Log. Univers.

DOI 10.1007/s11787-012-0058-5

Logica Universalis

On Rereading van Heijenoort's Selected Essays

Solomon Feferman

Abstract. This is a critical reexamination of several pieces in van Heijenoort's *Selected Essays* that are directly or indirectly concerned with the philosophy of logic or the relation of logic to natural language. Among the topics discussed are absolutism and relativism in logic, mass terms, the idea of a rational dictionary, and sense and identity of sense in Frege.

Mathematics Subject Classification. 03-03, 03A05.

Keywords. Jean van Heijenoort, philosophy of logic, history of logic.

1. Introduction

One of the books that I especially treasure in my library is the copy of Jean van Heijenoort's *Selected Essays* [62]. It was Van's own, lying unwrapped in his Stanford office, awaiting his return from Mexico in March, 1986. He no doubt had seen the proof sheets at some point; as a colleague remarked, it was a pleasant surprise to see an English language volume published in Italy with so few typographical errors.[1] But I am sure that he would have loved to hold the finished volume in his hands—as he never did—and I would have loved to then engage in exploring various of its ideas with him. We had not discussed the essays prior to that, and what I am perforce left with is a series of imaginary conversations.

Though I first met van Heijenoort at the Cornell Logic Symposium in 1957 and we came together from time to time over the subsequent years, his

[1] On the other hand, the review by Bell [3] concludes with the statement that the volume is marred by "many trivial typographical errors and infelicities of style", but he only notes one that should be corrected, namely on p. 80, line 6 up, replace 'a = 0' by 'a ≠ 0'. That one also leaped to my eyes, and I, too, noted some minor typographical errors; but I would not agree about the "infelicities of style"—on the contrary.

Published online: 30 June 2012

periods of stay at Stanford from 1981 to 1986 remain the most vivid for me.[2] That was not only because of his active involvement as co-editor of the project to publish a comprehensive edition of Gödel's *Collected Works* [21] (beautifully described in the memoir in this volume by John W. Dawson, Jr.),[3] but also because of the considerable personal time that my wife, Anita, and I were able to enjoy with him during his stays. We miss him greatly.

There have been a number of reviews and critical assessments of the *Selected Essays*, including those by Bell [3], de Rouilhan [10], and Padilla-Gálvez [39], as well as earlier reviews of some of its individual pieces including those of Gaifman [19], Hintikka [25], McDermott [32] and Smith [48].[4] Last but not least one should see Sec. 4.2 of the comprehensive work, [1]. For the present article, I reread the entire volume of van Heijenoort's essays, and found that the following view of it we had written in Feferman and Feferman [12, p. 6], to be still an apt overall characterization:

> Typically these papers are directed to a few significant points, but contain along the way many *aperçus*. The style is unhurried but succinct, precise but unfussy, clear and graceful; the point of view is definite, but the perspective is balanced. The reader who discovers these papers in the *Selected Essays* will become aware of a side of van Heijenoort... that has not been sufficiently or widely enough appreciated. These essays make abundantly evident his many excellent qualities, which, in their combination, made him unique.

The body of the *Selected Essays* consists of thirteen pieces on the history and philosophy of logic, often in combination, written between 1967 and 1985, together with one anomalous essay, "Friedrich Engels and mathematics", written in 1948 [62, pp. 123–151], of which van Heijenoort says in his foreword [62, p. 9] that "[w]ere I to write on this topic now, I would do it quite differently; but I let this old article of mine stand as it is." Other than the Engels essay, which runs twenty-eight pages, they range in length from one to nineteen pages, with an average of about seven pages each, and are thus relatively short. Instead of re-reviewing the volume in its entirety, what I decided to do here is concentrate on several of the essays that in whole or in part mainly concern the philosophy of logic and the possible applicability of logic to natural language; moreover, within that group I chose to limit myself to those that were of special interest to me, and where I thought more could and should have been said.

Throughout below, page references are as they appear in the 1985 volume.

---

[2] Van had also spent the spring quarter of 1970 at Stanford as a Visiting Professor of Philosophy according to the department records. (Anita Feferman [11, p. 321], placed that visit in 1971).

[3] See also Feferman [14].

[4] I'm grateful to Irving Anellis for providing this bibliographic information to me.

2. Logic, Mathematics and the Emergence of Model Theory

One of the essays that has had the most impact is "Logic as Calculus and Logic as Language" [52], with the allied "Absolutism and Relativism in Logic" [60] running a close second. These contrast the role of logic and the interpretation of associated formal languages in the work of Frege and Russell on the one hand and that in the work of Löwenheim—and those who followed him in the subsequent development of model theory—on the other. The essential difference is that in the Frege–Russell work, the individual variables are supposed to range over *everything*, including both concrete and abstract objects, while in that of the model-theoretic (or set-theoretic) approach, individual variables are taken to range over some specified domain of things, and the main notions are those of a formula being satisfied in the given domain and of being valid in some or all domains.[5] There is also some repetition of these ideas in the beginning of "Set-theoretic Semantics" [55]. The title of [52] explicitly follows Leibniz in his distinction between a *calculus ratiocinator* (a calculus by means of which to reason) and a *lingua characteri[sti]ca* (a universal language), but that is not so apt, since Frege's system provides both, while model theory by itself provides neither. One does, however, reason systematically in the latter *about* properties of structures over given domains with respect to those properties that are expressible in the formal languages of logic.

Following a brief description of Löwenheim's work, van Heijenoort writes of his results and methods that they were "entirely alien to the Frege–Russell trend in logic. So alien that it is quite puzzling how [he] came to think of his theorem."[6] [62, p. 15] He continues on p. 16, "[w]ith Löwenheim's paper we have a sharp break with the Frege–Russell approach to the foundations of logic and a return to, or at least connection with, pre-Fregean or non-Fregean logic", i.e. primarily the logic of relations due to Peirce and Schröder. The essay concludes as follows:

> The first reaction to Löwenheim's paper was *Skolem* [47], which still follows the set-theoretic approach to logic. Soon, however, the opposition between the two trends in logic dissolved. During the 1920s the work of Skolem, Herbrand and Gödel produced an amalgamation and also a *dépassement* of these two trends [62, p. 16]

As to this supposed amalgamation, van Heijenoort should also have referred to Hilbert and Ackermann [23], where the completeness problem for first-order logic with respect to validity in arbitrary non-empty domains was first explicitly and precisely raised. That was of course established by Gödel [20], though both Skolem and Herbrand could have used their work to come to the same conclusion had they regarded it in the proper light. But the use of the word

---

[5] The novelty of the approach taken in Löwenheim [30] compared to that of the Frege–Russell tradition was already signaled in the first paragraph of the introductory note to it in van Heijenoort [51].

