q]$.

The last of these is of course just Extensionality for predicates.

Models $\mathcal{M}_1$ of S are given by any non-empty universe of individuals U as the range of the individual variables, and the set $\{t, f\}$ (with $t \neq f$) as the range of the propositional variables. Furthermore each assignment to a k-ary predicate variable in $\mathcal{M}_1$ determines a propositional function P from U^k to $\{t, f\}$ as its extension, via the interpretation of the application function App_k. By Extensionality, we may think of the interpretation of the k-ary predicate variables in $\mathcal{M}_1$ as ranging over *some* collection of k-ary propositional functions. The interpretation of $\mathbf{t}^{(k)}$ is just the constant propositional function $\lambda(x_1,\ldots,x_k).t$ on U^k. In the following, all structures $\mathcal{M}_1$ considered are assumed to be models of S.

Each model $\mathcal{M}_2$ of the second order language L_2 may equally well be considered to be a model $\mathcal{M}_1$ of the first order language L_1 in the obvious way. Conversely, by extensionality each of the models $\mathcal{M}_1$ for L_1 may be construed to be a model $\mathcal{M}_2$ for L_2.

The essential difference lies in the way that formulas are formed and hence with how satisfaction is defined. In the first-order language, propositional terms are merely such, while they have also been taken to be atomic formulas in the second order language. Recall the abbreviation T(p) for p = **t** in L_1. Note that any assignment to the variables of L_2 in $\mathcal{M}_2$ counts equally well as an assignment to the variables of L_1 in $\mathcal{M}_1$. All of this goes over to the languages extended by **Q** and the corresponding interpretations of it in the respective models.

We define the translation of each formula A of the 2nd order language L_2, with or without **Q**, into a formula A↓ of the 1st order language L_1 by simply replacing each atomic formula τ of A (i.e. each propositional term) by T(τ). Thus, for example, the translation of the above formula characterizing the axiom and rule for universal quantification is simply

$$\forall p \forall r\{[\forall a(T(r) \to T(p(a))) \to (T(r) \to T(\mathbf{Q}(p)))] \wedge \forall a[T(\mathbf{Q}(p)) \to T(p(a))]\}.$$

Similarly, we obtain an inverse translation from any 1st order formula B of L_1 into a 2nd order formula B↑ of L_2 by simply removing each occurrence of 'T' that is applied to propositional terms. The atomic formulas $\pi_1 = \pi_2$ are replaced by $\pi_1 \leftrightarrow \pi_2$. These translations are inverse to each other (up to provable equivalence) and the semantical relationship between the two is given by the following lemma, whose proof is quite simple.

Lemma. Suppose $\mathcal{M}_1$ and $\mathcal{M}_2$ correspond to each other in the way described above. Then

(i) if A is a formula of L_2 and σ is an assignment to its free variables in $\mathcal{M}_2$ then

$\mathcal{M}_2 \models A[\sigma]$ iff $\mathcal{M}_1 \models A{\downarrow}[\sigma]$;

(ii) similarly, if B is a formula of L_1 and σ is an assignment to its free variables in $\mathcal{M}_1$ then $\mathcal{M}_1 \models B[\sigma]$ iff $\mathcal{M}_2 \models B{\uparrow}[\sigma]$.

Moreover, the same equivalences hold under the adjunction of **Q** throughout.

Now to prove the main theorem above, suppose A(**Q**) is a sentence of L_2(**Q**) such that over each model $\mathcal{M}_2$, Q_U is the unique operation restricted to the predicates of $\mathcal{M}_2$ that satisfies A(**Q**). Then it is also the unique operation that satisfies A(**Q**)↓ in $\mathcal{M}_1$. So now by the completeness theorem for many-sorted first-order logic, we have provability of

$$(A(\mathbf{Q})\!\downarrow \wedge\ A(\mathbf{Q}')\!\downarrow) \rightarrow (\mathbf{Q}(p_1,\ldots,p_n) = \mathbf{Q}'(p_1,\ldots,p_n))$$

in FOL, so that by Beth's definability theorem, which follows from the interpolation theorem for many-sorted logic (Feferman 1968a), the relation $\mathbf{Q}(p_1,\ldots,p_n) = \mathbf{t}$ is equivalent to a formula $B(p_1,\ldots,p_n)$ of L_1. Moreover, by assumption, in each model $\mathcal{M}_1$, B defines the relation Q_U restricted to the range of its predicate variables (considered as relations). Though B is a formula of L_1, it is not necessarily first-order in the usual sense since it may still contain quantified propositional and predicate variables; the remainder of the proof is devoted to showing how those may be eliminated.

