Math 177: Homework N2
Solutions

1. Recall that if X is a smooth vector field and f is a smooth function, the directional derivative of f along X is the smooth function Xf defined by $(Xf)(p) = df_p(X(p))$. This operation satisfies the product rule: $X(fg) = (Xf)g + f(Xg)$. Moreover, if Y is another smooth vector field, we have that $X = Y$ if and only if $Xf = Yf$ for all f.

a) For any smooth function f, we have

$$\begin{aligned}[X,[Y,Z]]f &= X([Y,Z]f) - [Y,Z](Xf)\\ &= X(Y(Zf) - Z(Yf)) - Y(Z(Xf)) + Z(Y(Xf))\\ &= (XYZ - XZY - YZX + ZYX)f.\end{aligned}$$

Similarly,

$$\begin{aligned}[Y,[Z,X]]f &= (YZX - YXZ - ZXY + XZY)f,\\ [Z,[X,Y]]f &= (ZXY - ZYX - XYZ + YXZ)f.\end{aligned}$$

Summing these up, we see that

$$([X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]])f = 0$$

for all f, which implies the Jacobi identity.

b) For any smooth function g, we have

$$\begin{aligned}[X,fY]g &= X(fYg) - fY(Xg)\\ &= (Xf)(Yg) + fX(Yg) - fY(Xg)\\ &= df(X)(Yg) + f[X,Y]g\\ &= (df(X)Y + f[X,Y])g.\end{aligned}$$

It follows that $[X,fY] = df(X)Y + f[X,Y]$.

2. To say that the flow of X is a symmetry of ℓ means that $(X^t)_*\ell = \ell$ for all t. More explicitly, this means that for any point $p \in \mathbb{R}^2$ and any vector $v \in \ell_p$, we have $d(X^t)_p(v) \in \ell_{X^t(p)}$. Since $\ell = \ker(\alpha)$, this is equivalent to $\alpha_{X^t(p)}(d(X^t)_p(v)) = 0$, which is the same thing as $((X^t)^*\alpha)_p(v) = 0$ by definition of the pullback. This shows that the flow of X is a symmetry of ℓ if and only if $\ker((X^t)^*\alpha) = \ker(\alpha)$ for all t. Two linear functionals have the same kernel if and only if they are proportional, so this is equivalent to the existence of a function $g = g(p,t)$ on $\mathbb{R}^2 \times \mathbb{R}$ such that $(X^t)^*\alpha = g\alpha$. Since X and α are smooth and α is nonvanishing, the function g is necessarily smooth. It is therefore sufficient to show that the following two conditions are equivalent:

1. $(X^t)^*\alpha = g\alpha$ for some smooth function $g = g(p,t)$ on $\mathbb{R}^2 \times \mathbb{R}$;

2. $L_X\alpha = f\alpha$ for some smooth function $f = f(p)$ on $\mathbb{R}^2$.

By definition of the Lie derivative, we have

$$\left.\frac{d}{dt}\right|_{t=0} (X^t)^*\alpha = L_X\alpha.$$

Since

$$\left.\frac{d}{dt}\right|_{t=0} (g\alpha) = \left.\frac{\partial g}{\partial t}\right|_{t=0} \alpha,$$

we have that 1 implies 2 with $f(p) = (\partial_t g)(p,0)$.

To show that 2 implies 1, first note that

$$\frac{d}{dt}(X^t)^*\alpha = (X^t)^*(L_X\alpha) = (X^t)^*(f\alpha) = \big((X^t)^*f\big)\big((X^t)^*\alpha\big) = (f\circ X^t)\big((X^t)^*\alpha\big).$$

