Indescribable cardinals and admissible analogues

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The following consists of precise formulations and several conjectures spelling out ideas that were suggested at the conclusion of Feferman (2009), "Operational set theory and small large cardinals." The aim is to have a straightforward and principled transfer of the notions of indescribable cardinals from current set theory (as based, say, on ZFC) to admissible ordinals.[1] Aczel and Richter (1972) pioneered the way for this, first in outline in that publication, and then with a number of follow-up details in Richter and Aczel (1974). However, those details were provided only for Π^1_1 indescribable cardinals, roughly as follows.

First, looked at within set theory, let κ be any regular uncountable cardinal (also called a 1-*regular* cardinal), and let f, g range over all functions from κ to κ, while F ranges over functionals F(f) = g of next higher type. F is said to be *bounded* if for every f: $\kappa \to \kappa$ and every $\xi < \kappa$, the value of F(f)(ξ) is determined by less than κ values of f. α is said to be a *witness* for F if for all f: $\kappa \to \kappa$, if the restriction of f to α maps α into α then the same holds for F(f). κ is said to be 2-*regular* if every F that is bounded has a witness. It is stated in Aczel and Richter (1972) that for functionals F of higher type the notions of being bounded and witnessed, and thence of being *n-regular*, "are defined in a similar spirit." However, no such definition was spelled out in Richter and Aczel (1974), and in fact the reader of that was referred back to the earlier article for precise definitions (cf. op.cit., p. 333). When I asked Richter years later for the details of the definition of what it means to be n-regular for n > 2 he could only give them to me for n = 3, and that proved to be quite complicated; I was never given the general definition and, as far as I know, that has never been published. At any rate, the main theorem stated by Aczel and Richter is that

[1] The difference in aims is that here I am concerned directly with properties of admissible ordinals rather than indirectly as in Feferman (2009) via the axiomatic theory OST.

κ is n+1-regular iff κ is strongly Π^1_n-indescribable.

This is proved for n = 1 in Richter and Aczel (1974), pp. 329-332. The notion of κ being *strongly* Π^1_n-*indescribable* was defined in Lévy (1971); it is given in terms of reflection of Π^1_n properties φ (with subsets of κ as possible parameters) from V_κ down to V_α, for some $\alpha < \kappa$. (Incidentally, the proof also uses the notion of κ being Π^1_n-*indescribable* (without the 'strongly') that is given in terms of reflection of such φ from κ down to some $\alpha < \kappa$.)

Now for the admissible analogues, Aczel and Richter consider admissible $\kappa > \omega$, and explain a notion of κ being n-*admissible* for n > 0 as, "roughly speaking [being] obtained from that of n-*regular* by replacing in the latter, *bounded* by [κ-]*recursive* and replacing the functions by their Gödel numbers." No motivation is given for this change, but presumably it is that κ-recursive functions are automatically bounded by the very nature of recursive definition. The main theorem stated for this is that for n > 1,

κ is n-admissible iff κ is Π^0_{n+1} reflecting.

This is proved for n = 2 only in Richter and Aczel (1974) pp. 332-333. At any rate, this statement leads them to propose the least Π^0_{n+1}-reflecting ordinal as the analogue of the first [strongly] Π^1_n-indescribable cardinal, and that analogy has been followed in the subsequent literature.

What I shall do here is propose an alternative to the Aczel-Richter notions of boundedness, witnessing and regularity that is easily spelled out for arbitrary n via generalizations to both set theory and admissible recursion theory of the notions of *continuous functionals of finite type* from ordinary recursion theory. The latter was developed in two equivalent but rather different looking ways by Kleene [1959] and Kreisel [1959]; Kleene called them *countable functionals.* Kreisel's formulation is in certain ways the conceptually superior one but his arguments for its main results are sketchy. By comparison, Kleene's formulation is notationally simpler and his proofs are given in full detail, so that is the one that I mainly follow here. A key point of difference from the Aczel and Richter approach is that we deal here only with objects of [*pure*] *type* *n over* κ, where the objects of type 0 are simply the ordinals less than κ, those of type 1 are the functions from κ to κ, and the objects F^n of type n for $n > 1$ are those functionals that map objects of type $n-1$ to objects of type 0.