[6] Alfred Tarski met Löwenheim during a visit to Berlin in 1938, and in an interview forty years later with Herbert Enderton, he said that "Löwenheim told me I was the first logician with academic status that he had met or talked to in his life." [13, p. 103].

"amalgamation" is puzzling here (though a bit ameliorated by the use of "*dépassement*") in view of van Heijenoort's emphatic assertions that meta-systematic questions such as those of consistency, completeness, independence of axioms, etc., cannot be raised in the approaches of Frege and Russell [62, pp. 13–14].[7]

In the essay, "Absolutism and Relativism in Logic" [60], van Heijenoort relates the logical work of Frege and Russell in a different way to an *absolutist* tendency in philosophical doctrines, while, by contrast, that of the model-theoretic or set-theoretic approach is allied with *relativistic* tendencies. In the case of quantificational logic, van Heijenoort brings in the immediate question, "... over what domain are the quantifiers supposed to range?"

> At this point the opposition between absolutism and relativism in logic strikes us with full force. For an absolutist, there is just one domain, a fixed and all-embracing universe (either on one level or hierarchized in several levels) which comprehends everything about which there can be any discourse. Such was the conception of Frege, such was also the conception of Russell, though for him this universe was stratified according to the theory of types. Under the name of *logica magna*, such a universal system has been the constant dream among logicians. Logicism is a modern form of *logica magna*. The well known difficulties with logicism have led contemporary logicians, for the most part, away from that dream. Rather than being a *logica magna*, present-day logic is a *logica utens*; systems are introduced, here and there, according to needs. Different domains are successively considered for interpretations. In that sense, relativism has at present the upper hand [62, pp. 79–80]

In both these essays, it would have been useful to make a further basic distinction between two types of model theory, as was well brought out in the survey by Vaught [63].[8] The first is *general* (or *theoretic*) model theory, while the second is *subject-specific* model theory. We may count under general model theory for first-order logic, such results as the Löwenheim-Skolem theorem (later extended in the "upward" direction by Tarski), the completeness theorem, the compactness theorem, the interpolation theorems, preservation theorems, the ultraproduct theorem, and so on. Under the subject-specific side one would count the study of models of arithmetic, of algebraic theories (groups, rings, fields, etc.), of analysis (including non-standard models of the reals), of set theory, and so on. The ground-breaking work of Löwenheim, Skolem, Herbrand and Gödel of course belongs to the general side, while the subject-specific side was (arguably) initiated by Langford [28] via the method of elimination of quantifiers for the theory of dense orders, which was then exploited systematically by Tarski, most notably for the theory of the ordered

[7] That is a rather controversial position that has led to much discussion in the literature; for a number of references, see note 26 on p. 444 of Mancosu, Zach and Badesa [31]. It has also been vigorously disputed publicly by Saul Kripke, though not in writing as far as I know.

[8] *Cf.* also [31].

field of real numbers [50], and his students, beginning with Presburger [44] for additive arithmetic.[9] I think the appellation *logica utens* would more properly be applied to the subject-specific side of model theory, though results from the general side are ubiquitous in that work. And one may take note that even relativists can be absolutists when it comes to methodology; for example, Tarski constantly promoted the development of model theory by purely model-theoretic means, to be carried out without any call on the completeness theorem or the proof theory of first-order logic.

On the other hand, the spirit that infused the Frege–Russell aim for a *logica magna* was transferred to the pursuit of a *lingua magna* for mathematics, namely that of set theory, and—more recently as a competitor—that of category theory. Van Heijenoort has nothing to say about these. It would have been useful to look into the commonly voiced idea that set theory (resp. category theory) provides a universal language for mathematics, since Tarski's theorem on the undefinability of truth for a language within that language would seem to provide an immediate counter-example. Moreover, consideration of set theory might have led to a reconsideration of what is said in the concluding paragraph of [60], namely that "[t]he failure of absolutism in logic is the failure of realism, that is, of a conception for which experience is transmuted into a reality independent of any process of knowledge. This is not a conception that the historical development of science seems to favor." [62, p. 83]. That is confuted on the one hand by the explicit realism of a number of set theorists, most notably Zermelo and Gödel, and the implicit realism of working mathematicians in general, including those like Tarski working on both set theory and model theory.

The essay [60] touches on a different kind of absolutism in logic, namely the view that logic is to be identified with *classical logic*, that is the logic of truth and falsity and the Law of the Excluded Middle. The departure from that in intuitionistic logic, or the logic of proof rather than truth, is said to introduce a kind of relativism, though again the promoters of intuitionism, beginning with Brouwer, were absolutists in their claims as to the nature of logic. On the side of relativism, van Heijenoort might also have mentioned the so-called *alternative* (or *deviant*)*logics*; he could at least have brought in modal logic in connection with his reference to Kripke models for intuitionistic logic on p. 82.[10]

I said above that van Heijenoort should also have mentioned in [52] the place of Hilbert in the amalgamation between the Frege–Russell axiomatic tradition and the Löwenheim *et seq.* purely model-theoretic tradition. Actually, he makes up for that by detailing that role as follows in the article on set-theoretic semantics [55].

---

[9] Instead of dating the subject-specific side of model theory to Langford [28], a case can be made that it goes back to the so-called American postulate theorists, most prominently through the work of Veblen and Huntington on the completeness and categoricity of various mathematical axiom systems; see [46] and [31, pp. 326–329].

[10] There is a brief dip into modal logic in the one page essays (1974) [62, p. 35], (1974*a*) [62, p. 37], and letter to Quine (1974*b*) [62, p. 39].

