The impact of the incompleteness theorems on mathematics

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In addition to this being the centenary of Kurt Gödel's birth, January marked 75 years since the publication (1931) of his stunning incompleteness theorems. Though widely known in one form or another by practicing mathematicians, and generally thought to say something fundamental about the limits and potentialities of mathematical knowledge, the actual importance of these results for mathematics is little understood. Nor is this an isolated example among famous results. For example, not long ago, Philip Davis wrote me about what he calls The Paradox of Irrelevance: "There are many math problems that have achieved the cachet of tremendous significance, e.g. Fermat, 4 color, Kepler's packing, Gödel, etc. Of Fermat, I have read: 'the most famous math problem of all time.' Of Gödel, I have read: 'the most mathematically significant achievement of the 20th century.' … Yet, these problems have engaged the attention of relatively few research mathematicians—even in pure math." What accounts for this disconnect between fame and relevance? Before going into the question for Gödel's theorems, it should be distinguished in one respect from the other examples mentioned, which in any case form quite a mixed bag. Namely, each of the Fermat, 4 color, and Kepler's packing problems posed a stand-out challenge following extended efforts to settle them; meeting the challenge in each case required new ideas or approaches and intense work, obviously of different degrees. By contrast, Gödel's theorems were simply unexpected, and their proofs, though requiring novel techniques, were not difficult on the scale of things. Setting that aside, my view of Gödel's incompleteness theorems is that their relevance to mathematical logic (and its offspring in the theory of computation) is paramount; further, their philosophical relevance is significant, but in just what way is far from settled; and finally, their mathematical relevance outside of logic is very much unsubstantiated but is the object of ongoing, tantalizing efforts. My main purpose here is to elaborate this last assessment.

Informal and formal axiom systems

One big reason for the expressed disconnect is that Gödel's theorems are about formal axiom systems of a kind that play no role in daily mathematical work. Informal axiom systems for various kinds of structures are of course ubiquitous in practice, viz. axioms for groups, rings, fields, vector spaces, topological spaces, Hilbert spaces, etc., etc.; these axioms and their basic consequences are so familiar it is rarely necessary to appeal to them explicitly, but they serve to define one's subject matter. They are to be contrasted with foundational axiom systems for the "mother" structures--the natural numbers (Peano) and the real numbers (Dedekind)--on the one hand, and for the general concepts of set and function (Zermelo-Fraenkel) used throughout mathematics, on the other. Mathematicians may make explicit appeal to the principle of induction for the natural numbers or the least upper bound principle for the real numbers or the axiom of choice for sets, but reference to foundational axiom systems in practice hardly goes beyond that.

One informal statement of the basic Peano axioms for the natural numbers is that they concern a structure (N, 0, s) where 0 is in N, the successor function s is a unary one-one map from N into N which does not have 0 in its range, and the Induction Principle is satisfied in the following form:

(IP) for any property P(x), if P(0) holds and if for all x in N,
P(x) implies P(s(x)) then for all x in N, P(x) holds.

But this is too indefinite to become the subject of precise logical studies, and for that purpose one needs to say exactly *which* properties P are admissible in (IP), and to do *that* one needs to specify a formal language L within which we can single out a class of well-formed formulas (wffs) A which express the admitted properties. And to do *that* we have to prescribe a list of basic symbols and we have to say which finite sequences of basic symbols constitute well-formed terms and which constitute wffs. Finally, we have to specify which wffs are axioms (both logical and non-logical), and which relations between wffs are instances of rules of inference. The wffs without free variables are those that constitute definite statements and are called the closed formulas or sentences of L. All of this is what goes into specifying a formal axiom system S.

In the case of a formal version of the Peano axioms, once its basic symbols are specified, and the logical symbols are taken to be $\neg$ ("not"), $\wedge$ ("and"), $\vee$ ("or"), $\rightarrow$ ("implies"), $\forall$ ("for all"), and $\exists$ ("there exists"), one puts in place of the Induction Principle an Induction Axiom Scheme:

(IA) $A(0) \wedge \forall x(A(x) \rightarrow A(s(x))) \rightarrow \forall x\, A(x)$,

where A is an arbitrary wff of the language L and A(t) indicates the result of substituting the term t for all free occurrences of the variable x in A.