Consider first the set-theoretical setting, again assuming κ to be an uncountable regular cardinal. Denote by $\kappa^{<\kappa}$ the set of sequences $s: \alpha \to \kappa$ for arbitrary $\alpha < \kappa$. Since κ is regular, we can choose a function $\pi: \kappa^{<\kappa} \to \kappa$ that is one-one and onto; in other words, π codes bounded sequences by ordinals, analogously to the coding in o.r.t. of finite sequences by sequence numbers. For g of type 1 over κ and $\alpha < \kappa$, write $g\lceil\alpha$ for the restriction of g to α; thus $\pi(g\lceil\alpha)$ is an ordinal that represents it. Now define by induction on $n > 0$, what it means for *a function f of type* 1 to be *an associate of a functional* F *of type n*, and what it means for F to be in the class B_n of *bounded functionals of type* n, as follows:

1. For n = 1, f is an associate of F iff f = F.

2. For n >1, f is an associate of F iff for every G in B_{n-1} and every associate g of G,
(i) $(\exists\alpha, \beta < \kappa)\ [f(\pi(g\lceil\alpha)) = \beta + 1]$, and
(ii) $(\forall\alpha, \beta < \kappa)\ [f(\pi(g\lceil\alpha)) = \beta + 1 \Rightarrow F(G) = \beta]$.

Then F is in B_n iff F has some associate f.

Next, we define for F in B_n and $\alpha < \kappa$, *α is a witness for F*, again by induction on n, as follows:

1. For n = 1, and F = f, α is a witness for F iff $f : \alpha \to \alpha$.

2. For n > 1, α is a witness for F iff $(\forall G \in B_{n-1})[\ \alpha$ a witness for $G \Rightarrow F(G) < \alpha\ ]$.

Finally, κ is defined to be *B_n-reg* for n > 1 iff every F in B_n has some witness $\alpha < \kappa$.

Conjectures:

(C1) For each $n > 1$, the predicate—f is an associate of some F in B_n—is definable in Π_{n-1} form.[2]

(C2) For each $n > 1$, κ is B_n-reg iff κ is strongly Π_{n-1}-indescribable.

Let's turn now to admissible $\kappa > \omega$. When formulating analogues in (κ-) recursion theory of statements concerning functions over a regular cardinal, we replace functions of type 1 by indices γ of (total) recursive functions $\{\gamma\}$. But then at type 2 we must restrict to those functions $\{\gamma\}$ that act on indices considered extensionally, that is that are such that if $\{\xi\} = \{\eta\}$ then $\{\gamma\}(\xi) = \{\gamma\}(\eta)$. In analogy to Kreisel (1959) p. 117, we define the class E_n of (κ-) *effective operations of type n* and the relation $\equiv_n$ by induction on $n > 0$, as follows:

1. E_1 consists of all indices γ of recursive functions; $\gamma \equiv_1 \delta$ iff for all ξ, $\{\gamma\}(\xi) = \{\delta\}(\xi)$.

2. For $n > 1$, E_n consists of all γ such that $\{\gamma\}: E_{n-1} \to \kappa$ and are such that for all ξ, η in E_{n-1}, if $\xi \equiv_{n-1} \eta$ then $\{\gamma\}(\xi) = \{\gamma\}(\eta)$; $\gamma \equiv_{n-1} \delta$ iff for all ξ in E_{n-1}, $\{\gamma\}(\xi) = \{\delta\}(\xi)$.

Note that the notions of the class B_n, of being an associate of a functional in B_n, and of being a witness α for such a functional, all make sense over the given admissible κ, taking for the sequence coding operation π a recursive function via a recursive one-one mapping of L_κ onto κ. In analogy to Kreisel (1959) p. 117,[3] we are thus led to make the following conjecture:

(C3) Every effective operation of type n is the restriction of a functional in B_n.

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[2] This is the analogue of Kleene (1959), Cor. 3, p.89.

[3] Kreisel's statement for the effective operations in o.r.t. is that each member of E_n is equivalent in a suitable sense to a continuous functional of type n. For $n = 2$ this is a modified form of the Myhill-Shepherdson theorem that was obtained in Kreisel, Lacombe and Shoenfield (1957). Kreisel says that the general case follows from that together with Theorem 1 of Kreisel, Lacombe and Shoenfield (1959), combined with the density theorem sketched in the appendix of Kreisel (1959). A full proof of the density theorem and thence of Kreisel's statement seems only to have been given in Harrison (1963). I would hope that a proof of (C3) could be carried out by analogous means.

Assuming (C3), if we translate the property of being B_n-reg into κ-recursion theory, the boundedness hypothesis is automatically satisfied. In other words, its translation simply comes down to saying that each effective operation has a witness. More precisely, we define by induction on n:

1. For γ in E_1, α is a witness for γ iff $\{\gamma\}: \alpha \rightarrow \alpha$.

2. For γ in E_n when $n > 1$, α is a witness for γ iff for each ξ in E_{n-1}, if α is a witness for ξ then $\{\gamma\}(\xi) < \alpha$.

Then one would call κ E_n-*admissible* if each γ in E_n has some witness $\alpha < \kappa$. But this is equivalent to κ being n-admissible in the sense of Aczel and Richter, and so one can use their result to conclude that for $n > 0$, κ is E_n-admissible iff it is Π^0_{n+1} reflecting.

Note that the present approach leaves open the question for admissible ordinals as to what is the proper analogue, if any, of a cardinal κ being Π^m_n-indescribable for $m > 1$.

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