WHICH QUANTIFIERS ARE LOGICAL?

A COMBINED SEMANTICAL AND INFERENTIAL CRITERION

Solomon Feferman

For the *Constructive in Logic and Applications* conference
in honor of the 60th birthday of Sergei Artemov

May 23 – May 25, 2012
Graduate Center, City University of New York

(Revised version of presentation at the ESSLLI Workshop on Logical Constants, Ljubljana, August 9, 2011)

What is Logic?

- It is the characterization of those forms of reasoning that lead invariably from true sentences to true sentences, independently of the subject matter.
- Sentences are analyzed according to their "logical" (as opposed to their grammatical) structure.

What is Logic? (cont’d)

- Generation of sentence parts by operations on propositions and predicates.
- Which of those operations are logical?
- Explained both by saying how truth of compounds is determined by truth of parts
- and by completely characterizing those forms of inference that preserve truth.

“The Problem of Logical Constants”

- Gomez-Torrente (2002)
- Mostly pursued via purely semantical or purely inferential approaches.
- Semantical criteria: Tarski (1986) going back to the 30s, Sher(1991), McGee (1996), etc. (critiqued in Feferman 1999, 2010).
- Inferential criteria: Gentzen (1936), Prawitz (1965), Hacking (1979), etc.

A combined Semantical and Inferential Partial Criterion

- Semantical part of the criterion for generalized quantifiers in the sense of Lindström (1966).
- Inferential part of the criterion first proposed by Zucker (1978): Uniquely characterize quantifiers via their axioms and rules of inference.

How is the Meaning of a Quantifier Specified?

- My view: Accept the Lindström explanation--as is done by workers in model-theoretic logics (cf. Barwise and Feferman 1985) and on quantifiers in natural language (cf. Peters and Westerståhl 2006)
- Zucker's view: The meaning of a given quantifier is specified by its axioms and rules, provided they uniquely determine it.

The Combined Criterion, and The Main Result

- The Combined Partial Criterion:
  A quantifier in Lindström's sense is logical only if it is uniformly uniquely characterized by some axioms and rules of inference over each universe of discourse.
- Main Theorem: A quantifier meets this criterion just in case it is definable in FOL.

Universes, Relations, and Propositional Functions

- Universe of discourse: non-empty U
- k-ary relations P on U are subsets of U^k; we may also identify such with k-ary "propositional" functions $P: U^k \to \{t, f\}$,
- Say that $P(x_1,\ldots,x_k)$ holds, or is true.

Global and Local Quantifiers

- Q is called a (global) quantifier of type $\langle k_1,\ldots,k_n\rangle$ if Q is a class of relational structures of signature $\langle k_1,\ldots,k_n\rangle$ closed under isomorphism.

- A typical member of Q is of the form $\langle U,P_1,\ldots,P_n\rangle$ where U is non-empty and P_i is a k_i-ary relation on U.

- Given Q, with each U is associated the (local) quantifier Q_U on U which is the relation $Q_U(P_1,\ldots,P_n)$ that holds between $P_1,\ldots,P_n$ just in case $\langle U,P_1,\ldots,P_n\rangle$ is in Q.

The Locality Principle

- Examples of quantifiers can be given in set-theoretical terms without restriction.
- Common examples: the uncountability quantifier of type $\langle 1 \rangle$, the equi-cardinality quantifier of type $\langle 1, 1 \rangle$, and the "most" quantifier of type $\langle 1, 1 \rangle$.
- Even though the definitions of those refer to the supposed totality of relations of a certain sort, all quantifiers satisfy the Locality Principle: The truth or falsity of $Q_U(P_1,\ldots,P_n)$ depends only on U and $P_1,\ldots,P_n$, and not on any such totalities.

Addition of Quantifiers to Given L

- Given any first-order language L with some specified vocabulary, we may add Q as a formal symbol to be used as a new constructor of formulas φ from given formulas ψ_i, I= 1,…,n:
- $\varphi(\underline{y}) = Q\underline{x}_1\ldots\underline{x}_n(\psi_1(\underline{x}_1,\underline{y}),\ldots,\psi_n(\underline{x}_n,\underline{y}))$
- The satisfaction relation for such in a given L model M is defined recursively: for an assignment $\underline{b}$ to $\underline{y}$ in U, $\varphi(\underline{b})$ is true in M iff $(U, P_1,\ldots,P_n)$ is in Q, where P_i = the set of k_i tuples $\underline{a}_i$ satisfying $\psi_i(\underline{a}_i,\underline{b})$ in M.