N. B. (IA) is not a single axiom but an infinite collection of axioms, each instance of which is given by some wff A of our language.

But what about the basic vocabulary of L? Besides zero and successor, nothing of number-theoretical interest can be derived without expanding it to include at least addition and multiplication. As shown by Dedekind, the existence of those operations as given by their recursive defining conditions can be established using (IP) applied to predicates P involving quantification over functions. But for a formal axiom system PA ("Peano Arithmetic") for elementary number theory in which one quantifies only over numbers, one needs to posit those operations at the outset. The basic vocabulary of PA is thus taken to consist of the constant symbol 0 and the operation symbols s, + and $\times$ together with the relation symbol =. Then the axioms indicated above for zero and successor are supplemented by axioms giving the recursive characterizations of addition and multiplication, namely: $x + 0 = x$, $x + s(y) = s(x + y)$, $x \times 0 = 0$, and $x \times s(y) = (x \times y) + x$.

Consistency, completeness and incompleteness

All such formal details are irrelevant to the working mathematician's use of arguments by induction on the natural numbers, but for the logician, the way a formal system S is specified can make the difference between night and day. This is the case, in particular, concerning the questions whether S is *consistent*, i.e. no contradiction is provable from S, and whether S is *complete*, i.e. every sentence A is *decided by S* in the sense that either S proves A or S proves $\neg$A. If neither A nor $\neg$A is provable in S then A is said to be *undecidable by S*, and S is said to be *incomplete.*

As an example of how matters of consistency and completeness can change dramatically depending on the formalization taken, consider the subsystem of PA obtained by restricting throughout to terms and formulas that do not contain the multiplication symbol $\times$. That system, sometimes called Presburger Arithmetic, was shown to be complete by Moses Presburger in 1928, and his proof of completeness also gives a finite combinatorial proof of its consistency.[1] Gödel's discovery in 1931 was that we have a radical change when we move to the full axiom system PA. What Gödel showed was that PA is *not* complete and that, unlike Presburger Arithmetic, its consistency *cannot* be established by finite combinatorial means, at least not those that can be formalized in PA. Before going into the mathematical significance of these results, let us take a closer look at how Gödel formulated and established them not only for PA, but also for a very wide class of its extensions S.[2] To do this he showed that the language of PA is much more expressively complete than appears on the surface. A primitive recursive (p.r.) function on N (in any number of arguments) is a function generated from zero and successor both by explicit definition and definition by recursion along N. A p.r. relation (which may be

[1] Presburger's work was carried out as an "exercise" in a seminar at the University of Warsaw run by Alfred Tarski. His proof applies the method of elimination of quantifiers to show that every formula is equivalent to a propositional combination of congruences. At its core it makes use of the Chinese Remainder Theorem giving conditions for the existence of solutions of simultaneous congruences.

[2] Gödel's initial statement of his results was for extensions of a variant P of the system of *Principia Mathematica*, but a year later he announced his results more generally for a system like PA in place of P; no new methods of proof were required. Nowadays it is known that much weaker systems than PA suffice for his results.

unary, i.e. a set) is a relation whose characteristic function is p.r. Gödel showed that every p.r. function is definable in the language of PA, and its defining equations can be proved there. For example, the operations of exponentiation, x^y, the factorial, x!, and the sequence of prime numbers, p_x, each of which is p.r., can all be represented in this way in PA, facts that are not at all obvious.[3] Each instance of a p.r. relation is decidable by PA; for example if R is a binary p.r. relation then for each n, m $\in$ N, either PA proves R(n, m) or it proves $\neg$R(n, m).