> Hilbert's position is somewhere between that of Frege–Russell and that of Peirce–Schröder–Löwenheim. Like the former, he works with his axioms and rules. With his mathematician's instinct, however, he is inclined to consider quantifiers ranging not over 'everything', but rather over well-defined collections of objects. . . . So, when the fusion of the two currents took place in the 1920s, Hilbert was one of its agents. The notion of a formal deductive system supplemented by an interpretation appears in Hilbert's logical writings, and in *Hilbert and Ackermann* [23] the problem of the completeness of first-order logic is posed in exactly the terms in which it will be solved in *Gödel* [20] [62, pp. 45–46]

This got me to thinking why Hilbert was not also the agent in the development of the subject-specific model theory of mathematical systems. There are some simple examples in Hilbert and Ackermann [23] of axiom systems for mathematical notions. But Hilbert did not call there on his experience with models of geometry in his work on the foundations of that subject. Moreover, the structural approach to mathematics that had its origins in the nineteenth century with the theories of groups and number fields[11] in which Hilbert was thoroughly versed was in full swing by the time he gave his lectures on logic in the period 1917–1922 on which his book with Ackermann is directly based. But perhaps it is because it was in that period that his finitist consistency program to secure the foundations of mathematics was being formulated and dominated his thoughts about logic. The main precondition for that program was the formalizability of mathematics in successive axiom systems for arithmetic, analysis and so on within a precise deductive basis given by the general work on logic. Indeed, Hilbert concludes his preface to the book with Ackermann by saying that it also serves to prepare one for the understanding of a book to come by him and P. Bernays on the foundations of mathematics, namely, as we know, Hilbert and Bernays [24].

3. Set-Theoretic Semantics and the Semantics of Natural Language

The first part of the essay "Set-theoretic Semantics" [55] traces the history of that subject back to ideas of Bolzano in the 1830s and forward to Tarski and the beginnings of systematic model theory a century later. There is some overlap in that respect with the essays discussed in the preceding section. But in the remainder of [55], interesting questions are raised about the problem of constructing "a semantics of ordinary language on a precise basis." [62, p. 47] These address first mass terms, then some limits of and objections to Montague semantics, and finally the idea of a *dictionnaire raisonné* (rational dictionary). I shall take up only the first and third of these, since the value and problems

[11] *Cf.* Corry [8].

of Montague semantics require getting into Montague's elaborate framework and have been much discussed elsewhere (*cf.* e.g., Partee and Hendricks [41] together with the references thereto).

In a mere two pages [55] [62, pp. 48–49], van Heijenoort brings out some of the central problems concerning the syntax and semantics of mass terms. Actually, this is mostly a reprisal of the last part of "Subject and predicate in Western logic" [53], but with the addition of a suggested approach that I shall explain below. I can do no better to explain van Heijenoort's account than to quote him directly.

> Just to question the universality of set-theoretic semantics I would like to raise here the question of mass terms. These terms, like *water*, *gold*, *dirt*, are opposed in their syntax to the so-called count terms, like *men*, *tables*, *chairs*. With them we use the question *How much*? instead of *How many*? ... Mass terms can, in a sentence, be used before the copula (*Iron is a metal*) or after (*This is gold*). They can be used with demonstratives (*This water is dirty*), possessives (*My coffee is cold*) or the definite article (*The wine that you gave me was sour*) to yield what I would call quasi-individual terms. They can also be used with so-called container words (*three buckets of water*, *two pounds of sugar*)... There is here a rich field for linguistic investigations. [62, p. 48]

It is also remarked that in addition to the problems of dealing with such concrete mass terms one also has those raised by abstract terms, like *courage*, *wisdom*, and so on, that behave linguistically like them. Of the literature on the proper logical means to treat mass terms, van Heijenoort refers only to work of Quine (presumably 1960 [45]) and Strawson (presumably 1959 [49]). He describes these as "brave attempts" that "can be pushed up to a certain point [but] then end up by doing violence to the linguistic facts." Van Heijenoort proposes instead that "[a]s we have a universal domain of individuals for count terms, we now consider a universal lump out of which we take slices."

> Since quantifiers cannot be interpreted except in a discrete domain, the proper treatment of mass terms requires a variable-free system. But several such systems are at hand today, and there does not seem to be any difficulty in adapting one of them to the semantics of the universal lump. One can only wonder why all that has not been done before. It is ironic that Tarski's famous example, 'Snow is white,' involves a mass term, and Tarski's semantics would have met with difficulties if sentences containing other uses of 'snow' had been considered. [62, p. 48]

What surprises me is that van Heijenoort was apparently ignorant of a growing body of literature in which real steps were being taken to deal with mass terms in a philosophically satisfactory and logically precise way.[12] Deserving special mention are the articles of Parsons [40], Burge [6], Moravcsik [36], Montague [34], Pelletier [42], and Bealer [2]. The articles of Moravcsik and Montague appeared in a collection, Hintikka et al. [27], devoted to approaches to natural language. And Bealer's article appeared in a special issue of the journal *Synthèse* (vol. 31, 1975) devoted specifically to mass terms; incidentally, all of its articles were later reprinted in the collection Pelletier [43], along with several additional pieces. Presumably, the issue of *Synthèse* in question would have been available to van Heijenoort by the time he wrote 1976 [55].

In his frequently cited 1970 article [40], Terence Parsons initiated a logically precise treatment of mass terms motivated by consideration of the following three sentences involving the mass term 'gold':

(a) My ring is gold.
(b) The element with atomic number 79 is gold.
(c) The particular bit of matter which makes up my ring is gold.

In Parsons' formal language, mass terms are given by individual constants s. Associated with each such are three relations, xCs, xQs and equality. xCs is read as "The object x is constituted of [or made of] the substance s", and xQs is read as "The bit of matter which makes up x is a quantity of s." Thus, if one takes 'g' for gold, 'r' for my ring, 'e' for the element with atomic number 79, and 'm' for the bit of matter which makes up my ring, the formal representations of (a)–(c) are given respectively by:

(a_1) rCg
(b_1) $\mathrm{e} = \mathrm{g}$
(c_1) mQg.