Representation of Axioms and Rules of Inference

- Back to Gentzen 1936; isolating the axioms and rules of inference separately for each operator.
- In the Natural Deduction calculi NJ and NK, use Introduction and Elimination Rules. In the Sequential Calculi LJ and LK, Right and Left Rules.
- Gentzen: "The [Introduction rules] represent, as it were, the 'definitions' of the symbols concerned."
- Prawitz' Inversion Principle (1965).

Implicit Completeness, not Meaning

- The Introduction and Elimination rules (Right and Left rules, resp.) for each basic operation of FOL are implicitly complete in the sense that any other operation satisfying the same rules is provably equivalent to it. Examples:
- (R→) r, p ⊦ q ⇒ r ⊦ p→q   (L→) p, p→q ⊦ q

  Given →′ satisfying the same rules as for →, infer from the left rule p→q, p ⊦ q the conclusion p→q ⊦ p→′q by taking p→q for r in (R→′).

Completeness (cont'd)

- $(R\forall)$ $r \vdash p(a) \Rightarrow r \vdash \forall x\, p(x)$ $(L\forall)$ $\forall x\, p(x) \vdash p(a)$.
- Given $\forall'$ that satisfies the same rules as $\forall$, we can derive $\forall x\, p(x) \vdash \forall' x\, p(x)$ by substituting $\forall x\, p(x)$ for r in $(R\forall')$.
- Hilbert-style formulation of the rules, assuming $\rightarrow$:
  $(R\forall)^H$ $r \rightarrow p(a) \Rightarrow r \rightarrow \forall x\, p(x)$

  $(L\forall)^H$ $\forall x\, p(x) \rightarrow p(a)$.

Formulation in a 2nd Order Metalanguage for Inferences

- A 2nd order language L_2 with variables for individuals, propositions and propositional functions and with the $\neg$, $\wedge$, $\rightarrow$, $\forall$ operators already granted.
- Example: treat universal quantification as a quantifier Q of type $\langle 1 \rangle$, given by:
- A(Q) $\forall p \forall r\{[\forall a(r \rightarrow p(a)) \rightarrow (r \rightarrow Q(p))] \wedge [\forall a(Q(p) \rightarrow p(a))]\}$.
- (Uniqueness) $A(Q) \wedge A(Q') \rightarrow (Q(p) \leftrightarrow Q'(p))$.

The Syntax of L_2

- Individual variables: a, b, c,…, x, y, z
- Propositional variables: p, q, r,…
- Predicate variables, k-ary: $p^{(k)}$, $q^{(k)}$, …; drop superscript k when determined by context.
- Propositional terms: the propositional variables p, q, r,… and the $p^{(k)}(x_1,\ldots,x_k)$ (any sequence of individual variables)
- Atomic formulas: all propositional terms
- Formulas: closed under $\neg, \wedge, \rightarrow, \forall$ applied to individual, propositional and predicate variables.

Models M_2 of L_2

- Individual variables range over a non-empty universe U. $M_2 = (U,\ldots)$
- Propositional variables range over {t, f} where $t \neq f$.
- Predicate variables of k arguments range over $Pred^{(k)}(M_2)$, a subset of $U^k \to \{t, f\}$.
- NB: In accord with the Locality Principle, predicate variables may be taken to range over any subset of the totality of k-ary predicates over U.

Satisfaction in M_2

- $M_2 \vDash \varphi[\sigma]$, for φ a formula of L_2 and σ an assignment to the free variables of φ in M_2, defined inductively as follows:
- For $\varphi \equiv p$, a propositional variable, `
  $M_2 \vDash \varphi[\sigma]$ iff $\sigma(p) = t$
- For $\varphi \equiv p(x_1,\ldots,x_k)$, p a k-ary predicate variable,
  $M_2 \vDash \varphi[\sigma]$ iff $\sigma(p)(\sigma(x_1),\ldots,\sigma(x_k)) = t$.
- Satisfaction is defined inductively as usual for formulas built up by $\neg, \wedge, \rightarrow$, and $\forall$.

Extension by a Quantifier

- Given a quantifier Q of arity $\langle k_1,\ldots,k_n\rangle$., the language $L_2(Q)$ adjoins a corresponding symbol Q to L_2.
- This is used to form propositional terms $Q(p_1,\ldots,p_n)$ where p_i is a k_i-ary variable. Each such term is then also counted as an atomic formula of $L_2(Q)$, with formulas in general generated as before.
- A model $(M_2, Q|M_2)$ of $L_2(Q)$ adjoins a function $Q|M_2$ as the interpretation of Q, with $Q|M_2$: $\mathrm{Pred}^{(k1)}(M_2) \times \ldots \times \mathrm{Pred}^{(kn)}(M_2) \rightarrow \{t, f\}$.