Gödel's incompleteness theorems

To apply these notions to the language and deductive structure of PA, Gödel assigned natural numbers to the basic symbols. Then any finite sequence σ of symbols gets coded by a number $\#\sigma$, say, using prime power representation; $\#\sigma$ is nowadays called the Gödel number (g.n.) of σ. A relation R between syntactic objects (terms, formulas, etc.) is said to be p.r. if the corresponding relation between g.n.'s is p.r. For example, with a basic finite vocabulary, the sets of terms and wffs are both p.r. Finally a formal system S for such a language is said to be p.r. if its set of axioms and its rules of inference are both p.r. The formal system PA and its subsystems defined above are all p.r.

Throughout the following, S is assumed to be any p.r. formal system that extends PA. Denote by $\mathrm{Proof}_S(x, y)$ the relation which holds just in case y is the g.n. of a proof in S of a wff with g.n. x. Then $\mathrm{Proof}_S(x, y)$ is p.r. Using its definition in PA, the formula $\exists y\, \mathrm{Proof}_S(x, y)$ expresses that x is the g.n. of a provable formula; this is denoted $\mathrm{Prov}_S(x)$. Finally, for each wff A, $\mathrm{Prov}_S(\#A)$ expresses in the language of PA that A is provable. By a diagonal argument, Gödel constructed a closed wff D_S which is provably equivalent in PA to $\neg\mathrm{Prov}_S(\#D_S)$; more loosely, "$D_S$ says of itself that it is not provable in S." And, indeed,

(1) if S is consistent, D_S is not provable in S.

[3] The way that is done might interest number-theorists; see Franzén (2004), Ch. 4, for an exposition.

Hence, in ordinary informal terms, D_S is true, so S cannot establish all arithmetical truths. This is one way that Gödel's first incompleteness is often stated, but actually (1) is only the first part of the way that *he* stated it. For that we need a few more slightly technical notions. A sentence A of the language of PA is said to be in ∃-form if, up to equivalence, it is of the form $\exists y\, R(y)$ where R defines a p.r. set, and A is said to be in ∀-form if, up to equivalence, $\neg A$ is in ∃-form, or what comes to the same, if A can be expressed in the form $\forall y\, R_1(y)$ with R_1 p.r.[4] Thus D_S is in ∀-form and its negation is in ∃-form. S is said to be *1-consistent* if every ∃-sentence provable in S is true. Automatically, every 1-consistent system is consistent, but the converse is not true: by (1), if S is consistent it remains consistent when we add $\neg D_S$ to it as a new axiom, and the resulting system is not 1-consistent. The following is Gödel's first incompleteness theorem:[5]

(2) if S is 1-consistent then D_S is undecidable by S; hence S is not complete.

It turns out that only the first part, (1), is needed for his second incompleteness theorem. Let C be the sentence $\neg(0 = 0)$, so S is consistent if and only if C is not provable in S; this is expressed by the ∀-sentence $\neg Prov_S(\#C)$, which is denoted Con_S. By formalizing the proof of (1) it may be shown that the formal implication $Con_S \rightarrow \neg Prov_S(\#D_S)$ is provable in PA. But since $\neg Prov_S(\#D_S) \rightarrow D_S$ in PA by the diagonal construction, we have $Con_S \rightarrow D_S$ too. Hence:

(3) if S is consistent then S does not prove Con_S.

That is Gödel's second incompleteness theorem. Its impact on Hilbert's consistency program has been much discussed by logicians and historians and philosophers of

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[4] Standard logical terminology for ∃-form and ∀-form is $\sum^0{}_1$-form and $\prod^0{}_1$-form, resp. It should be noted that the formulas in ∃-form are closed under existential quantification and those in ∀-form under universal quantification. That is, like quantifiers can be collapsed to a single one of that type.

[5] Gödel used a stronger, purely syntactic, assumption in place of 1-consistency, that he called ω-consistency.

mathematics and will not be gone into here, except to say that it is generally agreed that the program as originally conceived cannot be carried out for PA or any of its extensions. However, various modified forms of the program have been and continue to be vigorously pursued within the area of logic called proof theory, inaugurated by Hilbert as the tool to carry out his program. I recommend Zach (2003) (readily accessible online) for an excellent overview and introduction to the literature on Hilbert's program.