Note in this last that 'g' corresponds to van Heijenoort's "universal lump" of gold, and 'm' corresponds to a "slice" of that. As to his suggestion that the treatment of mass terms requires a variable-free system, while it is true that the mass terms themselves will be given by constants, the use of variables in relation to those constants with the above relations is essential to express statements such as "Gold is malleable," or more explicitly, "Every quantity of gold is malleable," rendered as $\forall x(xQg \rightarrow Mx)$. Parsons introduces further means to deal with the formation of complex mass terms, such as "unmined gold," as well as with quantities, such as "Most gold is unmined," among others. Montague [34] and Bealer [2] largely followed Parsons [40] with some elaborations. Whether their approach would have met with van Heijenoort's approval is of course something we cannot know, but it is a pity that he was not informed of it. At any rate, one fully worked out logically precise system for the syntax and semantics of mass terms only became available after the writing of 1976 through the work of Bunt [4,5].

[12] There are no references concerning mass terms in the article [55] itself, except, without citation, to Strawson and Quine, as already mentioned; however, we find explicit references to Strawson [49] and Quine [45] in the discussion of mass terms in the 1973 essay [53]. One also finds Cartwright [7] in the references to the volume, without any mention in either [53] or [55].

The last part of 1976 [55] starts off with the idea that a *dictionnaire raisonné* (rational dictionary) would set down various relations, such as those that relate *fox* and *vixen*, in the hope that "if we could codify these relations, we would be able to read off the logical entailments that obtain among sentences of a natural language like English. The entailement of (1) *No man lives forever*, by (2) *All men are mortal*, should emerge from the proper semantical analysis of *mortal*, *to live* and *forever*. And we should be able, progressively, to deal with more and more complex examples." (1976) [62, pp. 50–51] [55] But, van Heijenoort says, many problems immediately present themselves to the idea of such a dictionary; he does not think that such a work cannot be done, but that "before it can be done (perhaps with the aid of computers), certain fundamental questions have to be answered, questions that have not yet been formulated properly." One is that the meanings of words is affected by their context, they interact with each other; that might require a dictionary of expressions as well as of words. Then there is the problem of idioms, all too familiar to the translator. And so on. Sure, one has the experience of beginners' courses in logic, where sentences of natural language are translated into the language of quantification theory, where logical entailments can be examined in one way or another. But "[m]any things get bulldozed. Given a sentence that we consider to be logically atomic, we pick one, two or three parts in it, which, we decide, work as individual terms, and we push everything else into the predicate. Adverbs suffer, and many other things. But nevertheless, within the limits of what we intend to do, it works." That agrees with the experience of teachers of introductory logic, mine included. As to the general project, I do not know of any work since the writing of 1976 [55] that would come close to fulfilling van Heijenoort's conception of what a rational dictionary should be. But a significant initial approximation may be provided by Douglas Lenat's computerized language CycL (extending the language of first order quantification theory) begun in the mid-1980s that is intended to codify common sense reasoning in a vast knowledge data base encompassing hundreds of thousands of terms and rules of thumb, such as "If you are dead, you can't respond to mail." (*Cf.*, e.g., [29]).

Van Heijenoort's essay on set-theoretic semantics concludes with the following fervently expressed and in certain ways personally revealing paragraph.

> The problem is to pass from the local treatment of the semantics of a natural language to a global one. One feels that it is possible and that it should be done. Why? Because the handling of words is an exact and precise activity. Whoever writes with some care knows that replacing one word by another, apparently close, may change the meaning of a sentence appreciably; even a comma, added or taken away, has its effect. The syntactic codification of such facts cannot be the full answer. Unless God-given, syntactic rules must be grounded somewhere. And where, if not in meaning? But between the exact working of a natural languge, with the convolutions and intricacies of its meanings, and the logical skeleton representable

> in quantification theory there is a large gap, in which we are still groping. (1976) [62, p. 53].

4. Van Heijenoort on Frege's Sinn

At the beginning of "Sense in Frege" [56], van Heijenoort writes that "Frege's distinction between the *Bedeuting* and the *Sinn* of an expression is well known. Also well known are his reasons for introducing the distinction." Familiar examples provided by Frege are two senses of the number 4 given by '$2 + 2$' and '2^2', and two senses of the same person given by 'Mark Twain' and 'Samuel Clemens'. For Frege, the expressions which have a sense are simply those that are *names*.[13] In particular, sentences, according to him, are names of the True or the False, so one may well ask what is meant by the sense of a sentence; however, Frege never proposes an answer to this question. What van Heijenoort advances in this essay is a definition of the sense of sentences in the language of Frege's *Grundgesetze der Arithmetik* [16,17], setting aside concerns about the consistency of Frege's system. One critique of his approach was raised in a review by David Bell [3, p. 227]: "Although van Heijenoort has valuable things to say about Frege's place in the history of logic, he is much less reliable when he comes to look more closely at the details of Frege's thought. And in this respect one article, 'Sense in Frege', is particularly poor." I am not competent to judge the merits of the particular criticisms which support Bell's severe assessment, concerning as they do various specific features of the *Grundgesetze*.[14] But it seems to me that even if van Heijenoort is not faithful to Frege's system, there is still something of interest to be found in his approach when taken at face value.

One explanation of the sense of a sentence is the mode of presentation of its putative reference (*cf*. fn. 14), and that is determined by the way it is built up syntactically; looked at that way it may be considered to be a particular kind of finite tree. Then the sense of the sentence consists of an abstract finite tree together with an assignment of certain entities to the terminal nodes. The problem with this idea, as van Heijenoort sees it, is that

> [i]n the syntactic tree of a name, a terminal node may be occupied by a variable [but] there is, among Frege's primitive entities, none that can be assigned to such a node. To put it differently, Frege's

[13] According to Philippe de Rouilhan (personal e-mail communication, May 18, 2012), for Frege, names in ordinary language need not have a reference, in contrast to names in scientific language.

[14] In a personal e-mail communication (Feb. 16, 2012), Yiannis Moschovakis says that he disagrees with Bell's assertion that van Heijenoort got Frege wrong. His impression is rather that the fault in re these issues lies with Frege, "who was incomplete on the first (what senses are) and incoherent on the second (what are the identity conditions on them)."