The Criterion of Logicality for Q

- Axioms and rules of inference for a quantifier Q as, e.g., in LK can now be formulated directly by a sentence A(Q) in the language $L_2(Q)$, as was done above for the universal quantifier, by using the associated Hilbert-style rules as an intermediate auxiliary.
- The Semantical-Inferential Partial Criterion for Logicality. A global quantifier Q of type $\langle k_1,\ldots k_n\rangle$ is logical only if there is a sentence A(Q) in $L_2(Q)$ such that for each model $M_2 = (U,\ldots)$, Q_U is the unique solution of A(Q) when restricted to the predicates of M_2.

Difference from Usual Completeness

- One needs to be careful to distinguish completeness of a system of axioms in the usual sense, from (implicit) completeness in the sense of this criterion of a sentence A(Q) expressing formal axioms and rules for a quantifier Q.
- For example, Keisler proved the completeness of FOL extended by the uncountability quantifier K. His axioms for K are not uniquely satisfied by that, so K does not meet the above criterion for logicality.

The Main Theorem

- Main Theorem. Suppose Q is a logical quantifier according to the criterion. Then Q is equivalent to a quantifier defined in FOL.

- First proof idea:
  Apply a version of Beth's definability theorem to
  $A(Q) \wedge A(Q') \rightarrow (Q(p_1,\ldots,p_n) = Q'(p_1,\ldots,p_n))$
  in order to show $Q(p_1,\ldots,p_n)$ is equivalent to a formula in L_2 without Q.

- That was the basis for the proposed proof in Zucker (1978) of a related theorem with a different 2nd order language than here.

Two Problems with Zucker's Proof

- Problem 1: Beth's theorem is only stated in the literature for 1st order languages. It is plausible though that it applies to certain 2nd order languages with general ("Henkin") semantics, such as $L_2(Q)$.
- Problem 2: Even if Beth's theorem applies to $L_2(Q)$, we only get a definition of Q in the language L_2 with propositional and predicate variables.
- My way around these problems: Simulate $L_2(Q)$ in a 1st-order language $L_1(Q)$ to which Beth's theorem applies--and then use a further special reduction theorem--to obtain a FOL defn. of Q.

The Syntax of L_1

- Individual variables: a, b, c, …, x, y, z
- Propositional variables: p, q, r,…
- Propositional constants: t, f
- Predicate variables $p^{(k)}$ of k arguments for $k \geq 1$; where there is no ambiguity, we will drop the superscripts on these variables.
- Predicate constants $t^{(k)}$ of k arguments for each $k \geq 1$.

The Syntax of L_1 (cont'd)

- There is for each k a k+1-ary function symbol App_k for application of a k-ary predicate variable $p^{(k)}$ to a k-termed sequence of individual variables $x_1,\ldots,x_k$; write $p^{(k)}(x_1,\ldots,x_k)$ for $App(p^{(k)}, x_1,\ldots,x_k)$.
- The terms are the variables and constants of each sort, as well as the terms $p^{(k)}(x_1,\ldots,x_k)$ of propositional sort for each k-ary pred. variable $p^{(k)}$.
- The atomic formulas are $\pi_1 = \pi_2$, where π_1 and π_2 are terms of propositional sort. Formulas in general are built up usual, allowing quantification over each sort.

The Semantics of L_1

The following is a base set S of axioms for L_1:

(i) $\neg(t = f)$

(ii) $\forall p(p = t \vee p = f)$, ('p' a prop. variable)

(iii) $\forall x_1 \ldots \forall x_k(t^{(k)}(x_1,\ldots,x_k) = t)$ for each $k \geq 1$

(iv) (Extensionality)

$$\forall p,q[\forall x_1 \ldots \forall x_k(p(x_1,\ldots,x_k)=q(x_1,\ldots,x_k)) \rightarrow p=q],$$

for p, q k-ary predicate variables.

The Semantics of L_1 (cont'd)

- Models M_1 of S are given by any non-empty universe of individuals U as the range of the individual variables, and the set {t, f} (with $t \neq f$) as the range of the propositional variables. For each $k \geq 1$, we have a set $Pred^{(k)}(M_1)$ as the range of the k-ary predicate variables.
- Note that each member of $Pred^{(k)}(M_1)$ determines a propositional function P from U^k to {t, f} as its extension, via the interpretation of the application function App.
- By Extensionality, each such P is identified with a unique member of $Pred^{(k)}(M_1)$.

Syntax and Semantics of $L_1(Q)$

- The language $L_1(Q)$ is the extension of L_1 by a function symbol Q taking a sequence $(p_1,\ldots,p_n)$ of predicate variables (not necessarily distinct) as arguments where p_i is k_i-ary, to a term $Q(p_1,\ldots,p_n)$ of propositional sort.
- The semantics of $L_1(Q)$is a direct extension of that for L_1.
- For any term π of propositional sort, whether in the base language or this extension, we write $T(p)$ for $p = t$, to express that p is true.