First of all, we can replace any quantified propositional variable p in B by its instances **t** and **f**, so we need only eliminate the predicate variables. Next, given two models $\mathcal{M}_1 = (U,\ldots)$ and $\mathcal{M}_1' = (U',\ldots)$ of L_1, we write $\mathcal{M}_1 \leq \mathcal{M}_1'$ if $\mathcal{M}_1$ is a substructure of $\mathcal{M}_1'$ in the usual sense, but for which $U = U'$. The relation $(\mathcal{M}_1, Q|\mathcal{M}_1) \leq (\mathcal{M}_1', Q|\mathcal{M}_1')$ is defined in the same way, so that when this holds, $Q|\mathcal{M}_1$ is the restriction to the predicates of $\mathcal{M}_1$ of $Q|\mathcal{M}_1'$, in accordance with the Locality Principle. Suppose both structures are models of $A(\mathbf{Q})$; then by assumption, $Q|\mathcal{M}_1 = Q_U$ on the predicates in $\mathcal{M}_1$ and $Q|\mathcal{M}_1' = Q_U$ on the predicates in $\mathcal{M}_1'$. Moreover both are equivalent to B on the respective classes of predicates. Hence, given $P_1,\ldots,P_n$ predicates in $\mathcal{M}_1$, $B(P_1,\ldots,P_n)$ holds in $\mathcal{M}_1$ if and only if it holds in $\mathcal{M}_1'$. In other words, B is invariant under ≤ extensions in the sense of Feferman (1968b).[6] It follows from Theorem 4.2, p.47 of Feferman (1968b) that we can choose B to have quantifiers only over individuals; in addition, since we have a constant $\mathbf{t}^{(k)}$ of each propositional and predicate sort, we can take B to have no free variables other than $p_1,\ldots,p_n$. In other words, B is a first-order formula in the usual sense, with all quantified variables being of the individual sort, which defines Q_U in each $\mathcal{M}_1$ when restricted to the predicates of $\mathcal{M}_1$. Lifting B to $B\!\uparrow$ and $\mathcal{M}_1$ to the corresponding $\mathcal{M}_2$ gives, finally, the desired result.

[6] These are called outer extensions in Feferman (1968b), but in the case at hand they are just ordinary extensions with one sort fixed (or "stationary" in the language of that paper), namely the sort of individuals.

Discussion and Questions.

1. Comparison with Zucker (1978). Zucker considers formal quantifiers Q at every finite type level, within which he deals with first order quantifiers (i.e. those at type level 2 whose arguments are predicates of type level 1) as a special case. (The case of higher types uses different arguments with both positive and negative results.) He denotes by S_c ('c' for 'classical') the set [of operations] $\{\wedge, \neg, t, \forall\}$. By way of comparison, it is worth quoting him at some length as to his aims (the italics in the following are Zucker's):

> We are looking for an argument of the following form: given a proposed new 'logical operation' (say a quantifier), show that it is explicitly definable in terms of S_c. … Now what does it mean, to "propose a new quantifier Q for inclusion in the language?" Clearly, a symbol 'Q' by itself is useless: a *meaning* must be given along with it. … In fact a symbol 'Q' is never given alone: it is generally given together with a set of *axioms and/or inference rules*, proposed for incorporation in a logical calculus. Now we [make] the following *basic assumption*:
>
> > For Q to be considered as a *logical* constant, its'meaning' must be *completely contained* in these axioms and inference rules.
>
> In other words, it is quite *inadequate* to propose a quantifier Q for incorporation in the calculus as a logical constant, by giving its meaning in set theory, say (e.g., "there exist uncountably many"), and also axioms which are merely *consistent* with this meaning. The meaning of Q must be *completely determined* by the axioms (and rules) for it: they must carry the *whole weight* of the meaning, so to speak; the meaning must not be *imposed from outside* (by, e.g., a set-theoretical definition), for then we merely have a 'mathematical' or 'set-theoretical' quantifier, not a *logical* one. … Our basic assumption, then, gives a necessary condition for a proposed new constant to be considered as purely logical. We re-state it as a principle of *implicit definability*:
>
> > (ID) A logical constant must be defined implicitly by its axioms
>
> and inference rules.