Hence, if we fix a point $p \in \mathbb{R}^2$, a tangent vector v at p, and define a (smooth) function $h : \mathbb{R} \to \mathbb{R}$ by $h(t) = ((X^t)^*\alpha)_p(v)$, then h satisfies the ordinary differential equation

$$h'(t) = f(X^t(p))h(t)$$

with initial condition $h(0) = \alpha_p(v)$. This equation has a unique solution, namely

$$h(t) = \exp\left(\int_0^t f(X^s(p))\,ds\right)\alpha_p(v).$$

Since p and v were arbitrary, it follows that $(X^t)^*\alpha = g\alpha$ with

$$g(p,t) = \exp\left(\int_0^t f(X^s(p))\,ds\right).$$

3. It is enough to find such a curve in the fourth quadrant, since we can then obtain solutions in the other quadrants by reflecting along the coordinate axes. Note that a line in the plane will form a triangle with the coordinate axes in the fourth quadrant if and only if its slope is positive and its y-intercept is negative. It is then given by an equation of the form $y = px - q$ with $p > 0$, $q > 0$. Since we want the area of the triangle to be equal to $2a^2$, we must have

$$\frac{1}{2}q\frac{q}{p} = 2a^2,$$

i.e. $q = 2a\sqrt{p}$. Hence, we are looking for a curve which is tangent to the family of lines $\{y = px - g(p)\}$, where $g(p) = 2a\sqrt{p}$. Since g is strictly concave, its Legendre transform gives us such a curve (see Arnold's book, p. 20). It is given by

$$f(x) = \inf_p (xp - g(p)) = x\hat{p} - g(\hat{p})$$

where $\hat{p}$ satisfies $g'(\hat{p}) = x$. Since $g'(p) = a/\sqrt{p}$, we have $\hat{p} = a^2/x^2$, so that

$$f(x) = x\frac{a^2}{x^2} - 2a\frac{a}{x} = -\frac{a^2}{x}.$$

The curve $y = -a^2/x$ therefore gives us a solution in the fourth (and second, by symmetry) quadrant. By reflecting across the x-axis, we see that the curve $y = a^2/x$ is a solution in the first and third quadrant.

4. Let $\langle , \rangle$ and ω denote the standard inner product and the standard symplectic form on $\mathbb{R}^{2n}$ respectively, and let h denote the complex-valued bilinear form on $\mathbb{R}^{2n}$ corresponding to the standard Hermitian form on $\mathbb{C}^n$ under the identification

$$\mathbb{R}^{2n} \cong \mathbb{C}^n$$
$$(x, y) \mapsto x + iy$$

(where $x, y \in \mathbb{R}^n$). Let J denote the linear transformation $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ corresponding to multiplication by i on $\mathbb{C}^n$ under this identification. An easy computation shows that

$$h(v, w) = \langle v, w \rangle - i\omega(v, w)$$

and

$$g(v, w) = \omega(v, Jw), \quad \omega(v, w) = g(Jv, w)$$

for all $v, w \in \mathbb{R}^{2n}$.

Let $A \in U(n)$. By definition, this means that $h(Av, Aw) = h(v, w)$ for all $v, w \in \mathbb{R}^{2n}$. Comparing real and imaginary parts, we see that this is equivalent to $\langle Av, Aw \rangle = \langle v, w \rangle$ and $\omega(Av, Aw) = \omega(v, w)$. The first equation means that $A \in O(2n)$, and the second means that $A \in Sp(2n)$. Thus

$$U(n) = O(2n) \cap Sp(2n).$$

If $A \in O(2n) \cap GL(n, \mathbb{C})$, then A preserves $\langle , \rangle$ and $AJ = JA$, so

$$\omega(Av, Aw) = \langle JAv, Aw \rangle = \langle AJv, Aw \rangle = \langle Jv, w \rangle = \omega(v, w)$$

for all $v, w \in \mathbb{R}^{2n}$. Thus $A \in Sp(2n)$. Since we already know that $U(n) = O(2n) \cap Sp(2n)$ and $U(n) \subset GL(n, \mathbb{C})$, this shows that $U(n) = O(2n) \cap GL(n, \mathbb{C})$.