Gödel's theorems and unsettled mathematical problems

With this background in place we can now return to the question of the impact of the incompleteness theorems on mathematics; in that respect it is mainly the first incompleteness theorem that is of concern, and indeed only the first part of it, namely (1). A common complaint about this result is that it just uses the diagonal method to "cook up" an example of an undecidable statement. What one would really like to show undecidable by PA or some other formal system is a natural number-theoretical or combinatorial statement of prior interest. The situation is analogous to Cantor's use of the diagonal method to infer the existence of transcendental numbers from the denumerability of the set of algebraic numbers; however, that did not provide any natural example. The existence of transcendentals had previously been established by an explicit but artificial example by Liouville. Neither argument helped to show that e and π, among other reals, are transcendental, but they did at least show that questions of transcendence are non-vacuous. Similarly, Gödel's first incompleteness theorem shows that the question of decidability of sentences by PA or any one of its consistent extensions is non-vacuous. That suggests looking for natural arithmetical statements which have resisted attack so far to try to see whether that is because they are not decided by systems that formalize a significant part of mathematical practice, and in particular to look for such statements in $\forall$-form. An obvious candidate for a long time was the Fermat conjecture; now that we know it is true, it would be interesting to see just what principles are needed to establish it from a logical point of view. Some logicians have speculated that it has an elementary proof that can be formalized in PA, but we don't have any evidence so far to settle this one way or the other. Another obvious candidate is the Goldbach conjecture; indeed, Gödel often referred to his independent statements as being "of Goldbach type", by which he simply meant that they are both expressible in ∀-form. A far less obvious candidate is the Riemann Hypothesis; Georg Kreisel showed that this is equivalent to a ∀-statement (see Davis, Matijasevic and Robinson (1976), p. 335 for an explicit presentation of such a sentence). No example like these has been shown to be independent of PA or any of its presumably consistent extensions .

Unsolvable diophantine problems

Consider any system S containing PA that is known or assumed to be consistent and suppose that A is a ∀-sentence conjectured to be undecidable by S. It turns out that proving its undecidability would automatically show A to be true, since ¬A is equivalent to an ∃-sentence $\exists y\, R(y)$; thus if ¬A were true there would be an n such that R(n) is true, hence provable in PA and thence in S. So, finally, ¬A would be provable in S, contradicting the supposed undecidability of A by S. The odd thing about this is that if we want to show a ∀-sentence undecidable by a given S, we better expect it to be true. And if we can show it to be true by one means or another, who cares (other than the logician who is interested in exactly what depends on what) whether it can or can't be proved in S? Still, the first incompleteness theorem is tantalizing for its prospects in this direction. The closest that one has come is due to the work of Martin Davis, Hilary Putnam, Julia Robinson and Yuri Matiyasevich resulting, finally, in the latter's negative resolution of Hilbert's 10^{th} Problem on the algorithmic solvability of diophantine equations (cf. Matiyasevich 1993). It follows from their work that every ∃-sentence is equivalent in PA to the existence of natural numbers $x_1, \ldots, x_n$ such that $p(x_1,\ldots,x_n) = 0$, where p is a suitable polynomial with integer coefficients. So each ∀-sentence A is equivalent to the non-solvability of a suitable diophantine equation, in particular, sentences known to be undecidable in particular systems such as the Gödel sentences D_S. The trouble with this result compared to open questions in the literature about the solvability of specific diophantine equations in two or three variables or of low degree is that the best value known for the above representation in terms of number of variables is $n = 9$, and in terms of degree d with a much larger number of variables is $d = 4$ (cf. Jones 1982).

