THE OPERATIONAL PERSPECTIVE

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Advances in Proof Theory
In honor of Gerhard Jäger's 60th birthday
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Operationally Based Axiomatic Programs

- The Explicit Mathematics Program
- The Unfolding Program
- A Logic for Mathematical Practice
- Operational Set Theory (OST)

Foundations of Explicit Mathematics

- Book in progress with Gerhard Jäger and Thomas Strahm, with the assistance of Ulrik Buchholtz
- An online bibliography

The Unfolding Program

- Open-ended Axiomatic Schemata; language not fixed in advance
- Examples in Logic, Arithmetic, Analysis, Set Theory
- The general concept of unfolding explained within an operational framework

Aim of the Unfolding Program

- S an open-ended schematic axiom system
- Which operations on individuals--and which on predicates--and what principles concerning them ought to be accepted once one has accepted the operations and principles of S?

Results on (Full) Unfolding

- Non-Finitist Arithmetic (NFA);
  $|\mathcal{U}(\text{NFA})| = \Gamma_0$
- Finitist Arithmetic (FA):
  $\mathcal{U}(\text{FA}) \equiv \text{PRA}$, $\mathcal{U}(\text{FA} + \text{BR}) \equiv \text{PA}$
- (Feferman and Strahm 2000, 2010)

Unfolding of ID_1

- $|U(ID_1)| = \psi(\Gamma_{\Omega+1})$
  (U. Buchholtz 2013)
- Note: $\psi(\Gamma_{\Omega+1})$ is to $\psi(\varepsilon_{\Omega+1})$ as Γ_0 is to ε_0.

Problems for Unfolding to Pursue

- Unfolding of analysis
- Unfolding of KP + Pow
- Unfolding of set theory

Indescribable Cardinals and Admissible Analogues Revisited

- Aim: To have a straightforward and principled transfer of the notions of indescribable cardinals from set theory to admissible ordinals.
- A new proposal and several conjectures, suggested at the end of the OST paper.
- NB: Not *within* OST

Aczel and Richter Pioneering Work

- Aczel and Richter [A-R] (1972)
  Richter and Aczel [R-A] (1974)
- In set theory, assume κ regular > ω.
- Let f, g: $\kappa \rightarrow \kappa$; F(f) = g type 2 over κ.

[A, R]-2

- F is *bounded* $\Leftrightarrow$ $(\forall f: \kappa \to \kappa)(\forall \xi < \kappa)$

  [$F(f)(\xi)$ is det. by $< \kappa$ values of f]

- α is a *witness* for F $\Leftrightarrow$ $(\forall f: \kappa \to \kappa)$

  [$f : \alpha \to \alpha \Rightarrow F(f): \alpha \to \alpha$]

- κ is 2-*regular* iff every bounded F has a witness.

[A, R]-3

- Notions of bounded, witness, n-regular for $n > 2$ are "defined in a similar spirit", but never published.
- Theorem 1. κ is $n+1$-regular iff κ is strongly Π^1_n-indescribable.
- Proved only for $n = 1$ in [R-A](1974).

[A, R]-4

- Admissible analogues:
- Assume κ admissible > ω
- κ is n-*admissible*, obtained by replacing 'bounded' in the defn. of n-regular by 'recursive', functions by their Gödel indices, and functionals by recursive functions applied to such indices.

[A, R]-5

- Theorem 2. κ is n-admissible iff κ is Π^0_{n+1} reflecting.
- Proved only for n = 2 in [R-A](1974).
- Proposed:
  Least Π^0_{n+2}-reflecting ordinal ~ least [strongly] Π^1_n-indescribable cardinal.

A Proposed New Approach

- Directly lift to card's and admissible ord's notions of *continuous functionals of finite type* from o.r.t.
- Kleene (1959), Kreisel (1959)
- Deal only with objects of *pure type n*.
- $\kappa^{(0)} = \kappa$; $\kappa^{(n+1)}$ = all $F^{(n+1)}: \kappa^{(n)} \to \kappa$.

“Sequence Numbers” in Set Theory

- Assume κ a strongly inaccessible cardinal.
- Let $\kappa^{<\kappa}$ = all sequences s: $\alpha \rightarrow \kappa$ for arbitrary $\alpha < \kappa$.
- Fix $\pi: \kappa^{<\kappa} \rightarrow \kappa$, one-one and onto; so $\pi(g{\restriction}\alpha)$ is an ordinal that codes $g{\restriction}\alpha$.