Relationships between the two Languages

- Each model M_2 of the second order language L_2 may equally well be considered to be a model M_1 of the first order language L_1, and vice versa.
- The same holds for the extensions by Q.
- Each formula A of L_2, with or without Q, is translated into a formula $A\downarrow$ of L_1 by simply replacing each atomic formula α of A (i.e. each propositional term) by $T(\alpha)$.
- We have a simple inverse translation of B in L_1 (with or without Q) into a formula $B\uparrow$ of L_2 .

Proof of the Main Theorem

- Suppose $A(Q)$ is a sentence of $L_2(Q)$ such that over each model M_2, Q_U is the unique operation restricted to the predicates of M_2 that satisfies $A(Q)$. Then it is also the unique operation restricted to $Pred^{(k)}(M_1)$ that satisfies $A(Q)\downarrow$ in M_1.

- By the completeness theorem for many-sorted first-order logic, we have provability in FOL of $A(Q)\downarrow \wedge A(Q')\downarrow \rightarrow (Q(p_1,\ldots,p_n) = Q'(p_1,\ldots,p_n))$

- Thus the relation $Q(p_1,\ldots,p_n) = t$ is equivalent to a formula $B(p_1,\ldots,p_n)$ of L_1 by Beth's theorem for many-sorted FOL.

Proof of the Main Theorem (cont'd)

- The propositional variables can be eliminated from B by replacing them by their instances t, f.
- Next, to eliminate the predicate variables, given two models $M_1 = (U,\ldots)$ and $M_1' = (U',\ldots)$ of L_1, let $M_1 \leq M_1'$ if M_1 is a substructure of M_1' in the usual sense and if $U = U'$.
- Given $p_1,\ldots,p_n$ predicates in M_1, show $B(p_1,\ldots,p_n)$ holds in M_1 iff it holds in M_1', because it is the unique solution of A(Q) restricted to the "predicates" of each, and by the Locality Principle.

Proof of the Main Theorem (concluded)

- In other words, B is invariant under $\leq$ extensions in the sense of Feferman (1968), "Persistent and invariant formulas under outer extensions."
- Since the axioms of S are in universal form and we have a constant of each sort, it follows from Theorem 4.2 (ibid.) that B is equivalent to a formula without bound propositional and predicate variables, i.e. it is equivalent to a formula of FOL.

What is a N.A.S.C. for Logicality?

- Many mathematical notions *qua* Lindström quantifiers that are definable in FOL would not ordinarily be considered as logical.
- For example, let Q be all (U, P), P ternary, a group. Presumes =; could alternatively consider groups (U, P, E) a group w.r.t. the congruence relation E.
- This is why the semantical-inferential criterion here is only a necessary condition for logicality.
- To tighten to a n.a.s.c. need to tighten A(Q). How?

Questions

- Q1. It is shown in Feferman (1968) that the results from there needed for the proof of the Main Theorem hold equally well for the sublanguages L_A of the language with countably long conjunctions and disjunctions and ordinary quantification, for which A is an admissible set. Thus one should expect that the Main Theorem carries over directly to those languages.

- But now there is a new question that ought to be considered, namely whether all infinitary propositional operations that satisfy a criterion for logicality similar to the one taken here, are definable in L_A.

Questions (cont'd)

- Q2. Are there analogous results for intuitionistic logic?
- Which semantics are we talking about?
  (i) Using constructions and constructive proofs as primitives;
  (ii) or some form of realizability;
  (iii) or inferential semantics;
  (iv) or forcing in Kripke structures;
  (v) or other (?)
- The results here carry over to (iv).

Selected References

S. Feferman (1968), Persistent and invariant formulas for outer extensions, *Compositio Mathematica* 20, 29-52. (Or http://math.stanford.edu/~feferman/papers.html, #10.)

S. Feferman (2010), Set-theoretic invariance criteria for logicality, *Notre Dame J. of Formal Logic* 5, 3-20. (Or http://math.stanford.edu/~feferman/papers.html,#82.)

G. Gentzen (1969), Investigations into logical deduction (1936), in M. E. Szabo (ed.), *The Collected Papers of Gerhard Gentzen*, North-Holland, Amsterdam.

P. Lindström (1966), First order predicate logic with generalized quantifiers, *Theoria* 32, 186-195.

Selected References (Cont'd)

S. Peters and D. Westerståhl (2006), *Quantifiers in Language and Logic,* Clarendon Press, Oxford.

J. I. Zucker (1978), The adequacy problem for classical logic, *J. Philosophical Logic* 7, 517-535.

J. I. Zucker and R. S. Tragesser (1978), The adequacy problem for inferential logic, *J. Philosophical Logic* 7, 501-516.