> system is really a function calculus, and names are terms of this calculus.
>
> The difficulties in dealing with individual variables in a function calculus are well known and are precisely those that have led to combinatory logic. [62, p. 56]

Thus, what van Heijenoort does is list what he takes to be the eight primitive functions of Frege's system and translate every sentence of the language of that system, call it L_F, into a closed term of a certain typed combinatory language, call it L_C. The *types* (or *type symbols*) used in L_C are generated from the type ι of all objects (*qua* individuals), by closing under the formation of $(\alpha \to \beta)$, the type of all functions whose arguments are of type α and whose values are of type β; the truth values T and F are taken to be among the individuals. We can think of the translation of L_F into L_C as being mediated by the typed λ-calculus. Van Heijenoort gives $\forall x(x = x)$ and $\forall X \forall x(Xx \supset Xx)$ as examples, where 'x' ranges over individuals and 'X' over predicates, i.e. functions from individuals to truth values, thus of type $(\iota \to \iota)$. Among the primitive functions are the negation N, the conditional C, the identity I and the functions U and V for universal quantification over individuals and predicates, resp.; N is of type $(\iota \to \iota)$, C and I are both of type $(\iota \to (\iota \to \iota))$, U is of type $((\iota \to \iota) \to \iota)$, and V is of type $(((\iota \to \iota) \to \iota) \to \iota)$. The translation of $\forall x(x = x)$ is $U(\lambda x.Ixx)$, which can be rewritten as $U(WI)$ where W is the combinator given by $Wfx = fxx$. The translation of $\forall X \forall x(Xx \supset Xx)$ is $V(\lambda X.(U(\lambda x.WC(Xx)))$; this then takes the form $V(B^\alpha U(B^\beta(WC)))$ for suitable α, β where for various types τ, B^τ is the combinator given by $B^\tau uvw = u(vw)$. The functions W and the various B^τ are auxiliary to the eight primitive functions; the tree assigned as sense to $\forall X \forall x(Xx \supset Xx)$ thus has V, B^α, U, B^β, W and C at its terminal nodes and is binary branching at each function application in the term $V(B^\alpha U(B^\beta(WC)))$. The decision to identify the sense of a sentence A in L_F with the tree of its translation t_A into L_C such as given for this particular sentence seems to be motivated by the desire to have it appear as a mathematical object rather than as a syntactical expression. Note that there is some choice as to the auxiliary functions to be used. For example, as is well known, one can transform every closed term of the typed lambda calculus into one of the typed combinatory calculus that makes use of two basic combinators K and S in all appropriate types; in particular, each of W, B^α and B^β can be rewritten in terms of the K's and S's. There are other combinatory bases as well. Thus, the sense of an expression as defined by van Heijenoort will depend on the choice of the auxiliary functions; fewer such functions will in general lead to more complicated trees.

Note that van Heijenoort's procedure also serves to explain more generally the sense of the *names* ν of objects in Frege's system as just those which translate into closed terms t of type ι, with sentences among these being the names of the truth values. Van Heijenoort remarks in favor of his definition of sense that if a sentence A of L_F contains a name ν, then the sense of A contains the sense of ν as a part, namely as a subtree; this is supposed to be in accord with a principle asserted for *Sinn* by Frege. Furthermore, the *Bedeutung* of any name ν, as translated into a closed term t_ν of $\mathrm{L_C}$ can simply be taken to be the value of t_ν in the domain of all individuals.[15]

Van Heijenoort's definition of sense in 1977 [56] can be extended to a variety of other formal languages. The article concludes with the statement that the approach given there is "not well adapted to a natural language like English. For a natural language the definition has only a suggestive value and can have only piecemeal applications." The situation is said to be similar in this respect to the possible application of Tarski's notion of truth to natural language. As it stands, then, "[t]he limitation of our definition to formal languages must be clearly acknowledged, and applying it to other contexts is a tentative and fragmentary enterprise."

One obvious question to be raised about van Heijenoort's definition of sense in [56] is that no non-trivial criterion of identity of sense beyond identity of labelled trees is offered, or even raised as an issue. True, the question of such a criterion is taken up in the immediate follow-up essay, "Frege on Sense Identity" [57]. But instead of building on his own proposal in the preceding article, van Heijenoort spends most of it trying to understand Frege's view of the matter. The trouble is that Frege does not have much to say about it. Some trivial formal examples are quoted, such as those involving changes of bound variables, and there are some examples from natural language, such as transformation from the active to the passive voice; Frege also asserts that double negation is sense-preserving. Two longer passages from Frege are quoted, one from a 1906 letter to Husserl, and a related second one from a manuscript of the same year. In the first, "Frege is saying, fairly clearly, that A and B have the same *Sinn* if and only if the biconditional $A \equiv B$ can be established by 'purely logical laws'." ([62, p. 67]). Van Heijenoort points out that this has the defect that "two sentences not containing non-logical notions would have the same *Sinn* as soon as they have The same *Bedeutung*." [57] (see [62, p. 68]) In Frege's manuscript, on the other hand, it is also required that the truth of the biconditional be recognized "straight away" or "immediately." However, this has the defect that the relation of having the same sense is not transitive.

In an e-mail communication to me (Feb. 14, 2012), Yiannis Moschovakis made the following comments about Frege's letter to Husserl and the manuscript referred to by van Heijenoort. "The second paper [57] is much more interesting (than 1977 [56]) and a great deal more useful, because of the extensive quotes of the letter from Frege to Husserl, which (as JvH says) were not widely known. JvH tries in this paper to understand Frege, which is very difficult because the great man was basically incoherent and self-contradictory on the subject of synonymy. ... As I read the two long passages quoted by JvH, I get one coherent principle that Frege seems to expound, and this is that the sense of A should be (somehow) derived by (truth functional) logic alone not by 'poetic fragrance', etc. It is not so clear that this is what Frege is saying,

[15] The "computation" of this value is necessarily infinitary when ν involves quantification over individuals or predicates.

and he never says it directly, because, I suppose, it is not easy to say. I usually attribute this principle to Davidson who expressed it by saying (in effect) that the sense of A is contained in the 'truth conditions' [for] A. ... I am not certain that I read that in JvH's article, but his center-staging the quotes from Frege certainly amounted to a very useful contribution of his second article on sense in the Bibliopolis book."