Combinatorial independence results

Things look more promising if we consider $\forall\exists$-sentences, i.e. those that can be brought to the form $\forall x \exists y\, R(x, y)$ with R p.r.[6] The statement that there are infinitely many y's satisfying a p.r. condition P(y) is an example of a $\forall\exists$-sentence, since it is expressed by $\forall x \exists y(y > x \wedge P(y))$. In particular, the twin prime conjecture has this form. Again, no problems of prior mathematical interest that are in $\forall\exists$-form have been shown to be undecidable in PA or one of its extensions. However, in 1977, Jeff Paris and Leo Harrington published a proof of the independence from PA of a modified form of the Finite Ramsey Theorem. The latter says that for each n, r and k there is an m such that for every r-colored partition π of the n-element subsets of $M = \{1,\ldots,m\}$ there is a subset H of M of cardinality at least k such that H is homogeneous for π, i.e., all n-element subsets of H are assigned the same color by π. The Paris-Harrington modification consists in adding the condition that $\mathrm{card}(H) \geq \min(H)$. It may be seen that this statement, call it PH, is in $\forall\exists$-form. Their result is that PH is not provable in PA. But they also showed that PH is true, since it is a consequence of the Infinite Ramsey Theorem. The way that PH was shown to be independent of PA was to prove that it implies Con_{PA}; in fact, it implies the formal statement of the 1-consistency of PA. That work gave rise to a number of similar independence results for stronger systems S, in each case yielding a $\forall\exists$-sentence A_S which is a variant of a combinatorial result already in the literature such that A_S is true but unprovable from S on the assumption that S is 1-consistent. The proof consists in showing that A_S implies (and is in some cases equivalent to) the formal statement of its 1-consistency. However, no example is known of an unprovable $\forall\exists$-sentence whose truth has been a matter of prior conjecture.

Set theory and incompleteness

Gödel signaled a move into more speculative territory in footnote 48a (evidently an afterthought) of his 1931 paper:

[6] Standard logical terminology for these is $\prod^0_2$ sentences.

> As will be shown in Part II of this paper, the true reason for the incompleteness inherent in all formal systems of mathematics is that the formulation of ever higher types can be continued into the transfinite… For it can be shown that the undecidable propositions constructed here become decidable whenever appropriate higher types are added… An analogous situation prevails for the axiom system of set theory.[7]

The reason for this, roughly speaking, is that for each system S the notion of truth for the language of S can be developed in an axiomatic system S′ for the subsets of the domain of interpretation of S; then in S′ one can prove by induction that the statements provable in S are all true, and hence that S is consistent. Implicit in the quote is that S′ ought to be accepted if S is accepted. Later, in his famous article on Cantor's Continuum Problem (1947), Gödel pointed to the need for new set-theoretic axioms to settle specifically set-theoretic problems, such as the Continuum Hypothesis (CH).

At that point, it was only known as a result of his earlier work that AC (the Axiom of Choice) and CH are consistent relative to Zermelo-Fraenkel axiomatic set theory ZF.[8] In Gödel's 1947 article he argued that CH is a definite mathematical problem and, in fact, he conjectured that it is false while all the axioms of ZFC (= ZF + AC) are true. Hence CH must be independent of ZFC; he thus concluded that one will need new axioms to determine the cardinal number of the continuum. In particular, he suggested for that purpose the possible use of axioms of infinity not provable in ZFC:

[7] Part II of Gödel (1931) never appeared. Also promised for it was a full proof of the second incompleteness theorem, the idea for which was only indicated in Part I. He later explained that since the second incompleteness theorem had been readily accepted there was no need to publish a complete proof. Actually, the impact of Gödel's work was not so rapid as this suggests; the only one who immediately grasped the first incompleteness theorem was John von Neumann, who then went on to see for himself that the second incompleteness theorem must hold. Others were much slower to absorb the significance of Gödel's results (cf. Dawson 1997, pp. 72-75.) The first detailed proof of the second incompleteness theorem for a system Z equivalent to PA appeared in Hilbert and Bernays (1939).

[8] See Floyd and Kanamori (this issue).

> The simplest of these … assert the existence of inaccessible numbers… The latter axiom, roughly speaking, means nothing else but that the totality of sets obtainable by exclusive use of the processes of formation of sets expressed in the other axioms forms again a set (and, therefore, a new basis for a further application of these processes). Other axioms of infinity have been formulated by P. Mahlo. … These axioms show clearly, not only that the axiomatic system of set theory as known today is incomplete, but also that it can be supplemented without arbitrariness by new axioms which are only the natural continuation of those set up so far. (Gödel 1947, as reprinted in 1990, p. 182).