> Hence in order to prove the adequacy of S_c, it will be sufficient to show that any constant which is *implicitly definable* (by its axioms and rules) is also *explicitly definable* from S_c." (Zucker 1978, pp. 518-519)

There follow three notes (ibid.). The first is that (ID) is only proposed as a necessary (but not necessarily sufficient) condition for logicality. The second is that the inference rules for the new constant need not be of the natural deduction kind. Third, it is assumed that the status of the members of S_c as logical constants is not in doubt.

As noted in the introductory discussion above, one essential difference I have with Zucker is that I regard the meaning of a quantifier to be provided from the outside so to speak, i.e., to be given in model theoretic terms prior to the consideration of any rules of inference that may be in accord with it. For me, the significance of the condition ID is to specify completely its role as an inferential agent.

2. What is a necessary and sufficient condition for logicality? Taking for granted that the standard operations of FOL are logical, it is at first sight plausible that any quantifier defined in terms of them should also be considered logical. However, in a personal discussion following a presentation of this material,[7] Lauri Hella questioned this. He pointed out that many mathematical notions considered as Lindström quantifiers that would not ordinarily be considered logical are definable in FOL. For example, we can thus define what it is for (U, P) to be a group, where P is a ternary relation for the product relation of the group. Note that the relation of equality is used in this definition, and it is a matter of some contention whether equality is a logical notion (cf. Quine (1986), pp. 61ff and Feferman (1999), p. 44). We can side-step that issue by considering the definition in FOL without equality of all (U, P, E) where E is a congruence relation with respect to a product relation P under which the structure forms a group. The collection Q of all such (U, P, E) would still not ordinarily be considered to be a logical quantifier. In any case, that is the reason why the combined semantical and inferential criterion considered here is only proposed as a necessary condition. In order for such to be tightened to a necessary and sufficient condition, we would have to be explicit about

[7] At the 2011 Workshop on Logical Constants in Ljubljana referred to in fn. 1.

what would constitute axioms and rules of inference for a quantifier Q that determine it uniquely. The work of Zucker and Tragesser (1978) pp. 10-15 is a start on that for a formulation in natural deduction terms, but that needs to be generalized and, if possible, simplified.

3. Extension to countable admissible languages. It is shown in Feferman (1968a, 1968b) that the results from those articles needed for the proof here of the Main Theorem hold equally well for the sublanguages L_A of the language with countably long conjunctions and disjunctions and ordinary quantification, and for which A is an admissible set. Thus one should expect that the Main Theorem carries over directly to those languages. But now there is a new question that ought to be considered, namely whether all infinitary *propositional* operations that satisfy a necessary criterion for logicality similar to the one taken here, are definable in L_A.

4. Are there analogous results for intuitionistic FOL? There are several possible options to consider for the semantics of general quantifiers looked at constructively: the most familiar ones are the (so-called BHK) interpretation in terms of primitive notions of construction and constructive proof, realizability interpretations, inferential semantics, and Kripke models. It is an open question how Lindström quantifiers might be treated with respect to either of the first two of these. As to the third, one would take the work of Zucker and Tragesser (1978) as a point of departure as suggested at the end of item 2 above; it is shown there (under certain natural hypotheses about the forms of inferences) that every formal quantifier given by introduction rules is equivalent to one definable in intuitionistic FOL. Finally, given any Lindström quantifier Q viewed classically, one can extend its semantics to arbitrary Kripke structures $(W, \preccurlyeq, \langle U_w\colon w \in W\rangle, \ldots)$ for which $w \preccurlyeq v$ implies $U_w \subseteq U_v$, by taking a formula $Q\underline{x}_1\ldots\underline{x}_n(\psi_1(\underline{x}_1,\underline{y}),\ldots,\psi_n(\underline{x}_n,\underline{y}))$ to be satisfied by $\underline{b}$ in $(U_w, \ldots)$ just in case $(U_v, P_1,\ldots,P_n)$ is in Q for each $v \succcurlyeq w$, where P_i is the set of all k_i-tuples a_i in U_v such that $\psi_i(\underline{a}_i, \underline{b})$ is true at v. Then the definition of forcing works as usual. Since Kripke semantics reduces to classical semantics on worlds W having a single element, the Main Theorem can be applied to show that any Lindström quantifier on Kripke structures dealt with in this way and that satisfies the criterion considered here is definable in classical FOL. This leaves open whether some more intrinsic version of the Main Theorem holds for Kripke structures and intuitionistic FOL.

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