Curiously, van Heijenoort only mentions his proposed definition of sense in [56] in the very last sentences of [57], where he says that "[s]ince Frege's system has the features of a function calculus... the problem of *Sinn* identity is that of the intensional identity of functions." As it stands, two sentences A and B of $\mathrm{L_F}$ have the same sense according to 1977 [56] if and only if their translations t_{A} and t_B in $\mathrm{L_C}$ are syntactically identical. This would not, for example, give $\neg\neg A$ and A the same sense. Given the extensive literature on λ-calculi and combinatory calculi available to him at that time, one would have expected van Heijenoort to follow up his translation with a criterion of sense identity for terms in the combinatory language $\mathrm{L_C}$ in terms of a standard notion of equivalence for such languages. Namely, a suitable reflexive and transitive reduction relation $s \geq t$ would be introduced for terms s and t of $\mathrm{L_C}$, with respect to which a term t is said to be in normal form if it is irreducible. Then an obvious proposal would be to define $s \equiv t$ to hold just in case s and t are both reducible to a term r in normal form. The crucial question would be, how is $\geq$ to be defined? This is standard for the auxiliary combinators K and S in the various types, with $\mathrm{K}st \geq s$ and $Sstu \geq su(tu)$. It would also be natural to take $NT \geq F$ and $NF \geq T$ for the negation operator; this satisfies that NNt and t reduce to a common normal form for t reducible to T or F. The reduction relation for the conditional C would also be given by its truth table. But what about the identity I and the quantifiers U and V (not to mention the other primitive functions)? The semantic specifications of these are infinitary; presumably one would use instead, versions of the instantiation schemes for the quantifiers. However, that would not serve to reduce every sentence to T or F, and, unless further *ad hoc* conditions on $\geq$ are added, we would not have, for example, $\neg\neg A \equiv A$.[16]

Apropos of all this is the interesting work of Moschovakis [37,38] which proposes a substantial theory of meaning (sense) and synonymy (identity of sense) for certain formal languages and which unfortunately van Heijenoort did not live to see. In the earlier paper, this is carried out for a language for mathematical algorithms. As Moschovakis writes in the introduction to his 2006 paper [38], in the earlier work he argued that "the meaning of a term A can be faithfully modeled by its *referential intension* $\mathrm{int}(A)$, an (abstract, idealized, not necessarily implementable) algorithm which computes the denotation of A." The object $\mathrm{int}(A)$ is given in an extension of the typed λ-calculus by a reduction of A to a canonical form using a certain formal language of recursion, FLR; then A and B are taken to be synonymous if $\mathrm{int}(A) = \mathrm{int}(B)$. The 2006 article extends this procedure to "reasonably large" fragments of

[16] Of course, one might disagree with Frege that double negation is sense preserving.

natural language via the work of Montague [33,35]. That comes by translating (or rendering) the natural language expressions in those fragments into another extension of the typed λ-calculus, to which the methods of FLR are also applicable. Moschovakis notes that "[s]ome would argue that the [*rendering*] operation is the most important part of the extraction of meaning from linguistic expressions," with which he agrees. But he argues that "what comes next" via FLR is what is needed to give technical substance to the approach and in terms of which many specific examples can be treated in an illuminating way.

5. Van Heijenoort as a Philosopher of Logic

In rereading van Heijenoort's essays with concentration on the ones discussed above, I was repeatedly struck by the disparity between the many stimulating questions raised and their scant development. That surprised me especially on the relations of formal logic and semantics with the workings of natural language, given his broad historical knowledge of the former combined with his wide linguistic talents and concerns with precision of expression. Still, as I have tried to show above, for a full engagement, his interests and insights would only serve to be the starting point for a substantive pursuit of the topics in question. It may be questioned whether van Heijenoort was equipped for such deeper work given the lateness in his career at which the given issues were raised. We know that he had a genuine interest in philosophy going back to his school days, but his study of it as an adult seems mostly to have been self-directed, with that in the philosophy of logic stimulated indirectly by his work on the history of logic.

Other reasons for the lack of deeper immersion in these questions may be found in the many-pronged directions of van Heijenoort's life and work during the period 1967–1985 that is the span of these essays. In this respect we have two excellent sources, the biography of him by Anita Feferman [11] and the compendium of his career as an academic and as a researcher by Irving Anellis [1].[17] From 1965 until his retirement in 1977, van Heijenoort was a Professor of Philosophy at Brandeis University, and his teaching and work with graduate students certainly took up considerable time. And within a year after the publication of *From Frege to Gödel* [51] that had taken almost a decade of effort to bring to completion, he edited and published *Jacques Herbrand. Écrits logiques* [22]. Then during a good part of the 1970s in connection with his regular lectures on logic at Brandeis and a few special lectures in Mexico City and Paris, he devoted a great amount of attention (perhaps to the point of obsession) to the application of the Beth tableaux and falsifiability-trees methods to the soundness and completeness of various systems of logic. That

[17] See Anita B. Feferman [11], p. 307, for a bibliography of van Heijenoort's books and monographs referred to below, and [11, pp. 308–309] for his articles on logic and on politics (many under pseudonyms); see [1, pp. 313–324] for a complete bibliography of his works, reviews, and summary of the contents of his *Nachlass*.

resulted in two monographs, *El desarollo de la teoría de la cuantificación* in 1976 [54] and *Introduction à la sémantique des logiques non-classiques* in 1979 [59]. Meanwhile, in continued dedication to his past life in politics, he carried on his unremitting work on the Trotsky archives at Harvard,[18] wrote his own account of the years with Trotsky—published in 1978 as *With Trotsky in Exile* [58]—and in 1980 published an edition of the correspondence between Leon and Natalia Trotsky from the 1930s [61]. Finally, his personal life took a substantial new turn in 1969 with his marriage to Ana María Zamora, a transition that led him to spend more and more time in Mexico in addition to his work in Cambridge and Brandeis and his regular trips to France.[19] But then, after their divorce and subsequent remarriage, that relationship became increasingly tortured in the period from 1981 on, the same period during which he added to his responsibilities and transits the co-editorship of the Gödel *Collected Works* [21] at Stanford.[20] All of this can serve as an explanation for van Heijenoort's not carrying through his ideas with reference to the contemporary literature available to him and/or with a systematic development on his own. In any case, it is our loss that he did not manage to do so.