Similarly, if $A \in Sp(2n) \cap GL(n, \mathbb{C})$, we have

$$\langle Av, Aw \rangle = \omega(Av, JAw) = \omega(Av, AJw) = \omega(v, Jw) = \langle v, w \rangle$$

for all $v, w \in \mathbb{R}^{2n}$, so that $A \in O(2n)$. It follows that $U(n) = Sp(2n) \cap GL(n, \mathbb{C})$.

We have shown that

$$U(n) = O(2n) \cap Sp(2n) = O(2n) \cap GL(n, \mathbb{C}) = Sp(2n) \cap GL(n, \mathbb{C}).$$

To only thing left to prove is that these equalities remain true if we replace $O(2n)$ by $SO(2n)$. This follows from the fact that every element of $GL(n, \mathbb{C})$ preserves the standard orientation of $\mathbb{R}^{2n}$ and hence has positive determinant when viewed as a real $2n \times 2n$ matrix.

5. We want to solve the quasilinear equation

$$x\partial_x u + y\partial_y u = u - xy$$

with initial condition $u(2, y) = 1 + y^2$. The characteristic vector field is given by $A(x, y, u) = (x, y, u - xy)$ and the initial submanifold is $\Gamma = \{(2, y_0, 1 + y_0^2) \mid y_0 \in \mathbb{R}\}$ (see Arnold's book, §7). This gives us the system of ordinary differential equations

$$\dot{x} = x$$
$$\dot{y} = y$$
$$\dot{u} = u - xy$$

with initial conditions $x(0) = 2$, $y(0) = y_0$, $u(0) = 1 + y_0^2$. From the first two equations we see that $x = 2e^t$, $y = y_0 e^t$. The third equation can then be rewritten as

$$\dot{u} - u = -2y_0 e^{2t}.$$

Multiplying both sides by e^{-t}, we obtain the equation

$$\frac{d}{dt}(e^{-t}u) = -2y_0 e^t,$$

whose general solution is $u = -2y_0 e^{2t} + Ce^t$, $C \in \mathbb{R}$. The initial condition $u(0) = 1 + y_0^2$ implies that $C = 1 + 2y_0 + y_0^2 = (1 + y_0)^2$. Thus

$$u = -2y_0 e^{2t} + (1 + y_0)^2 e^t = -xy + \left(1 + \frac{2y}{x}\right)^2 \frac{x}{2} = -xy + \frac{x}{2} + 2y + \frac{2y^2}{x}.$$

6. We want to solve the quasilinear equation

$$u_t + uu_x = -x$$

with initial condition $u|_{t=0} = 0$. The characteristic vector field is $A(t, x, u) = (1, u, -x)$ and the initial submanifold is $\Gamma = \{(0, x_0, 0) \mid x_0 \in \mathbb{R}\}$. Hence, we need to solve the system of ordinary differential equations

$$\begin{aligned} \dot{t} &= 1 \\ \dot{x} &= u \\ \dot{u} &= -x \end{aligned}$$

with initial condition $t(0) = 0$, $x(0) = x_0$, $u(0) = 0$. The first equation implies that $t(s) = s + C$ for some constant C, and the initial condition tells us that $C = 0$. The second and third equations imply that $x''(s) = -x(s)$; the general solution to this equation is

$$x(s) = c_1 \cos(s) + c_2 \sin(s),$$

and we have $u(s) = x'(s) = c_2 \cos(s) - c_1 \sin(s)$. The initial conditions for x and u imply that $c_1 = x_0$ and $c_2 = 0$. Thus $u = -x_0 \sin(s) = -x_0 \sin(t)$ and $x = x_0 \cos(s) = x_0 \cos(t)$. It follows that

$$u = -x_0 \cos(t) \frac{\sin(t)}{\cos(t)} = -x \tan(t).$$

The function $\tan(t)$ is well-defined for $t \in [0, \pi/2)$, but it goes to ∞ when $t \to \pi/2$. Thus $\pi/2$ is the largest value of T for which the solution to our Cauchy problem can be extended to the interval $[0, T)$.