Continuous Functionals and Their Associates

- Inductive definition of $F \in C^{(n)}$, and of *f is an associate of F*, where f is of type 1:
- For n = 1, f is an associate of F iff f = F.
- For $F \in \kappa^{(n+1)}$, f is an associate of F iff for every G in $C^{(n)}$ and every associate g of G,

Continuous Functionals and Their Associates (cont'd)

- (i) $(\exists\alpha, \beta < \kappa)(\forall\gamma)[\alpha \leq \gamma < \kappa \Rightarrow f(\pi(g{\restriction}\gamma)) = \beta + 1]$, and
- (ii) $(\forall\gamma, \beta < \kappa)\ [f(\pi(g{\restriction}\gamma)) = \beta + 1 \Rightarrow F(G) = \beta]$.
- F is in $C^{(n+1)}$ iff F has some associate f.

Witnesses

- For F in $C^{(n)}$ and $\alpha < \kappa$, define α is a witness for F, as follows:
- For n = 1, and F = f, α is a witness for F iff $f : \alpha \to \alpha$.
- For $F \in C^{(n+1)}$, α is a witness for F iff $(\forall G \in C^{(n)})[\ \alpha$ a witness for $G \Rightarrow$

  $F(G) < \alpha\ ]$.

$C^{(n)}$-Regularity; Conjectures

- κ is *$C^{(n)}$-reg* for n > 1 iff every F in $C^{(n)}$ has some witness α < κ.

- Conjecture1. For each n ≥ 1, the predicate *f is an associate of some F in $C^{(n+1)}$*, is definable in Π^1_n form.

- Conjecture 2. For each n ≥ 1,
  κ is $C^{(n+1)}$-reg iff κ is strongly Π^1_n-indescribable.

- Conj-2 holds for n = 1 by [R-A] proof.

Analogues over Admissibles

- Consider admissible $\kappa > \omega$.
- For analogues in (κ-) recursion theory replace functions of type 1 by indices ζ of (total) recursive functions $\{\zeta\}$.
- But then at type 2 (and higher) we must restrict to those functions $\{\zeta\}$ that act extensionally on indices.

Effective Operations over Admissibles

- Following Kreisel (1959), define the class E_n of *(κ-) effective operations of type n*, and the relation $\equiv_n$ by induction on $n > 0$:
- E_1 consists of all indices ζ of recursive functions;
  $\zeta \equiv_1 \nu$ iff for all ξ, $\{\zeta\}(\xi) = \{\nu\}(\xi)$.

Effective Operations over Admissibles (cont'd)

- $\zeta \in E_{n+1} \Leftrightarrow \{\zeta\}: E_n \rightarrow \kappa$ and

  $(\forall \xi, \eta \in E_n)[\ \xi \equiv_n \eta \Rightarrow \{\zeta\}(\xi) = \{\zeta\}(\eta)];$

  $\zeta \equiv_{n+1} \nu \Leftrightarrow (\forall \xi \in E_n)[\{\zeta\}(\xi) = \{\nu\}(\xi)].$

- Conjecture 3. Every type n+1 effective operation is the restriction of a functional in $C^{(n+1)}$.

- This would show why can drop the boundedness hypothesis in analogue.

Witnesses for Effective Operations

- For ζ in E_1, α is a witness for ζ iff $\{\zeta\}: \alpha \to \alpha$.
- For ζ in E_{n+1} when $n \geq 1$, α is a witness for $\zeta \Leftrightarrow (\forall \xi \in E_n)$

  [α a witness for $\xi \Rightarrow \{\zeta\}(\xi) < \alpha$].
- κ is *E_n-admissible* if each ζ in E_n has some witness $\alpha < \kappa$. (Equiv. to [A, R] n-admiss.)

Further Work

- Settle the conjectures.
- (Scott)The partial equivalence relation approach to types in λ-calculus models over $\mathcal{P}$(N) gives a "clean"definition of the Kleene-Kreisel hierarchy. Can this idea be generalized to $\mathcal{P}(\kappa)$? [What about effective operations?]

Further Work (cont'd)

- The present approach leaves open the question as to what is the proper analogue for admissible ordinals--if any--of a cardinal κ being Π^m_n-indescribable for $m > 1$.

The End