Acknowledgements

I have benefited greatly from the helpful comments on a draft of this article by John Dawson, Anita Feferman, Paolo Mancosu, Yiannis Moschovakis and Philippe de Rouilhan.

References

[1] Anellis, I.H.: Van Heijenoort: Logic and Its History in the Work and Writings of Jean van Heijenoort. Modern Logic Publishing, Ames (1994)

[2] Bealer, G.: Predication and matter. Synthèse **31**, 493–508 (1975). Reprinted in: [43]. pp. 279–294 (1979)

[3] Bell, D.: Review of [62]. History Philos. Logic **7**, 226–228 (1986)

[4] Bunt, H.C.: Ensembles and the formal semantic properties of mass terms. In: [43]. pp. 249–277 (1979)

[5] Bunt, H.C.: Mass Terms and Model-Theoretic Semantics. Cambridge University Press, Cambridge (1985)

[6] Burge, T.: Truth and mass terms. J. Philos. **69**, 263–282 (1972)

[7] Cartwright, H.: Quantities. Philos. Rev. **79**, 25–52 (1970)

[8] Corry, L.: Modern Algebra and the Rise of Mathematical Structures. Birkhäuser Verlag, Basel (2004)

[9] Dawson, J.W. Jr.: Jean van Heijenoort and the Gödel Editorial Project. THIS ISSUE (2012)

[18] See Ch. 14, "The archivist in spite of himself," of Anita Feferman's biography [11].

[19] See Ch. 15, "The door to life. Mexico: 1960–1980" of the biography [11].

[20] See Ch. 16, "The end. 1981–1986" of the biography [11].

[10] De Rouilhan, P.: De l'univérsalité de la logique. In: Bouveresse, J. (ed.), L'âge de la science: lectures philosophiques, 4: Philosophie de la logique et philosophie du langage. Éditions Odile Jacob, Paris, pp. 93–119 (1991). Revised English translation in THIS ISSUE (2012)

[11] Feferman, A.B.: Politics, Logic and Love. The Life of Jean van Heijenoort. A. K. Peters Ltd., Wellesley (1993). Reprinted as: From Trotsky to Gödel. The Life of Jean van Heijenoort. A. K. Peters Ltd., Natick (2001)

[12] Feferman, A.B., Feferman, S.: Jean van Heijenoort (1912–1986). In: Paris Logic Group (eds.) Logic Colloquium '85. North-Holland, Amsterdam, pp. 1–7 (1987). Reprinted with revisions: Modern Logic **2**, 231–238 (1992)

[13] Feferman, A.B., Feferman, S.: Alfred Tarski: Life and Logic. Cambridge University Press, Cambridge (2004)

[14] Feferman, S.: The Gödel editorial project: A synopsis. Bull. Symb. Logic **11**, 132–149 (2005). Reprinted in: [15], pp. 3–20 (2010)

[15] Feferman, S., Parsons, C., Simpson, S.G. (eds.): Kurt Gödel. Essays for his Centennial, Lecture Notes in Logic **33**, Association for Symbolic Logic, Cambridge University Press, Cambridge (2010)

[16] Frege, G.: Grundgesetze der Arithmetik, begriffsschriftlich abgeleitet, vol. 1, Pohle, Jena (1893). Partial translation in: [18]

[17] Frege, G.: Grundgesetze der Arithmetik, begriffsschriftlich abgeleitet. vol. 2, Pohle, Jena (1903). Partial translation in: [18]

[18] Frege, G.: (Furth, M., trans. and ed.) Basic Laws of Arithmetic. University of California Press, Berkeley/Los Angeles (1964)

[19] Gaifman, H.: Review of [55]. Math. Rev. **58**, #10270 (1979)

[20] Gödel, K.: Die Vollständigkeit der Axiome des logischen Funktionenkalküls. Monatshefte für Mathematik und Physik **37**, 349–360 (1930). English translation in: [51], pp. 582–591 (1967); reprinted, with English translation, in: [21]. vol. I, pp. 102–123 (1986)

[21] Gödel, K.: (Feferman, S., et al. eds.), Collected Works, 5 vols. Oxford University Press, Oxford/New York (1986–2003)

[22] Herbrand, J.: (van Heijenoort, ed.), Écrits logiques. Presses Universitaire de France, Paris (1968)

[23] Hilbert, D., Ackermann, W.: Grundzüge der theoretischen Logik. Springer, Berlin (1928)

[24] Hilbert, D., Bernays, P.: Grundlagen der Mathematik, vol. I. Springer, Berlin (1934)

[25] Hintikka, J.: On the development of the model-theoretic viewpoint in logical theory, Synthèse **77**, 1–36 (1988). Reprinted in: [26], pp. 104–139 (1997)

[26] Hintikka, J.: Lingua Universalis vs. Calculus Ratiocinator: An Ultimate Presupposition of Twentieth-Century Philosophy. Kluwer Academic Publishers, Dordrecht (1997)

[27] Hintikka, K.J.J., Moravcsik, J.M.E., Suppes, P. (eds.): Approaches to Natural Language. D. Reidel Publishing Co., Dordrecht (1973)

[28] Langford, C.H.: Some theorems on deducibility. Ann. Math. **28**, 16–40, 459–471 (1927)

[29] Lenat, D.B.: Building Large Knowledge Based Systems. Addison-Wesley Publishing Co., Boston (1989)

[30] Löwenheim, L.: Über Möglichkeiten im Relativkalkul, Mathematische Annalen **76**, 447–470 (1915). English translation in: [51], pp. 228–251 (1967)

[31] Mancosu, P., Zach, R., Badesa, C.: The development of mathematical logic from Russell to Tarski, 1900–1935. In: Haaparanta, L. (ed.) The Development of Modern Logic, pp. 318–470. Oxford University Press, Oxford (2009)

[32] McDermott, C.: Marginalia on van Heijenoort's "Subject and predicate" (from an Indological and a "Lagadogical" point of view). Philos. East West **24**, 269–274 (1974)