Whether or not one agreed with Gödel about the ontological underpinnings of set theory and in particular about the truth or falsity of CH, in the following years it was widely believed to be independent of ZFC; that was finally demonstrated in 1963 by Paul Cohen using a new method of building models of set theory. And, contrary to Gödel's expectations, it has subsequently been shown by an expansion of Cohen's method that CH is undecidable in every plausible extension of ZFC that has been considered so far, at least along the lines of Gödel's proposal (cf. Martin 1976 and Kanamori 2003). For the most recent work on CH, see the conclusion of Floyd and Kanamori (2006).

But what about arithmetical problems? For a number of years, Harvey Friedman has been working to produce mathematically perspicuous finite combinatorial $\forall\exists$-statements A whose proof requires the use of many Mahlo cardinals and even stronger axioms of infinity (such as the so-called subtle cardinals) and has come up with a variety of candidates; for a fairly recent report, including work in progress, see Friedman (2000).[9] The strategy for establishing that such a statement A needs a system S incorporating the strong axioms in question is like that above: one shows that A is equivalent to (or in certain cases is slightly stronger than) the 1-consistency of S. In my discussion of this in Feferman (2000), p. 407, I wrote: "In my view, it is begging the question to claim this

[9] More recently, Friedman has announced the need for such large cardinal axioms in order to prove a certain combinatorial statement A that can be expressed in $\forall$-form; see the final section of Davis (this issue). Here, A implies Con(S) and is itself provable in S′ for S and S′ embodying suitable large cardinal axioms.

shows we need axioms of large cardinals in order to demonstrate the truth of such A, since this only shows that we 'need' their 1-consistency. However plausible we might find the latter for one reason or another, it doesn't follow that we should accept *those axioms themselves* as first-class mathematical principles." (Cf. also op. cit. p. 412).

Prospects and practice

As things stand today, these explorations of the set-theoretical stratosphere are clearly irrelevant to the concerns of most working mathematicians. A reason for this was also given by Gödel near the outset of his 1951 Gibbs lecture (posthumously published in 1995), where he said that the "phenomenon of the inexhaustibility of mathematics" follows from the fact that "the very formulation of the axioms [of set theory over the natural numbers] up to a certain stage gives rise to the next axiom. It is true that in the mathematics of today the higher levels of this hierarchy are practically never used. It is safe to say that 99.9% of present-day mathematics is contained in the first three levels of this hierarchy." In fact, modern logical studies have shown that the system corresponding to the second level of this hierarchy, called second-order arithmetic or analysis and dealing with the theory of sets of natural numbers, already accounts for the bulk of present-day mathematics. Indeed, much weaker systems suffice, as is demonstrated in Simpson (1999). Even more, I have conjectured and verified to a considerable extent that all of current scientifically applicable mathematics can be formalized in a system that is proof-theoretically no stronger than PA (cf. Feferman 1998, Ch. 14).

Whether or not the kind of inexhaustibility of mathematics discovered by Gödel is relevant to present-day pure and applied mathematics, there is a different kind of inexhaustibility which is clearly significant for practice: no matter which axiomatic system S is taken to underlie one's work at any given stage in the development of our subject, there is a potential infinity of propositions that can be demonstrated in S, and at any moment, only a finite number of them have been established. Experience shows that significant progress at each such point depends to an enormous extent on creative ingenuity in the exploitation of accepted principles rather than essentially new principles. But Gödel's theorems will always call us to try to find out what lies beyond them.

Note to the reader: Gödel's incompleteness paper (1931) is a classic of its kind; elegantly organized and clearly presented, it progresses steadily and efficiently from start to finish, with no wasted energy. The reader can find it in the German original along with a convenient facing English translation in Vol. I of his *Collected Works* (1986). I recommend it highly to all who are interested in this landmark in the history of our subject.

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