[33] Montague, R.: The proper treatment of quantification in ordinary English. In: [27], pp. 221–224 (1973). Reprinted in: [35], pp. 247–270 (1974)

[34] Montague, R.: Reply to Moravcsik. In: [27], pp. 289–294 (1973)

[35] Montague, R.: (Thomason, R.H., ed.) Formal Philosophy, Selected Papers of Richard Montague. Yale University Press, New Haven (1974)

[36] Moravcsik, J.M.: Mass terms in English. In: [27], pp. 263–285 (1973)

[37] Moschovakis, Y.N.: Sense and denotation as algorithm and value. In: Väänänen, J., Oikkonen, J. (eds.) Logic Colloquium '90. ASL Summer Meeting in Helsinki, July 15–22, 1990. Lecture Notes in Logic **2**. Springer-Verlag, Berlin/New York and Association for Symbolic Logic, Natick, pp. 210–249 (1993)

[38] Moschovakis, Y.N.: A logical calculus of meaning and synonymy. Linguist. Philos. **29**, 27–89 (2006)

[39] Padilla-Gálvez, J.: Lógica, lenguaje y teoría de la cuantificación (a propósito de los 'Selected Essays' de Jean van Heijenoort), Modern Logic **2**, 260–280 (1992)

[40] Parsons, T: An analysis of mass and amount terms. Found. Lang. **6**, 363–388 (1970). Reprinted in: [43], pp. 137–166 (1979)

[41] Partee, B.H., Hendricks, H.W.: Montague grammar. In: van Benthem, J., ter Meulen, G.B.A. (eds.) Handbook of Logic and Language, pp. 5–91. Elsevier, Amsterdam (1997)

[42] Pelletier, F.J.: On some proposals for the semantics of mass terms. J. Philos. Logic **3**, 87–108 (1974)

[43] Pelletier, F.J. (ed.): Mass Terms: Some Philosophical Problems. D. Reidel Publishing Co., Dordrecht (1979)

[44] Presburger, M.: Über die Vollständigkeit eines gewissen Systems der Arithmetik ganzer Zahlen, in welchem die Addition als einzige Operation hervortritt. In: Leva, F. (ed.) Sprawozdanie z 1. Kongresu matematyków krajów s lowiańskich/Comptes-rendus du I Congrès des Mathématiciens des Pays Slaves, Varsovie 1929. Książnica atlas t.n.s.w., Warszawa, etc. pp. 92–101 (1930). English translation in: History and Philosophy of Logic **12**, 225–233 (1991)

[45] Quine, W.V.O.: Word and Object. MIT Press, Cambridge (1960)

[46] Scanlan, M.: Who were the American postulate theorists? J. Symb. Logic **56**, 981–1002 (1991)

[47] Skolem, Thoralf (1920), Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit oder Beweisbarkeit mathematischer Sätze nebst einem Theoreme über dichte Mengen, Videnskapsselskapets Skrifter, I. Matematisk-Naturvidenskabelig Klasse 4 (1920). English translation in: [51], pp. 252–263 (1967)

[48] Smith, P.B.: Review of [56] and [57]. Math. Rev. **58**, ##4990a-b (1979)

[49] Strawson, P.F.: Individuals. Methuen, London (1959)

[50] Tarski, A.: A Decision Method for Elementary Algebra and Geometry. RAND Corp., Santa Monica (1948); 2nd ed., University of California Press, Berkeley (1951)

[51] Van Heijenoort, J.: From Frege to Gödel. A Source Book in Mathematical Logic, 1879–1931. Harvard University Press, Cambridge (1967)

[52] Van Heijenoort, J.: Logic as calculus and logic as language. In: Cohen, R.S., Wartofsky, M.W. (eds.) Boston Studies in the Philosophy of Science 3: In Memory of Russell Norwood Hansen, Proceedings of the Boston Colloquium for Philosophy of Science 1964/1965. Reidel, Dordrecht, pp. 440–446 (1967). Reprinted as (1967*a*) in: [62], pp. 11–16 (1985)

[53] Van Heijenoort, J.: Subject and predicate in Western logic. Phil. East West **24**, 253–268 (1974); reprinted as (1973) in: [62], pp. 17–34 (1985)

[54] Van Heijenoort, J.: El desarrollo de la teoría de la cuantificación, Universidad Nacional Autónoma de México, Mexico City (1976)

[55] Van Heijenoort, J.: Set-theoretic semantics. In: Gandy, R.O., Hyland, J.M.E. (eds.) Logic Colloquium '76. North-Holland, Amsterdam (1977). Reprinted as (1976) in: [62], pp. 43–53 (1985)

[56] Van Heijenoort, J.: Sense in Frege. J. Philos. Logic **6**, 93–102 (1977). Reprinted as (1977) in: [62], pp. 55–63 (1985)

[57] Van Heijenoort, J.: Frege on sense identity. J. Philos. Logic **6**, 103–108 (1977). Reprinted as (1977a) in: [62], pp. 65–69 (1985)

[58] Van Heijenoort, J.: With Trotsky in Exile: From Prinkipo to Coyoacán. Harvard University Press, Cambridge (1978)

[59] Van Heijenoort, J.: Introduction à la sémantique des logiques non-classiques. Collection de l'École Normale Supérieur des Jeunes Filles, no. 16, Paris (1979)

[60] Van Heijenoort, Jean (1979), Absolutism and relativism in logic, English draft of a paper read in Spanish at the Tercer Coloquio nacional de filosofía, Puebla. Reprinted as (1979a) in: [62], pp. 75–83 (1985)

[61] Van Heijenoort, J. (ed.): Leon et Natalia Trotsky: Correspondance 1933–1938. Gallimard, Paris (1980)

[62] Van Heijenoort, J.: Selected Essays. Bibliopolis, Naples (1985)

[63] Vaught, R.L.: Model theory before 1945. In: Henkin, L., et al. (eds.): Proceedings of the Tarski Symposium, Proceedings of Symposia in Pure Mathematics, vol. XXV, American Mathematical Society, Providence (1974)

Solomon Feferman
Stanford University, Stanford, CA, USA
e-mail: `feferman@stanford.edu`

Received: June 1, 2012.
Accepted: June 6, 2012.